by the same argument as above, we have

$$\overline{\lim_{n\to\infty}}\ \varphi_{x_n+y_n}(\alpha) \le \varphi_y(\alpha).$$

Combining this with above result, we get

$$\lim_{n\to\infty} \varphi_{x_n+y_n}(\alpha) = \varphi_y(\alpha).$$

This completes the proof.

Theorem 5. If $x(t)$ has the unit asymptotic distribution function and $y(t)$ has an asymptotic distribution function φ_y, then $x(t)+y(t)$ has also an asymptotic distribution function φ_y.

Proof. By the same way as in the proof of Theorem 4, we have

$$\overline{\lim_{T\to\infty}}\ \frac{1}{2T} m E_t[-T\le t\le T, x(t)+y(t)>\alpha]$$
$$\le 1-\varphi_y(\alpha-\varepsilon)+1-\varphi_x(\varepsilon) = 1-\varphi_y(\alpha-\varepsilon).$$

Since $\varepsilon>0$ is arbitrary, if α is a continuity point of φ_y, then we have

$$\overline{\lim_{T\to\infty}}\ \frac{1}{2T} m E_t[-T\le t\le T, x(t)+y(t)>\alpha]$$
$$\le 1-\varphi_y(\alpha),$$

that is

$$1-\underline{\lim_{T\to\infty}}\ \frac{1}{2T} m E_t[-T\le t\le T, x(t)+y(t)<\alpha]$$
$$\le 1-\varphi_y(\alpha),$$

or

$$\underline{\lim_{T\to\infty}}\ \frac{1}{2T} m E_t[-T\le t\le T, x(t)+y(t)<\alpha] \ge \varphi_y(\alpha).$$

Analogously we have

$$\overline{\lim_{T\to\infty}}\ \frac{1}{2T} m E_t[-T\le t\le T, x(t)+y(t)<\alpha] \le \varphi_y(\alpha)$$

That is

$$\lim_{T\to\infty} \frac{1}{2T} m E_t[-T\le t\le T, x(t)+y(t)<\alpha] = \varphi_y(\alpha).$$

This completes the proof.

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Tokyo High Normal School.

A NOTE ON GENERATORS OF COMPACT LIE GROUPS

By Hiraku TOYAMA and
Masatake KURANISHI.

H. Auerbach has obtained the following theorem [1] :

THEOREM: Let $\mathfrak{G}$ be a (connected) compact Lie group, and for any integer k let

$$M(x,y,k) = \{ p;\ p = \prod_{i=1}^{k} v_i,\ v_i = x^{n_i} \text{ when } i \text{ is odd},\ v_i = y^{n_i} \text{ when } i \text{ is even} \}$$

$$M(x,y) = \bigcup_{k=1}^{\infty} M(x,y,k)$$

Then there exist x and y such that $\mathfrak{G} = \overline{M(x,y)}$.

Here arises a question: Is there any integer k such that $\mathfrak{G} = \overline{M(x,y,k)}$. The affirmative answer for this question can easily be obtained. Let $f(\mathfrak{G})$ be the minimum of such k. The next problem, to determine $f(\mathfrak{G})$ for each compact Lie group, is not yet solved for the writers, but it can be seen

$$f(\mathfrak{G}) \geqslant \dim(\mathfrak{G})/\mathrm{rank}(\mathfrak{G})$$

where rank$(\mathfrak{G})$ is the dimension of a maximal abelian subgroup of $\mathfrak{G}$.

This note will contain the proofs of these two propositions.

For any element x of $\mathfrak{G}$, let $T(x)$ be the abelian closed subgroup of $\mathfrak{G}$ generated by x, and put

(1) $\quad H(x,y,k) = \{ p;\ p = \prod_{i=1}^{k} w_i,\ w_i \in T(x) \text{ when } i \text{ is odd and } w_i \in T(y) \text{ when } i \text{ is even} \}$

(2) $\quad H(x,y) = \bigcup_{k=1}^{\infty} H(x,y,k)$

Then it is clear that

(3) $\quad H(x,y,k) \subseteq \overline{M(x,y,k)}$

If $\mathfrak{G} = \overline{M(x,y)}$ and if $T(x)$ and $T(y)$ are connected, we shall say that x and y constitute a pair of generators of $\mathfrak{G}$. The existence of such x and y is proved in [1].

