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ON THE MULTIPLE VALUES OF ALGEBROID FUNCTIONS*

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Abstract

For any ν-valued algebroid function of finite order $\rho > 0$ in $|z| < \infty$, we prove the existence of the sequence of filling disks and Borel direction dealing with its multiple values.

1. Introduction

Valiron [1] conjectured that there exists at least a Borel direction for any ν-valued algebroid function of order ρ $(0 < \rho < \infty)$. Rauch [2] proved that there exists a direction such that the corresponding Borel exceptional values form a set of linear measure zeros. Toda [3] proved that there exists a direction such that the set of corresponding Borel exceptional values is countable. Later Lü and Gu [4] proved that there exists a direction such that the number of Borel exceptional values is equal to 2ν at most. However, it was not discussed whether there exists a Borel direction dealing with its multiple values. In the present paper we investigate this problem.

Let $w = w(z)$ be a ν-valued algebroid function in $|z| < \infty$ defined by irreducible equation

$$A_\nu(z)w^\nu + A_{\nu-1}(z)w^{\nu-1} + \cdots + A_0(z) = 0, \tag{1}$$

where $A_\nu(z), \ldots, A_0(z)$ are entire functions without any common zero. The single valued domain of definition of $w(z)$ is a ν-sheeted covering of z-plane, a Riemann surface, denoted by $\tilde{R}_z$. A point in $\tilde{R}_z$ whose projection in the z-plane is z, is denoted by $\tilde{z}$. The part of $\tilde{R}_z$, which covers a disk $|z| < r$, is denoted by $|\tilde{z}| < r$. Let $n(r, a)$ be the number of the zeros, counted according to their multiplicities, of $w(z) - a$ in $|\tilde{z}| \le r$, $\bar{n}^{l)}(r, a)$ be the number of distinct zeros with multiplicity $\le l$ of $w(z) - a$ in $|\tilde{z}| \le r$. Let

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$$S(r,w) = \frac{1}{\pi} \iint\limits_{|\tilde{z}| \le r} \left(\frac{|w'(z)|}{1 + |w(z)|^2} \right)^2 d\omega,$$

$$T(r,w) = \frac{1}{\nu} \int_0^r \frac{S(r,w)}{r} dr.$$

$S(r,w)$ is called the mean covering number of $|\tilde{z}| \le r$ into w-sphere under the mapping $w = w(z)$. $T(r,w)$ is called the characteristic function of $w(z)$. Let

$$N(r,a) = \frac{1}{\nu} \int_0^r \frac{n(r,a) - n(0,a)}{r} dr + \frac{n(0,a)}{\nu} \log r,$$

$$m(r,w) = \frac{1}{2\pi\nu} \int\limits_{|\tilde{z}|=r} \log^+ |w(re^{i\theta})|\, d\theta, \quad z = re^{i\theta},$$

where $|\tilde{z}| = r$ is the boundary of $|\tilde{z}| \le r$. We have

$$T(r,w) = m(r,w) + N(r,\infty) + O(1).$$

The order of algebroid function $w(z)$ is defined by

$$\rho = \overline{\lim_{r \to \infty}} \frac{\log^+ T(r,w)}{\log r}.$$

In this paper, we suppose that $0 < \rho < \infty$. Let $n(r, \tilde{R}_z)$ be the number of the branch points of $\tilde{R}_z$ in $|\tilde{z}| \le r$, counted with the order of branch. Write

$$N(r,\tilde{R}_z) = \frac{1}{\nu} \int_0^r \frac{n(r,\tilde{R}_z) - n(0,\tilde{R}_z)}{r} dr + \frac{n(0,\tilde{R}_z)}{\nu} \log r.$$

By [5]

$$N(r,\tilde{R}_z) \le 2(\nu - 1) T(r,w) + O(1). \tag{2}$$

We define angular domain:

$$\Delta(\theta_0,\delta) = \{z \,|\, |\arg z - \theta_0| < \delta\}, \quad 0 \le \theta_0 < 2\pi, \; 0 < \delta < \frac{\pi}{2}.$$

The part of $\tilde{R}_z$ which lie over $\Delta(\theta_0,\delta)$ is denoted by $\tilde{\Delta}(\theta_0,\delta)$. Let $n(r,\Delta(\theta_0,\delta),a)$ be the number of the zeros of $w(z) - a$ in $\tilde{\Delta}(\theta_0,\delta) \cap \{|\tilde{z}| \le r\}$ and $n(r,\Delta(\theta_0,\delta),\tilde{R}_z)$ be the number of the branch points in the same region. Similarly as above, we can define $\bar{n}^{l)}(r,\Delta(\theta_0,\delta),a)$.

