THE LIPSCHITZ CONDITION OF A FUNCTION AND FEJER MEANS OF FOURIER SERIES

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1. Let $f(x)$ be a continuous function satisfying the Lipschitz condition of order α, $(0<\alpha\leq 1)$

$$(1.1)\qquad f(x+h)-f(x)=O(|h|^{\alpha})$$

(uniformly for small h), which we shall denote as $\mathrm{Lip}\,\alpha \ni f(x)$ and let its Fourier series be

$$(1.2)\qquad f(x)\sim \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

If $\sigma_n(x)$ denotes the Fejér mean of (1.2) and $0<\alpha<1$, then by the well known S. Bernstein's theorem (1), it holds that

$$(1.3)\qquad f(x)-\sigma_n(x)=O(n^{-\alpha})$$

uniformly. But this does not hold generally if $\alpha=1$ and we have only to see (2) that

$$(1.4)\qquad f(x)-\sigma_n(x)=O(n^{-1}\log n).$$

This is also well known as Bernstein's result. Regarding with this matter, Prof. A. Zygmund has recently proved the following theorem. (3)

I. If $f(x)$ satisfies the Lipschitz condition

$$(1.5)\qquad f(x+h)-f(x)=O(|h|)$$

and its Fourier series be of power series type, then

$$(1.6)\qquad f(x)-\sigma_n(x)=O(n^{-1}).$$

In § 2 of the present paper we shall discuss the condition for the validity of (1.6) and give a slightly general theorem (Theorem 5).

Recently R. Salem and A. Zygmund has shown that (4)

II. If $\alpha>0$ and

$$(1.7)\qquad f(x)-s_n(x)=O(n^{-\alpha})$$

uniformly, where $s_n(x)$ denotes the n-th partial sum of (1.2), then

$$(1.8)\qquad \bar{f}(x)-\bar{s}_n(x)=O(n^{-\alpha}),$$

$\bar{f}(x)$, $\bar{s}_n(x)$ being respectively the conjugate function of $f(x)$ and the partial sum of the conjugate series of $f(x)$

$$(1.9)\qquad \sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx).$$

We now ask the following question: under the assumption that

$$(1.10)\qquad f(x)-\sigma_n(x)=O(n^{-\alpha})\ \text{uniformly},$$

does it hold that

$$(1.11)\qquad \bar{f}(x)-\bar{\sigma}_n(x)=O(n^{-\alpha})\ \text{uniformly},$$

where $\bar{\sigma}_n(x)$ is the Fejér means of (1.9)?

If $0<\alpha<1$, then the question is affirmatively answered. For, by the well known de la Vallée Poussin theorem(5), $f(x)$ belongs to $\mathrm{Lip}\,\alpha$, if (1.10) holds, $0<\alpha<1$. Then by the Privaloff's theorem(6) $\bar{f}(x)$ also belongs to $\mathrm{Lip}\,\alpha$, and then (1.11) hold by the S. Bernstein's theorem.

But if $\alpha=1$, then the above fact fails to be true. This is seen by the following simple example;

$$(1.12)\qquad f(x)\sim\sum_{n=1}^{\infty}(\sin nx)/n^2,$$

$$(1.13)\qquad \bar{f}(x)\sim\sum_{n=1}^{\infty}(\cos nx)/n^2.$$

For denoting partial sums and Fejér means of (1.12) and (1.13) by $s_n(x)$, $\bar{s}_n(x)$ and $\sigma_n(x)$, $\bar{\sigma}_n(x)$ respectively, we have

$$f(x)-\sigma_n(x)=\frac{1}{n}\sum_{k=1}^{n}(f(x)-s_k(x))$$
$$=\frac{1}{n}\sum_{k=1}^{n}\sum_{\nu=k+1}^{\infty}\frac{\sin\nu x}{\nu^2}$$

$$= \frac{1}{n}\sum_{\nu=2}^{n}(\nu-1)\frac{\sin \nu x}{\nu^2}+\sum_{\nu=n+1}^{\infty}\frac{\sin \nu x}{\nu^2}$$

$$= O(\tfrac{1}{n}),$$

Since $\sum_1^n \sin \nu x/\nu$ is uniformly bounded, and on the other hand,

$$\bar{f}(x)-\bar{\sigma}_n(x)=\frac{1}{n}\sum_{k=1}^{n}(\bar{f}(x)-\bar{S}_k(x))$$

$$=\frac{1}{n}\sum_{k=1}^{n}\sum_{\nu=k+1}^{\infty}\frac{\cos \nu x}{\nu^2}$$

$$=\frac{1}{n}\sum_{\nu=2}^{n}(\nu-1)\frac{\cos \nu x}{\nu^2}+\sum_{\nu=n+1}^{\infty}\frac{\cos \nu x}{\nu^2},$$

the second term of the right hand side being $O(1/n)$ uniformly, while the first term is of the order $n^{-1}\log n$, at $x=0$. We shall now prove that if $f(x)\in$ Lip 1 and (1.10) holds for $\alpha=1$. then (1.11) holds and $\bar{f}(x)\in$ Lip 1. (Theorem 4).

