ARC-WISE CONNECTED SUBGROUP OF A VECTOR GROUP

By Tsuyoshi HAYASHIDA.

A theorem that an arc-wise connected subgroup of the n-dimentional (real) vector group is also a vector group, was proved recently by Iwamura, Kuranishi, and Homma and Minagawa, each by different methods. In this paper I shall give another proof. I shall show, in addition, some integral equality concerning periodic functions.

It is sufficient to prove the following

Theorem 1. Let in the n-dimensional Euclidean space E^n there be n continuous curves $f_i(t)$, $0 \leqq t \leqq 1$ (vector functions) $i=1,2,\cdots,n$, $f_i(t)$ joining the origin to the end of the i-th unitvector e_i. And let each coordinate be taken modulo 1. Then the vector sum $\{f_1(t_1)+\cdots+f_n(t_n);\ 0 \leqq t_i \leqq 1\}$ covers the whole space.

Proof. The whole space E^n, each coordinate being taken modulo 1, is a torus T^n. Vector sum $F=\{f_1(t_1)+\cdots+f_n(t_n)$ (mod. 1); $0\leqq t_i \leqq 1\}$ is compact in T^n, for $f_i(t)$ are continuous in $0 \leqq t \leqq 1$ and T^n is a continuous image of E^n. If F did not cover T^n, the rest would be open and contain an open sphere (of radius δ). So if we substitute $f_i(t)$ by slightly different continuous curves $g_i(t)$ so that $|f_i(t)-g_i(t)|<\delta/n$, $G=\{g_1(t_1)+\cdots+g_n(t_n)\ (\mathrm{mod.}\ 1);\ 0\leqq t_i \leqq 1\}$ would not cover T^n, either. In particular we may take for $g_i(t)$ partially linear curves, each having a finite number of corners:

$$g_i(t)=\frac{1}{N}({}_1g_i+\cdots+{}_{k-1}g_i)+{}_kg_i\left(t-\frac{k-1}{N}\right);\ \frac{k-1}{N}\leqq t\leqq\frac{k}{N},\ k=1,\cdots,n,$$

${}_kg_i$ being constant vectors and $\frac{1}{N}({}_1g_i+\cdots+{}_Ng_i)=e_i$. In that case we shall get the following identity. Let $A(x^1,\cdots,x^n)$ be any continuous function on T^n, then

$$(1)\quad \int_{T^n}A(x^1,\cdots,x^n)\,dx^1\cdots dx^n=\int_{0\leqq t_i\leqq 1}A(x^1,\cdots,x^n)\,dg_1(t_1)\cdots dg_n(t_n)$$

where on the right side $x^j=g_1^j(t_1)+\cdots+g_n^j(t_n)$. The integral on the right side is to be taken as the sum of N^n Riemannian integrals of the type

$$I(k_1,\cdots,k_n)=\int_{\frac{k_i-1}{N}\leqq t_i\leqq\frac{k_i}{N}}A(x^1,\cdots,x^n)\,\det|{}_{k_i}g_i^j|\,dt_1\cdots dt_n$$

We have only to show (1) in special cases where we take $A(x^1,\cdots,x^n)=e^{2\pi i(l_1x^1+\cdots+l_nx^n)}$, ($\{l_i\}$ being any set of integers), on account of the completeness of the trigonometric functions (for an arbitrarily given ε, $A(x^1,\cdots,x^n)$ is approximated uniformly by some trigonometric polynomial). But these are verified by straightforward calculation:

(we put $\sum_{j=1}^{n} l_j\,{}_{k_i}g_i^j={}_{k_i}h_i$ and $\sum_{k=1}^{t}{}_kh_i={}_tH_i$).

$$\int_{0\leqq t_i\leqq 1}e^{2\pi i(l_1x^1+\cdots+l_nx^n)}\,dg_1(t_1)\cdots dg_n(t_n)=\sum_{\substack{k_i=1,2,\cdots N\ (i\ \text{fixed})\\ i=1,2,\cdots,n}}I(k_1,\cdots,k_n)$$

$$=\sum_{k_i}\int_{0\leqq t_i\leqq\frac{1}{N}}e^{2\pi i\sum_{j=1}^{n}\sum_{i=1}^{n}l_j\{\frac{1}{N}({}_1g_i^j+\cdots+{}_{k_i-1}g_i^j)+{}_{k_i}g_i^j t_i\}}\det|{}_{k_i}g_i^s|\,dt_1\cdots dt_n$$

$$=\sum_{k_i}\frac{e^{2\pi i\sum_{i=1}^{n}\frac{1}{N}{}_{k_i}H_i}-e^{2\pi i\sum_{i=1}^{n}\frac{1}{N}{}_{k_i-1}H_i}}{\prod_{i=1}^{n}(2\pi i\,{}_{k_i}h_i)}\det|{}_{k_i}g_i^s|=\sum_{k_i}\det\left|\frac{e^{2\pi i\frac{{}_{k_i}H_i}{N}}-e^{2\pi i\,{}_{k_i-1}H_i}}{2\pi i\,{}_{k_i}h_i}{}_{k_i}g_i^s\right|$$