(1) When $\mathfrak{G}$ is simply connected: Take a pair of generators x, y of $\mathfrak{G}$. Then $H(x,y)$ is an arc-wise connected subgroup of $\mathfrak{G}$ and everywhere dense in $\mathfrak{G}$. It follows from these that $H(x,y) = \mathfrak{G}$ (for the proof see [2]). From

(2) and (3), by the sum theorem of dimension(1), there exists k such that $\dim(\mathfrak{G}) = \dim(H(x,y,k))$. Because $\mathfrak{G}$ is locally euclidean, $H(x,y,k)$ must contain an interior point p, i.e., there exists a neighborhood U of e such that $Up \subset H(x,y,k)$. As $p^{-1} \in H(x,y,k+1)$, it follows that $U \subset H(x,y,2k+1)$. Since $\mathfrak{G}$ is compact, we have $U^{\ell} = \mathfrak{G}$, for some integer ℓ.

Then $$\mathfrak{G} = H(x,y,2k\ell+\ell) = \overline{M(x,y,2k\ell+\ell)}$$

(II) When $\mathfrak{G}$ is a direct product of a simply connected compact Lie group $\mathfrak{G}_1$ and an abelian compact Lie group $\mathfrak{G}_2$. $\mathfrak{G}_2$ is generated by an element z, i.e., $\mathfrak{G}_2 = T(z)$. x and y be a pair of generators of $\mathfrak{G}_1$. Since $xz = zx$, $T(x)\cdot T(z)$ is an abelian Lie group, and so it is generated by an element w. Let $\mathfrak{G}_1 = \overline{M(x,y,k)}$. As $T(x) \subset \overline{T(x)\cdot T(z)} = \overline{T(w)}$, it is clear that $\mathfrak{G}_1 \subset \overline{M(x,y,k)} \subset \overline{M(w,y,k)}$. On the other hand $\mathfrak{G}_2 \subset T(w) \subset \overline{M(w,y,1)}$, we can see

$$\mathfrak{G} = \mathfrak{G}_2\cdot\mathfrak{G}_1 = \overline{M(w,y,k+1)}$$

(III) It is well known that any compact Lie group $\mathfrak{G}$ can be identified with a factor group $\widetilde{\mathfrak{G}}/\mathfrak{Z}$, where $\widetilde{\mathfrak{G}}$ is a direct product of a simply connected semi-simple compact Lie group and an abelian compact Lie group. Take a pair of generators x and y of $\widetilde{\mathfrak{G}}$. From (II) there exists an integer k such that $\widetilde{\mathfrak{G}} = \overline{M(x,y,k)}$. Let the images of x and y by the natural mapping $\widetilde{\mathfrak{G}} \to \widetilde{\mathfrak{G}}/\mathfrak{Z}$ be x' and y' respectively. Then it is clear that

$$\mathfrak{G} = \overline{M(x',y',k)}$$

We can introduce in the local group $\mathfrak{G}_{\ell}$ of $\mathfrak{G}$, a canonical system of coordinates of the first kind, such that

$$T(x) \cap \mathfrak{G}_{\ell} = \{p;\ p=(p_1,\dots,p_s,0,\dots,0)\}$$
$$T(y) \cap \mathfrak{G}_{\ell} = \{q;\ q=(0,\dots,0,q_i,\dots,q_{i+r},0,\dots,0)\}$$

Put $N(k) = N_1 \times N_2 \times \cdots \times N_k$, where $N_i = T(x) \cap \mathfrak{G}_{\ell}$ when i is odd, $N_i = T(y) \cap \mathfrak{G}_{\ell}$ when i is even. We can introduce a coordinates-system in $N(k)$ by the above mentioned coordinates-system of N_i. Suppose $\mathfrak{G} = H(x,y,k)$. The mapping

$$n_1 \times n_2 \times \cdots \times n_k \to n_1\cdot n_2\cdot\cdots\cdot n_k$$

of $N(k)$ on $\mathfrak{G}$ is an analytic function with respect to the coordinate-systems of $N(k)$ and of $\mathfrak{G}$. Therefore

$$\dim(\mathfrak{G}) \leqq \dim(N(k)) \leqq k \times \mathrm{rank}(\mathfrak{G})$$

i.e., $$f(\mathfrak{G}) \geqq \dim(\mathfrak{G})/\mathrm{rank}(\mathfrak{G}).$$

There are a few groups, for which the exact value of $f(\mathfrak{G})$ is easily obtained, e.g.,

(1) 3-dimensional orthogonal group:

$$f(\mathfrak{G}) = 3$$

(2) 2-dimensional unitary group [3] :

$$f(\mathfrak{G}) = 3$$

(*) Received March 5, 1949.

(1) Note that $H(x,y,k)$ is compact.

(1) H. Auerbach, Sur les groupes linéaires bornés (III). Studia Math. V. p.43-49.
(2) T. Iwamura and M. Kuranishi: On arc-wise connected subgroups of Lie groups (Forthcoming shortly).
(3) H. Tôyama, Zur Theorie der hyperabelschen Funktionen, III. Proc. Imp. Acad. Tokyo, 20 (1944), p.557-559.

Tokyo Institute of Technology.