If $f(x)\in$ Lipα, $0<\alpha<1$, then $\bar{f}(x)\in$ Lipα. This is the Privaloff theorem above used. This does not holds when $\alpha=1$. We shall give the necessary and sufficient condition for the validity of this fact if $\alpha=1$. (Theorem 2).

It is to be remarked that the above three questions are very closely connected and the condition that

$$(1.14)\qquad \int_\eta^\pi \frac{\varphi(x,t)}{t}dt=O(1),\ \text{as}\ \eta\to 0$$

uniformly with respect to x, $\varphi(x,t)=\frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}$ plays the central role.

2. We shall prove the following theorem ~~that~~ the proof of which is essentially same as ~~the~~ that of I. But for completeness sake, we shall give it.

Theorem 1. In order that

$$(2.1)\qquad f(x)-\sigma_n(x)=O(n^{-1})$$

uniformly, under the assumption that $f(x)\in$ Lip 1, it is necessary and sufficient that the condition (1.14) holds uniformly in x.

We have

$$\sigma_n(x)-f(x)=\frac{1}{\pi}\int_0^\pi \varphi(x,t)\frac{\sin^2(n+1)t/2}{(n+1)\sin^2 t/2}dt$$

$$(2.2)\qquad =\frac{1}{\pi}\int_0^{\pi/n}+\frac{1}{\pi}\int_{\pi/n}^{\pi}=I_1+I_2,$$

say. Then since $f(x)\in$ Lip 1. $\varphi(x,t)=O(|t|)$ and

$$(2.3)\qquad I_1=\frac{1}{\pi}\int_0^{\pi/n}O(t)\frac{n^2t}{nt}dt=O(\tfrac{1}{n})$$

Therefore

$$\sigma_n(x)-f(x)-\frac{1}{\pi(n+1)}\int_{\pi/n}^{\pi}\frac{\varphi(x,t)}{2\sin^2 t/2}dt$$

$$=O(\tfrac{1}{n})-\frac{1}{2\pi(n+1)}\int_{\pi/n}^{\pi}\varphi(x,t)\frac{\cos(n+1)t}{\sin^2 t/2}dt$$

$$(2.4)\qquad =O(\tfrac{1}{n})-I_3,$$

say. We shall now estimate the values of I_3, which is done by the precisely similar arguments in Zygmund theorem I.

Since $f(x)\in$ Lip 1, $f(x)$ is absolutely continuous and $|f'(x)|\le M$ almost everywhere. And then we have

$$(2.5)\quad |I_3|\le\left|\frac{1}{2\pi(n+1)}\left[\varphi(x,t)\int_t^\pi\frac{\cos(n+1)t}{\sin^2 t/2}dt\right]_{\pi/n}^{\pi}+\frac{1}{2\pi(n+1)}\int_{\pi/n}^{\pi}\{f'(x+t)+f'(x-t)\}dt\int_t^\pi\frac{\cos(n+1)u}{\sin^2 u/2}du\right|.$$

The second mean value theorem shows

$$\left|\int_t^\pi\frac{\cos(n+1)u}{\sin^2 u/2}du\right|\le\left|\frac{1}{\sin^2 t/2}\int_\xi^\pi\cos(n+1)u\,du\right|$$

$$=O(\tfrac{1}{nt^2}),$$

$(t<\xi<\pi)$. Hence putting this into the right hand side of (2.5), we get

$$(2.6)\qquad I_3=O(\tfrac{1}{n})+O(\tfrac{1}{n}\int_{\pi/n}^{\pi}\frac{dt}{nt^2})=O(\tfrac{1}{n}).$$

Since it is easily verified that the last term of the left hand side of (2.4) differs from

$$\int_{\pi/n}^{\pi}\frac{\varphi(x,t)}{t^2}dt$$

by the term of order $O(1/n)$, putting (2.6) into (2.4) we finally get

$$\sigma_n(x)-f(x)=\frac{2}{\pi(n+1)}\int_{\pi/n}^{\pi}\frac{\varphi(x,t)}{t^2}dt+O(\tfrac{1}{n}).$$

Thus the necessary and sufficient condition for the validity of (2.1) is that

$$\int_{\pi/n}^{\pi}\frac{\varphi(x,t)}{t^2}dt=O(1)$$

uniformly in x, which is equivalent to that (1.14) holds uniformly. Hence our theorem is proved.

