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ON VALUE DISTRIBUTION OF ENTIRE MAPS OF C^2 TO C^2

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§0. Introduction

In 1935, Thullen [T] studied about essential singular surfaces of analytic functions of n complex variables and obtained a generalization of the big Picard theorem. He studied about value distribution of such functions by considering continuation of level surfaces to essential singular surfaces.

And from 1968, Nishino studied about value distribution of level surfaces of entire functions of n complex variables systematically. He studied especially about algebraic type entire functions of two vaiables and obtained a beautiful theorem (see Theorem 1.12 in this paper) in [N2].

Now we study about value distribution of entire maps of C^2 to C^2 with being based on the above studies. We obtain analogies of Nishino's theorem (Theorem 2.8).

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§1. Value distribution of entire functions of C^2

Firstly, after some preliminalies, we prove Lemma 1.5 which is important in this paper.

DEFINITION 1.1. If C is an irreducible curve in C^2 (that is, irreducible 1-dimensional analytic subset of C^2) which may be transcendental and the normalization of C is isomorphic to a Riemann surface whose genus is $g < \infty$ and boundaries are n punctured points, we call C is a curve of algebraic type and of (g, n) type.

DEFINITION 1.2. If A is an irreducible algebraic curve in C^2 and its normalization is of (g, n) type where $2g - 2 + n > 0$, we call A is an algebraic curve of general type.

DEFINITION 1.3. If A is an irreducible algebraic curve in C^2 such that either its genus >0 or it intersects the line at infinity with more than two different points, we call A is a hyperbolic algebraic curve.

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LEMMA 1.4. *Let C be a curve of algebraic type and A be an algebraic curve of general type in $\boldsymbol{C}^2$. If f is a nonconstant holomorphic map of C to A, f covers A finite times.*

Proof. Let R be a compact Riemann surface and $R_0 = R - \{p_1, \dots, p_m\}$, where R_0 is isomorphic to the normalization of C. The normalization $\tilde{A}$ of A is a hyperbolic manifold. The map f induces the holomorphic map $\tilde{f} : R_0 \to \tilde{A}$ and $\tilde{f}$ can be extended to a holomorphic map of R to $\hat{A}$ where $\hat{A}$ is the compactification of $\tilde{A}$ by the big Picard theorem. So f covers A finite times. □

LEMMA 1.5. *Let A be a hyperbolic algebraic curve, C be a curve of algebraic type in $\boldsymbol{C}^2$ and the set $C \cap A$ consists of at most finite points. Then C is an algebraic curve.*

Proof. Let $\pi : R_0 \to C$ be the normalization of C, where $R_0 = R - \{p_1, \dots, p_m\}$ with some compact Riemann surface R. Then π is a holomorphic map of R_0 to C in $\boldsymbol{C}^2(x, y)$. Let $t \in \Delta$ be a local coordinate at p_i in R and $\pi = (x(t), y(t))$, $t \in \Delta^*$ where $\Delta^* = \{z; 0 < |z| < 1\}$. Suppose C is a transcendental curve in $\boldsymbol{C}^2$, there exists a point p_i such that $x(t)$ or $y(t)$ has an essential singularity at 0 and there is a sufficiently small $\rho > 0$ such that $\pi(\Delta^*(\rho)) \cap A = \emptyset$ where $\Delta^*(\rho) = \{z; 0 < |z| < \rho\}$. Let $B = \bar{A} \cup L_\infty$ where L_∞ is the line at infinity. Then, π defines a holomorphic map of $\Delta^*(\rho)$ into $\boldsymbol{P}^2 - B$ and the cluster set $D = \bigcap_{0<\rho<1} \overline{\pi(\Delta^*(\rho))} \subset L_\infty$ of π at 0 satisfies $D \subset L_\infty \subset B$ where $\overline{\pi(\Delta^*(\rho))}$ is the closure of $\pi(\Delta^*(\rho))$ in $\boldsymbol{P}^2$.