$$=\det\left|\sum_{k=1}^{n}\frac{e^{2\pi i\frac{1}{N}{}_kH_i}-e^{2\pi i\frac{1}{N}{}_{k-1}H_i}}{2\pi i\,{}_kh_i}{}_kg_i^s\right|=\prod_s\frac{1}{2\pi i l_s}\cdot\det\left|\sum_{k=1}^{n}\frac{e^{2\pi i\frac{1}{N}{}_kH_i}-e^{2\pi i\frac{1}{N}{}_{k-1}H_i}}{2\pi i\,{}_kh_i}\times 2\pi i\,l_s\,{}_kg_i^s\right|=0$$

(when all l_i are not zero),

since in the last determinant sums of components which are in the same column are all zeros. When $l_1=l_2=\cdots=l_n=0$, the value of the integral is 1, as readily be seen. Hence (1) is proved.

If G did not cover T^n, the rest would be an open set H. If we put for $A(x^1,\cdots,x^n)$ a continuous function that is positive in H and is zero outside of H, then the left side of the equality (1) is positive and the right side would be zero. That is a contradiction.

We can also prove the following

Theorem 2. If in Theorem 1 $f_i(t)$ belong to $C^{(1)}$ class, then for an arbitrary continuous function $A(x^1,\cdots,x^n)$ on T^n,

$$(2)\quad \int_{T^n}A(x^1,\cdots,x^n)\,dx_1\cdots dx_n=\int_{0\leqq t_i\leqq 1}A(x^1,\cdots,x^n)\,df_1(t_1)\cdots df_n(t_n)$$

where on the right side $x^j = f_1^j(t_1) + \cdots + f_n^j(t_n)$.

Proof. The integral on the right side is

$$\int_{0 \leq t_i \leq 1} A(x^1, \cdots, x^n)\, det. \left| \frac{d f_i^j(t_i)}{d t_i} \right| dt_1 \cdots dt_n .$$

Let us substitute $f_i(t)$ by partially linear curves $g_i(t)$ whose corners are $g_i(k/N) = f_i(k/N)$, $k = 1, 2, \cdots, N$. For an arbitrarily given positive number δ , we can choose N sufficiently large such that $|f_i^j(t) - g_i^j(t)| < \varepsilon$ and $\left| \frac{d f_i^j(t)}{dt} - \frac{d g_i^j(t)}{dt} \right| < \varepsilon$;

consequently we get

$$\left| A(x(f))\, det. \left| \frac{d f_i^j(t_i)}{d t_i} \right| - A(x(g))\, det. \left| \frac{d g_i^j(t_i)}{d t_i} \right| \right| < \delta$$

and error of the integral (2) and that substituted $g_i(t_i)$ thereinto is less than δ . Then, on account of (1) our result follows.

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VECTOR-GROUP IN REAL EUCLIDEAN SPACE

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We shall describe in this paper an elementary proof of the theorem which has also been proved in this volume by Prof. Iwamura, Messrs. M. Kuranishi and T. Hayashida.

We denote "free vectors" in an n-dimensional real euclidean space R_n by x, y, z, ..., a, b, c ,....., and the corresponding points in R_n by the same symbols, i.e., "a point x" means the point which is located by the free vector x starting from the original point o previously determined in R_n . The distance between any two points x and y is defined by the euclidean one, i.e., $|x-y|$. We shall prove in this paper the following Theorem and Corollary.

THEOREM. Let M be a real euclidean vector-group in R_n and contain a continuum K . Then M contains the whole straight-line through any two distinct points of K .

COROLLARY. Let M be a real euclidean vector-group in R_n and let any two points of M be connected by a continuum in M . Then M coincides with a real linear vector-group.

We shall prove the theorem by the induction with respect to the dimension-number n of R_n . If $n = 1$, the theorem is evident. Suppose $n > 1$.

LEMMA 1. Let K be any continuum in M . We define K' as the aggregate of all the points $x - y + z$, where x, y and z run throughout K . Then K' is also a continuum in M and $K \subset K'$.

The proof is immediate. We are going to prove that the straight-line segment joining any two distinct points a and b of K is contained in $K'' = (K')'$. As K is connected, a and b can be connected for any positive ε by an ε-chain with its points of joint all belonging to K . This chain can be represented by

$$x(t) ; \quad 0 \leq t \leq 1 ,$$

where $x(t)$ is a continuous curve in $0 \leq t \leq 1$, with its points of joint $x(t_i)$: $0 = t_0 < t_1 < t_2 < \ldots < t_m = 1$ all belonging to K and the parts $x(t)$, $t_i \leq t \leq t_{i+1}$, $i = 0, 1, 2, \ldots, m-1$ are all straight-line segments. Moreover $|x(t_{i+1}) - x(t_i)| < \varepsilon$, for $i = 0, 1, 2, \ldots, m-1$.