3. Theorem 2. Let $f(x) \in$ Lip 1 Then the necessary and sufficient condition that the conjugate function $\bar{f}(x)$ also belongs to Lip 1 is that (1.14) holds uniformly in x.

The conjugate function of $f(x)$ is by definition

$$\bar{f}(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(x+t)-f(x)}{2\tan t/2}dt$$

and hence

$$\bar{f}(x+h) = -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(x+t)-f(x+h)}{2\tan(t-h)/2}dt.$$

Since $f(x) \in$ Lip 1, we have

$$(3.1)\qquad \int_{-2h}^{2h}\frac{f(x+t)-f(x)}{2\tan t/2}dt = \int_{-2h}^{2h}O(1)dt = O(h)$$

$$(3.2)\qquad \int_{-2h}^{2h}\frac{f(x+t)-f(x+h)}{2\tan(t-h)/2}dt = \int_{-2h}^{2h}O(1)dt = O(h).$$

Hence

$$(3.3)\qquad \bar{f}(x+h)-\bar{f}(x) = -\frac{1}{\pi}\left(\int_{-\pi}^{-2h}+\int_{2h}^{\pi}\right)\{f(x+t)-f(x)\}\cdot\{\cot\tfrac{1}{2}(t-h)-\cot\tfrac{t}{2}\}\cdot dt$$
$$+[f(x+h)-f(x)]\int_{2h}^{\pi}\{\cot\tfrac{1}{2}(t-h)-\cot\tfrac{1}{2}(t+h)\}dt + O(h).$$

The first term of the right hand side equals to

$$-\int_{2h}^{\pi}\{f(x+t)-f(x)\}\{\cot\tfrac{1}{2}(t-h)-\cot\tfrac{t}{2}\}dt$$
$$-\int_{2h}^{\pi}\{f(x-t)-f(x)\}\{\cot\tfrac{t}{2}-\cot\tfrac{1}{2}(t+h)\}dt$$
$$= -\int_{2h}^{\pi}\varphi(x,t)\{\cot\tfrac{1}{2}(t-h)-\cot\tfrac{t}{2}\}dt$$
$$+\int_{2h}^{\pi}\{f(x-h)-f(x)\}\{\cot\tfrac{1}{2}(t-h)+\cot\tfrac{1}{2}(t+h)-2\cot\tfrac{t}{2}\}$$
$$\cdot dt = J_1 + J_2,$$

say. Then we have easily

$$(3.4)\qquad J_2 = O\left(\int_{2h}^{\pi}h^2\frac{t^2}{(t-h)(t+h)}dt\right) = O(h^2)$$

For J_1, we have

$$J_1 = -h\int_{2h}^{\pi}\frac{\varphi(x,t)}{t^2}dt + O\left(\int_{2h}^{\pi}\left|\varphi(x,t)\left(\frac{h}{t^2}-\frac{1}{t-h}+\frac{1}{t}\right)\right|dt\right)$$
$$= -h\int_{2h}^{\pi}\frac{\varphi(x,t)}{t^2}dt + O\left(h^2\int_{2h}^{\pi}\frac{1}{t(t-h)}dt\right)$$

$$(3.5)\qquad = -h\int_{2h}^{\pi}\frac{\varphi(x,t)}{t^2}dt + O(h).$$

The second term of the right of (3.3) is clearly

$$(3.6)\qquad O(h)\int_{2h}^{\pi}\frac{h}{(t-h)(t+h)}dt = O(h).$$

Putting (3.4), (3.5) and (3.6) into (3.3) we have finally

$$\bar{f}(x+h)-\bar{f}(x) = -h\int_{2h}^{\pi}\frac{\varphi(x,t)}{t^2}dt + O(h)$$

from which our theorem results.

4. In this section we shall give, in the case $\alpha = 1$, the sufficient condition for the validity of (1.11) under the assumption (1.10). Before it we shall prove the following.

Theorem 3. If

$$(4.1)\qquad f(x)-\sigma_n(x) = O\left(\frac{1}{n}\right)$$

uniformly, then

$$(4.2)\qquad \bar{f}(x)-\bar{\sigma}_n(x) = O\left(\frac{\log n}{n}\right)$$

holds uniformly.