Let ψ be a holomorphic map of Δ^* into $\boldsymbol{P}^2 - E$, where E is an algebraic curve in $\boldsymbol{P}^2$. The list of possible figures of E, when the cluster set $F = \bigcap_{0<\rho<1} \overline{\psi(\Delta^*(\rho))}$ of ψ at 0 satisfies $F \subset E$ and F contains at least two points, is determined in Kizuka [K2], Theorem 1, applying the theory of the cluster set due to Nishino-Suzuki [N-S]. Here we apply this theorem to ψ. Since B contains an irreducible component of genus >0 or there are two hyperbolic components $L_\infty - (L_\infty \cap \bar{A})$ and A, we can see that B is not in the Kizuka's list hence D consists of one point. Therefore C is an algebraic curve, which completes the proof. □

Applying above lemma, we have following

THEOREM 1.6. *Let $f(x, y)$ be an entire function, $\{f(x, y) = 0\}$ be an algebraic curve which contains a hyperbolic component and $\{f(x, y) = 1\}$ be at most finite number of irreducible curves of algebraic type. Then f is a polynomial.*

Proof. In case $\{f(x, y) = 1\} = \emptyset$, f is a polynomial (constant) by Thullen's theorem. So, we may assume $\{f(x, y) = 1\} \neq \emptyset$. Let A be a hyperbolic component of $\{f(x, y) = 0\}$ and C be any irreducible component of $\{f(x, y) = 1\}$. Since $C \cap A = \emptyset$, C is an algebraic curve by Lemma 1.5. So, f is a polynomial by Thullen's theorem. □

Remark 1.7. A statement obtained by replacing the phrase "a hyperbolic component" in Theorem 1.6 with a phrase "a component of general type" is no longer true. For example, $f(x,y) = \{x(x-1)y-1\}e^x$ is a transcendental entire function such that $\{f(x,y)=0\}$ is an algebraic curve of type $(0,3)$ and $\{f(x,y)=1\}$ is an irreducible curve of algebraic type.

Secondly, we explain Theorem 1.12 which is the key tool of §3.

An analytic automorphism T of the space $\boldsymbol{C}^2(x,y)$ is given by two entire functions $f(x,y)$ and $g(x,y)$ in the form $x' = f(x,y)$, $y' = g(x,y)$. We say that T is algebraic if both f and g are polynomials of x, y. The group of analytic automorphisms of $\boldsymbol{C}^2$ is denoted by $Aut(\boldsymbol{C}^2)$ and the group of algebraic automorphisms of $\boldsymbol{C}^2$ is denoted by $Aut_{alg}(\boldsymbol{C}^2)$. It is well known that if $T \in Aut_{alg}(\boldsymbol{C}^2)$, $T^{-1} \in Aut_{alg}(\boldsymbol{C}^2)$.

Let $P(x,y)$ be a nonconstant polynomial. It is well known that every irreducible component of all level curves of P is the same type except for a finite number of them. So, if the type of general irreducible components of level curves of P is (g,n), we call P a polynomial of type (g,n). It is well known that if the type of an exceptional irreducible component of level curves of P is (g',n'), then $g' \leq g$ and $g'+n' \leq g+n$. (Theorem I in [N1]). A polynomial P is called primitive if $\{P=\alpha\}$ is irreducible except for at most a finite number of $\alpha \in \boldsymbol{C}$.

Definition 1.8. If $P(x,y)$ is a polynomial of type (g,n) and $2g-2+n>0$, we call P is a polynomial of general type and otherwise we call P is a polynomial of exceptional type.

Theorem 1.9 (Theorem 1 in Kizuka [K1]). *Let T be an automorphism of $\boldsymbol{C}^2$. If there is a polynomial P of general type such that $P \circ T$ is a polynomial, then T is algebraic.*

Theorem 1.10 (cf. Proposition and Theorem 3 in [K1]). *Let A be an algebraic curve in $\boldsymbol{C}^2$ which contains a hyperbolic component. An automorphism T transforms A onto an algebraic curve if and only if T is algebraic.*

