matrices:

$$\begin{pmatrix} 1 & \lambda b_{12} & \lambda^2 b_{13} \\ 0 & 1 & \lambda b_{23} \\ 0 & 0 & 1 \end{pmatrix},$$

where b_{12}, b_{13}, b_{23} are integers and λ a real number. Then G can be approximated by a sequence g_λ, when $\lambda \longrightarrow 0$.

The second example: G consists of matrices:

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

where a_{11}, a_{33} are real positive and a_{12} real. Such a matrix is denoted simply by (a_{11}, a_{12}, a_{33}). If (a_{11}, a_{12}, a_{33}) $(a_{11} \neq 1)$ belongs to a discrete subgroup g, then a suitable conjugate subgroup cgc^{-1} contains $a = (a_{11}, 0, a_{33})$. If cgc^{-1} contains an elements $b = (b_{11}, b_{12}, b_{33})$ $(b_{12} \neq 0)$ we make a commutator of b and a^n,

$$b a^n b^{-1} a^{-n} = (1, b_{12}(1 - a_{11}^n), 1)$$

Let n tend to $-\infty$ in the case $a_{11} > 1$, and to $+\infty$ in the case $a_{11} < 1$, then it converges to $(1, b_{12}, 1)$. Hence cgc^{-1} is not discrete and the same for g. Therefore g does not contain such a element b $(b_{12} \neq 0)$, and is commutative, which cannot approximate G.

However, our problem is completely solved for compact Lie groups:

Theorem 2. Every non-commutative compact Lie group is not approximable by finite subgroups.

Proof is easily established, if we consider the Levi decomposition of Lie groups, and the commutativity of solvable compact Lie groups.

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(1) L.Pontrjagin, Topological groups, p.170.
(2) loc.cit.p.187.
(3) loc.cit.p.236.

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AN ELEMENTARY METHOD TO DERIVE THE NORMAL FORM OF N-DIMENSIONAL REAL EUCLIDEAN ROTATION

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It is a well-known theorem that n-dimensional real orthogonal matrix A can be transformed into a direct-product of several 2-dimensional rotations and, if it exists, one reflection. In this paper an elementary geometric method is explained. The essential point of this method is to find the fixed planes of the rotation.

Let $x, y, z, \ldots, a, b, c, \ldots$ be real vectors in n-dimensional real euclidean space R_n, and let $A, B, C, \ldots$ be real orthogonal (n,n)-matrices, while small Greek letters mean real numbers. We use the ordinary symbols of matrix-calculus, i.e., a' and A' mean the transposed ones, A^{-1} is its inverse and E is the unit-matrix.

THEOREM. Let A be any real orthogonal matrix, i.e., $A' = A^{-1}$. Suppose that $x'Ax$ with $|x| = 1$, attains the maximum value λ for $x = a$, where $|a| = 1$. Then $|\lambda| \leq 1$ and

a. if $\lambda = 1$, $Aa = a$;

b. if $\lambda = -1$, $A = -E$;

c. if $-1 < \lambda < 1$, $A^2 a - 2\lambda A a + a = 0$,

i.e., a and Aa span a fixed plane of A.

Proof. Put

$$\lambda = \max_{|x|=1} x'Ax. \tag{1}$$

Then it is evident that $|\lambda| = 1$. Since the unit-sphere S^n in R_n is compact, there exists at least one vector a with $|a| = 1$, where

$$\lambda = a'Aa. \tag{2}$$

We know that $|x| \cdot |y| = |xy|$ if and only if x and y are linearly dependent. Therefore if $\lambda = 1$, $Aa = a$, i.e., a is a fixed point. If $\lambda = -1$, we have $x'Ax = -1$ for any x, $|x| = 1$, i.e., $A = -E$. Finally if $-1 < \lambda < 1$, two vectors a and Aa are linearly independent. So there exists in the plane spanned by a and Aa a non-vanishing vector $b = Aa - \lambda a$ orthogonal to a. Put

(3) $$x = a + \rho b, \quad -\infty < \rho < +\infty.$$

Then we have

(4) $$x'Ax = a'Aa + 2\rho a'(A+A')b + \rho^2 b'Ab.$$

Since $|b| \leq 2$ and $|b'Ab| \leq 2$, $a'(A+A')b$ is to be zero. For if it were not zero we could easily construct a vector y, where $|y| = 1$ and $y'Ay > \lambda$. This contradicts the fact that $\lambda = \max\limits_{|x|=1} x'Ax$ in (1). The equality $a'(A+A')b = 0$ is equivalent to

(5) $$a'A^2a - 2\lambda a'Aa + a'a = 0.$$

Put

(6) $$z = A^2a - 2\lambda Aa + a.$$

Then we have by (5)

(7) $$a'z = 0, \quad (Aa)'z = 0, \quad (A^2a)'z = 0,$$

i.e., $z'z = 0$ and consequently $z = 0$. This means that the plane spanned by a and Aa is a fixed plane. Now we get the results that we can transform A by a real orthogonal matrix into the following three forms according to the cases a., b. and c.;

a. $\lambda = 1$,

$$T'AT = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & B \\ 0 & \end{pmatrix}$$

b. $\lambda = -1$, $A = -E$

c. $-1 < \lambda < 1$,

$$T'AT = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \cdots 0 \\ \sin\theta & \cos\theta & 0 \cdots 0 \\ 0 & 0 & \\ \vdots & \vdots & C \\ 0 & 0 & \end{pmatrix}$$

where $\cos\theta = \lambda$. We can repeat the processes upon B or C to get the final form:

$$\begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & \cos\theta & -\sin\theta & & & \\ & & & \sin\theta & \cos\theta & & & \\ & & & & & \ddots & & \\ & & & & & & \begin{matrix} \cos\zeta & -\sin\zeta \\ \sin\zeta & \cos\zeta \end{matrix} & \\ & & & & & & & -1 \end{pmatrix}$$

where the final -1 which means the reflection with respect to the hyperplane appears which and only when $|A| = -1$ if every $\begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix}$ is denoted by $\begin{matrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{matrix}$.

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