Take an arbitrary number $0<\beta<1$, and consider the series

$$(4.1)\qquad \sum_{n=1}^{\infty} n^{\beta}A_n(x),$$

where $A_n(x)$ is the n-th term of the Fourier series of $f(x)$. We denote the (C.1) mean of (4.1) as $\tau_n(x)$ and write

$$R_n(x) = \sum_{k=n+1}^{\infty}A_k(x) = f(x) - s_n(x).$$

Now we take α and α' such that $0<\beta<\alpha<\alpha'<1$. From (4.1), by de la Vallee Poussin's theorem(7), $f(x) \in$ Lip α', and hence by the well known theorem

$$f(x)-s_n(x) = O(n^{-\alpha'}\log n) = O(n^{-\alpha}).$$

Thus

$$\sum_{k=m}^{n}k^{\beta}A_k(x) = \sum_{k=m}^{n}k^{\beta}(R_k(x)-R_{k+1}(x))$$
$$= m^{\beta}R_m(x) - n^{\beta}R_{n+1}(x) + \sum_{k=m+1}^{n}\{k^{\beta}-(k-1)^{\beta}\}R_k(x)$$
$$= O\left(\frac{1}{m^{\alpha-\beta}}\right) + O\left(\frac{1}{n^{\alpha-\beta}}\right)$$
$$+\sum_{k=m+1}^{n}O(k^{\beta-1})\,O\left(\frac{1}{k^{\alpha}}\right)$$

which tends to 0 as $n\to\infty$, $m\to\infty$. Thus the series (4.1) is uniformly convergent and we denote its sum $\varphi(x)$. Writing the partial sum of (4.1) as $t_n(x)$, we have

$$\varphi(x) - \tau_n(x) = \frac{1}{n}\sum_{k=1}^{n}\{\varphi(x) - t_k(x)\}$$

$$= \frac{1}{n}\sum_{k=1}^{n}\sum_{\nu=k+1}^{\infty}\nu^{\beta}A_{\nu}(x)$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left[(k+1)^{\beta}R_{k+1}(x) + \sum_{\nu=k+1}^{\infty}(\nu^{\beta}-(\nu-1)^{\beta})R_{\nu}(x)\right].$$

Since

$$\frac{1}{k}\sum_{\nu=1}^{k}R_{\nu+1}(x) = \frac{1}{k}\sum_{\nu=1}^{k}\{f(x) - s_{\nu}(x)\} = f(x) - \sigma_k(x),$$

we can write,

$$\varphi(x) - \tau_n(x) = \frac{1}{n}\sum_{k=1}^{n}\{f(x) - \sigma_k(x)\}\,\Delta(k+1)^{\beta}$$
$$+ \frac{(n+1)^{\beta+1}}{n}\{f(x) - \sigma_n(x)\} + \frac{1}{n}\sum_{k=1}^{n}\Big\{\sum_{\nu=k+1}^{\infty}\nu\{f(x) - \sigma_{\nu}(x)\}\Delta^2\nu^{\beta}$$
$$+ \nu\{f(x) - \sigma_{\nu}(x)\}\,\Delta k^{\beta}\Big\}$$
$$= \frac{1}{n}\sum_{k=1}^{n}O(\Delta k^{\beta}) + \frac{1}{n^{1-\beta}} + \frac{1}{n}\sum_{k=1}^{n}\Big\{\sum_{\nu=k+1}^{\infty}O(\Delta^2\nu^{\beta})$$
$$+ O(\Delta k^{\beta})\Big\}$$

by (4.1), which is

$$= O\left(\frac{1}{n^{1-\beta}}\right).$$

Therefore by de la Vallee Poussin s theorem $\varphi(x) \in Lip(1-\beta)$ and by Privaloff s theorem the conjugate function $\bar{\varphi}(x)$ belongs to Lip($1-\beta$), from which it result that

(4.2) $\quad \bar{\varphi}(x) - \bar{\tau}_n(x) = O(n^{-1+\beta})$

holds uniformly, $\bar{\tau}_n(x)$ belong the (C.1) means of the conjugate Fourier series of $\varphi(x)$.