Proof. If T is algebraic, it is trivial that T transforms any algebraic curve onto an algebraic curve. So, we prove the converse statement. If a hyperbolic component of A intersects the line at infinity with more than two different points, theorem is true by Proposition in [K1], p. 560. We assume that genus of a hyperbolic component of A is positive. We assume that T transforms A onto an algebraic curve. Let P be a polynomial such that $A = \{P=0\}$. Since $\{P \circ T^{-1} = 0\}$ is an algebraic curve and $\{P \circ T^{-1} = 1\}$ is at most finite number of curves of algebraic type, $P \circ T^{-1}$ is a polynomial from Theorem 1.6. Therefore, T^{-1} is algebraic from Theorem 1.9 because P is a polynomial of general type. Then T is algebraic. □

Definition 1.11. Let $f(x,y)$ be a nonconstant entire function. Then, we call f is an entire function of algebraic type, if each irreducible component of nonempty level curves of f is a curve of algebraic type.

THEOREM 1.12 (principal theorem in Nishino [N2]). *Every entire function $f(x,y)$ of algebraic type is reduced to a polynomial $P(x,y)$, that is, there are an automorphism $T \in Aut(\boldsymbol{C}^2)$ and an entire function $\varphi(z)$ of one complex variable z such that $f = \varphi \circ P \circ T$.*

Applying above theorem, we have following

THEOREM 1.13. *Let $f(x,y)$ be an entire function of algebraic type. If there is a hyperbolic algebraic component A of a level curve of f, every irreducible component of level curves of f is an algebraic curve.*

Proof. By Theorem 1.12, there are a polynomial $P(x,y)$, $T \in Aut(\boldsymbol{C}^2)$ and an entire function φ of one variable such that $f = \varphi \circ P \circ T$. It is easy to see that T transforms A onto an algebraic curve. So T is algebraic by Theorem 1.10 and $P \circ T$ is a polynomial. □

§2. Value distribution of entire maps of C^2 to C^2

The first result of this section is Theorem 2.5. In order to prove it, we need a big Picard theorem (Theorem 2.3) which was proved in [A1]. So, we begin by preparing the notation for Theorem 2.3.

Let X be a complex manifold, M a relatively compact domain of X and d_M the Kobayashi pseudodistance of M. In [A-S] we extended d_M to $\overline{M}$, the closure of M in X, as follows:

For $p, q \in \overline{M}$, we define

$$\bar{d}_M(p,q) = \liminf_{p' \to p,\, q' \to q} d_M(p',q'), \quad p', q' \in M.$$

It is clear that $0 \leq \bar{d}_M \leq \infty$ and the function $\bar{d}_M$ does not necessarily satisfy the triangle inequality. (see p. 386 in [A2]) So $\bar{d}_M$ is not always a pseudodistance on $\overline{M}$.

DEFINITION 2.1. We call $p \in \overline{M}$ a degeneracy point of $\bar{d}_M$ if there exists a point $q \in \overline{M} - \{p\}$ such that $\bar{d}_M(p,q) = 0$. By $S_M(X)$ we denote the set of degeneracy points of $\bar{d}_M$ on $\overline{M}$ and call it the degeneracy locus of $\bar{d}_M$ in X.

THEOREM 2.2 (Theorem 2 in [A-S]). *The set $S_M(X)$ is a pseudoconcave subset of order* 1 *in X.* (*About the definition of a pseudoconcave set of order* 1, *see Definition* 1.2 *in* [A2])

From above theorem, if $S_M(X)$ is contained in a curve, $S_M(X)$ is a curve or an empty set.

THEOREM 2.3 (Theorem 12.1 in [A1]). *Let N be an arbitrary complex manifold and B be a proper analytic subset of N. Let A be a curve in $\boldsymbol{P}^2$ with $l (l \geq 4)$ irreducible components. Assume that $S_{\boldsymbol{P}^2 - A}(\boldsymbol{P}^2)$ is contained in a curve in*

P^2 and let F be a meromorphic map of $N - B$ to $P^2 - A$, then F can be extended to a meromorphic map of N to P^2 or $F(N - B) \subset S_{P^2-A}(P^2)$.

Corollary 2.4. *Let F be a nondegenerate entire map of C^2 to C^2* in the sense that *$F(C^2)$ contains an open set of C^2, A be an algebraic curve in C^2 with $l(l \geq 3)$ irreducible components and $S_{P^2-A'}(P^2)$ be contained in a curve in P^2 where $A' = \bar{A} \cup L_\infty$. If $F^{-1}(A)$ is contained in an algebraic curve B in C^2, then F is a polynomial map.*

Proof. Let $\bar{B} \cup L_\infty = B'$. Then F is a nondegenerate holomorphic map of $P^2 - B'$ to $P^2 - A'$. So F is extended to a meromorphic map of P^2 to P^2 by Theorem 2.3. Therefore F is a polynomial map. □

Now, we will prove Theorem 2.5.

Theorem 2.5. *Let F be a nondegenerate entire map of C^2* to *C^2. Suppose that there is a polynomial $Q(x.y)$ of general type such that $Q \circ F$ is a polynomial. Then F is a polynomial map.*

Proof. Let $A_i = \{Q = \alpha_i\}(i = 1, 2, 3)$ be general level curves of Q. Then $F^{-1}(A_i) = \{Q \circ F = \alpha_i\}$ is an algebraic curve. Set $A' = \overline{A_1} \cup \overline{A_2} \cup \overline{A_3} \cup L_\infty$. Then $S_{P^2-A'}(P^2)$ is contained in a curve in P^2 by Proposition 8.4 in [A1]. From Corollary 2.4 F is a polynomial map. □

Definition 2.6. Let F be a nondegenerate entire map of C^2 to C^2. If for every algebraic curve A in C^2, $F^{-1}(A) \neq \emptyset$ and every irreducible component of $F^{-1}(A)$ is of algebraic type, we call F is algebraic type.

The other result of this section is Theorem 2.8. After preparing Lemma 2.7, we will prove Theorem 2.8.

Lemma 2.7. *Let $\Phi = (\varphi(x), \psi(y))$ be an entire map of C^2 to C^2, where $\varphi(x)$ and $\psi(y)$ are nonconstant entire functions. If Φ is an algebraic type map, then $\varphi(x)$ and $\psi(y)$ are polynomials.*

Proof. If both $\varphi(x)$ and $\psi(y)$ are not polynomials, we may assume without loss of generality that $\varphi(x)$ is a transcendental entire function such that $\{\varphi = a\}$ has infinitely many solutions and $\{\psi = b\} \neq \emptyset$. Let $P(x, y) = (x - a)(y - b)$. Then $\{P \circ \Phi = (\varphi(x) - a)(\psi(y) - b) = 0\}$ consists of infinitely many lines which intersect at infinitely many points. It is absurd because $P \circ \Phi = \xi \circ Q \circ T$ where ξ is an entire function of one variable, $Q(x, y)$ is a polynomial and $T \in Aut(C^2)$ by Theorem 1.12 and the isolated singular points of $\{\xi \circ Q \circ T = 0\}$ are at most finite points. □

THEOREM 2.8. *Let $F : \boldsymbol{C}^2 \to \boldsymbol{C}^2$ be a nondegenerate entire map of algebraic type. Then, we have $F = F_0 \circ T$ where F_0 is a polynomial map and $T \in Aut(\boldsymbol{C}^2)$.*

Proof. Take any primitive polynomial P of (g, n) type with $g > 0$, then by Theorem 1.12, there exist an entire function φ of one variable, a polynomial $P'(x, y)$ and $T \in Aut(\boldsymbol{C}^2)$ with $P \circ F \circ T^{-1} = \varphi \circ P'$. We will show that $F \circ T^{-1}$ is a polynomial map by dividing the following argument into 4 parts (a)–(d).

(a) Take a general value $\alpha \in \boldsymbol{C}$ of P such that there is a component A of $\{P \circ F \circ T^{-1} = \alpha\}$ with nonconstanant $F \circ T_{|A}^{-1} : A \to \{P = \alpha\}$. Then A is algebraic because A is a component of $\{P' = \alpha'\}$ by some α' with $\varphi(\alpha') = \alpha$, and the genus of A is >0 because the map $F \circ T_{|A}^{-1}$ is nonconstant holomorphic map of A to the irreducible curve of positive genus $\{P = \alpha\}$.