Now, denoting n-th term of the conjugate Fourier series of $f(x)$ as $B_n(x)$,

$$\bar{f}(x) - \bar{\sigma}_n(x) = \frac{1}{n}\sum_{k=1}^{n}\sum_{\nu=k+1}^{\infty}B_{\nu}(x)$$

$$= \frac{1}{n}\sum_{k=1}^{n}\sum_{\nu=k+1}^{\infty}\nu^{\beta}B_{\nu}(x)\frac{1}{\nu^{\beta}}$$

$$= \frac{1}{n}\sum_{k=1}^{n}\sum_{\nu=k+1}^{\infty}\frac{1}{\nu^{\beta}}(T_{\nu}(x) - T_{\nu+1}(x)), \quad T_k(x) = \sum_{\nu=k}^{\infty}\nu^{\beta}B_{\nu}(x)$$

$$= \frac{1}{n}\sum_{k=1}^{n}\Big\{-\frac{1}{(k+1)^{\beta}}T_{k+1}(x) + \sum_{\nu=k+2}^{\infty}\Delta\left(\frac{1}{\nu^{\beta}}\right)T_{\nu}(x)\Big\}$$

$$= \frac{1}{n}\Big\{\frac{1}{(n+1)^{\beta}}(n+1)\{\bar{\varphi}(x) - \bar{\tau}_{n+1}(x)\}$$

$$+ \frac{1}{n}\sum_{k=1}^{n}(k+1)\{\bar{\varphi}(x) - \bar{\tau}_{k+1}(x)\}\,\Delta\left(-\frac{1}{(k+1)^{\beta}}\right)$$

$$+ \frac{1}{n}\sum_{k=1}^{n}\Big\{\Delta\left(\frac{1}{(k+2)^{\beta}}\right)\cdot(k+2)(\bar{\varphi}(x) - \bar{\tau}_{k+2}(x))$$

$$+ \sum_{\nu=k+3}^{\infty}\Delta^2\left(\frac{1}{\nu^{\beta}}\right)\cdot\nu\cdot(\bar{\varphi}(x) - \bar{\tau}_n(x))\Big\}$$

which is, by (4.2)

$$= O\left(\frac{1}{n^{\beta}}\,\frac{1}{n^{1-\beta}}\right) + \frac{1}{n}\sum_{k=1}^{n}O\left(k\cdot\frac{1}{k^{1-\beta}}\,\frac{1}{k^{1+\beta}}\right)$$

$$+ \frac{1}{n}\sum_{k=1}^{n}\Big\{O\left(\frac{1}{k^{1+\beta}}\cdot k\cdot\frac{1}{k^{1-\beta}}\right) + \sum_{\nu=k+3}^{\infty}O\left(\frac{1}{\nu^{2+\beta}}\cdot\nu\cdot\frac{1}{\nu^{1+\beta}}\right)$$

$$= O\left(\frac{1}{n}\right) + \frac{1}{n}\sum_{k=1}^{n}O\left(\frac{1}{k}\right) = O\left(\frac{1}{n}\log n\right)$$

which proves the theorem.

Next we now consider the additional condition for that

(4.3) $\quad \bar{f}(x) - \bar{\sigma}_n(x) = O\left(\frac{1}{n}\right)$

Then we get

Theorem 4. If $f(x) \in$ Lip 1 and (4.1) holds uniformly, then (4.3) holds uniformly and further $\bar{f}(x) \in$ Lip 1.

By theorem 1, (1.14) holds uniformly and thus by theorem 2 $\bar{f}(x) \in$ Lip 1. This proves the latter part of the theorem. Next since $f(x)$ and $\bar{f}(x)$ belongs to Lip 1, and the conjugate function of $\bar{f}(x)$ is $f(x)$, again by theorem 2

$$\int_{\eta}^{\pi}\frac{\bar{f}(x+t) + \bar{f}(x-t) - 2\bar{f}(x)}{t^2}dt = O(1), \quad \eta \to 0$$

uniformly in x and hence by Theorem 1, (4.3) holds.

Lastly we mention that by using theorem 2, we have the following generalization of Zygmund s Theorem I.

Theorem 5. If $f(x)$ and $\bar{f}(x)$ belong to Lip 1, then (1.6) holds.

For if $f(x)$ and $\bar{f}(x)$ belong to Lip 1, then by Theorem 2 (1.14) holds uniformly, and hence Theorem 1 shows our conclusion.

(*) Received March 1. 1949.

(1) A.Zygmund, Trigonometrical series, Warsaw, 1935, p.62.

(2) loc. cit.

(3) A.Zygmund, On the degree of approximation of functions by Fejer means, Bull.Amer. Soc., 51 (1945).

(4) R.Salem and A.Zygmund, The approximation by partial sums of Fourier series, Trans. Amer.Math.Soc., 59 (1946), 14-22.

(5) De la vallee Poussin, Lecons sur lapproximation des fonction dune variable reelle, Paris, 1919, p.58.

(6) A.Zygmund, Trigonometrical series, p.156.

(7) A. Zygmund, Trigonometrical series, p.156-157.

(8) De la vallee Poussin, loc. cit.

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