(b) Take a polynomial Q and $\beta \in \boldsymbol{C}$ arbitrarily such that $\{P = \alpha\} \cap \{Q = \beta\}$ consists of at most finite points. We will show that each irreducible component C of $\{Q \circ F \circ T^{-1} = \beta\}$ is algebraic. Since F is of algebraic type, C is of algebraic type. Further, since $F \circ T_{|A}^{-1} : A \to \{P = \alpha\}$ is finite to one by Lemma 1.4, $F \circ T_{|A}^{-1}(A \cap C) \subset \{P = \alpha\} \cap \{Q = \beta\}$ and that $\{P = \alpha\} \cap \{Q = \beta\}$ is finite, so $A \cap C$ is also finite. Therefore, by Lemma 1.5, C is algebraic.

(c) By Theorem 1.12, there are entire functions $\varphi(x)$, $\psi(y)$ and polymomials $P_i(x, y)$ and $T_i \in Aut(\boldsymbol{C}^2) (i = 1, 2)$ such that $F \circ T^{-1} = (\varphi \circ P_1 \circ T_1, \psi \circ P_2 \circ T_2)$. Take two distinct complex numbers α_1 and α_2 and let $a_i = \varphi(\alpha_i)$. Then, (b) applied to the function $Q(x, y) = x$ and the constant $\beta = a_i$ shows that every irreducible component of $\{x \circ F \circ T^{-1} = \varphi \circ P_1 \circ T_1 = a_i\}$ is algebraic, so every irreducible component of $\{P_1 \circ T_1 = \alpha_i\}$ is also algebraic. Hence, by Thullen's theorem, $P_1 \circ T_1$ is a polynomial. By the similar argument, $P_2 \circ T_2$ is also a polynomial.

(d) Define a polynomial map $F_1 = (P_1 \circ T_1, P_2 \circ T_2)$ and an entire map $\Phi = (\varphi(x), \psi(y))$. Then $F \circ T^{-1} = \Phi \circ F_1$. We will show that Φ is a polynomial map, which will comlete the proof of the theorem. If we take any algebraic curve B, then every irreducible component of $F \circ T^{-1}(B)$ is algebraic from (b). Then every irreducible component B' of $\Phi^{-1}(B)$ is algebraic because B' is the image of an algebraic curve by F_1. Thus, by Lemma 2.7, Φ is a polynomial map. □

REFERENCES

[A1] Y. ADACHI, A generalization of the big Picard theorem, Kodai Math. J., **18** (1995), 408–424.

[A2] Y. ADACHI, On the relation between tautly imbedded space modulo an analytic subset S and hyperbolically imbedded space modulo S, Publ. Res. Inst. Math. Sci. Kyoto Univ., **33** (1997), 385–392.

[A-S] Y ADACHI AND M. SUZUKI, Degeneracy points of the Kobayashi pseudodistances on complex manifolds, Several Complex Variables and Complex Geometry (E. Bedford et al. eds.), Proc. Sympos. Pure Math., **52-2**, Amer. Math. Soc., Providence, 1991, 41–51.

[K1] T. KIZUKA, Analytric automorphisms and algebraic automorphisms of C^2, Tôhoku Math. J., **31** (1979), 553–565.

[K2] T. KIZUKA, Analytic automorphisms of the complement of an algebraic curve in the complex projective plane, Tôhoku Math. J., **33** (1981), 77–89.

[N1] T. NISHINO, Nouvelles recherches sur les fonctions entières de plusieurs variables comlexe (IV), Types de surfaces premièrs, J. Math. Kyoto Univ., **13** (1973), 217–272.

[N2] T. NISHINO, Nouvelles recherches sur les fonctions entières de plusieurs variables complexe (V), Fonctions qui se réduisent aux polynômes, J. Math. Kyoto Univ., **15** (1975), 527–553.

[N-S] T. NISHINO AND M. SUZUKI, Sur les singularités essentielles et isolés des applications holomorphes à valeurs dans une surface complexe, Publ. Res. Inst. Math. Sci. Kyoto Univ., **16** (1980), 461–497.

[T] P THULLEN, Über die wesentlichen singularitäten analytishen funkutionen und flächen in räume von n komplexen veränderlichen, Math. Ann., **111** (1935), 137–157

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