Definition. Let $w = w(z)$ be a ν-valued algebroid function of order ρ $(0 < \rho < \infty)$ defined by (1) in $|z| < \infty$, and $l \ge 3$ be a position integer. For arbitrary $\delta > 0$ $(0 < \delta < \pi/2)$, if

$$\overline{\lim_{r \to \infty}} \frac{\log^+ \bar{n}^{l)}(r,\Delta(\theta_0,\delta),a)}{\log r} = \rho$$

holds for any complex value a except at most finite possible exceptions, then the half line $B : \arg z = \theta_0$ $(0 \leq \theta_0 < 2\pi)$ is called a Borel direction about multiple values of $w(z)$.

In this paper, the Riemann sphere of diamte 1 is denoted by V, C is a positive constant and it may be of different meaning when it appears in different position.

2. Some lemmas

Let F_1 be a connected domain on V, the boundary of F_1 is denoted by ∂F_1, which consists of a finite number of mutually disjoint circular curves $\{\wedge_j\}$, and the spherical distance between any two circular curves $\wedge_i$ and $\wedge_j$ is $d(\wedge_i, \wedge_j) \geq \delta \in (0, 1/2)$ $(i \neq j)$.

Let F be a finite covering surface of F_1, F is bounded by a finite number of analytic closed Jordan curves, its boundary is denoted by ∂F. We call the part of ∂F, which lies the interior of F_1, the relative boundary of F, and denote its length by L.

Let D be a domain on F_1, its boundary consists of a finite number of points or analytic closed Jordan curves, and $F(D)$ be the part of F, which lies above D. We denote the area of $F, F_1, F(D)$ and D by $|F|, |F_1|, |F(D)|$ and $|D|$ respectively. We call

$$S = \frac{|F|}{|F_1|}, \quad S(D) = \frac{|F(D)|}{|D|}$$

the mean covering number of F relative to F_1, D respectively.

Under the above hypotheses, Sun Daochun [6] estimated the constants of Ahlfors' covering theorem and Ahlfors' fundamental theorem [7] on unit sphere, and obtained the following results:

LEMMA 1.

$$|S - S(D)| < \max\left\{\frac{2}{\delta}, \frac{\pi^2}{|D|}\right\} L.$$

LEMMA 2.

$$\rho^+(F) \geq \rho(F_1)S - 2^5\pi^2\delta^{-3}L(F_1),$$

where $\rho(F), \rho(F_1)$ is Eulers' characteristic of F, F_1 respectively, $\rho^+ = \max\{\rho, 0\}$, $L(F_1)$ is the length of relative boundary of F with respect to F_1.

Let $D_j (j = 1, 2, \ldots, q)$ be $q (q \geq 3)$ disjoint circles on V, $d(D_i, D_j) \geq \delta \in (0, 1/2)$ $(i \neq j)$.

We take off $\{D_j\}$ from V and let F_0 be the remaining surface, then $\rho(F_0) = q - 2$.

Now $F(D_j)$ consists of a finite number of connected surfaces

$$F(D_j) = \bigcup_k F^{\iota}_{j,k} + \bigcup_k F^{z}_{j,k},$$

where $F^{\iota}_{j,k}$ has no relative boundary, with respect to D_j, which is called an island and $F^{z}_{j,k}$ has such one, which is called a peninsula.

In the following, we shall give two Lemmas for later use, their proof methods belong to Tsuji [7].

LEMMA 3. *Let F be m connected covering surfaces on the unit sphere V, $\rho(F) = \tilde{n} - m$($\tilde{n}$ is a nonegative integer), $\{D_\nu\}$ be $q(q \geq 3)$ disjoint disks on V, where the spherical distance $d(D_i, D_j) \geq \delta \in (0, 1/2)$ $(i \neq j)$.*

If n_j is the number of simply connected islands in $F(D_j)$, then

$$\tilde{n} + \sum_{j=1}^{q} n_j \geq (q-2)S - \frac{C}{\delta^3} L, \tag{3}$$

where $C > 0$ is a constant and L is the length of the boundary of F.

Proof. We take off from F all peninsulas $\{F^{z}_{j,k}\}$ above $\{D_j\}$, and let F' be the remaining surface:

$$F' = F - \bigcup_{j=1}^{q} \bigcup_k F^{z}_{j,k}.$$

Since the peninsulas involve only the part of the boundary of F, then $\rho(F') \leq \rho(F)$. Suppose that F' consists of $N(F')$ of connected surfaces, then $N(F') \leq m$. Hence

$$\rho(F') \leq \rho(F) = \tilde{n} - m \leq \tilde{n} - N(F'). \tag{4}$$

Next we take off from F' all islands $\{F^{\iota}_{j,k}\}$ above $\{D_j\}$ and let F'' be the remaining surface:

$$F' = \bigcup_{j=1}^{q} \bigcup_k F^{\iota}_{j,k} + F''.$$

F'' consists of a finite number of connected surfaces:

$$F'' = \bigcup_\mu F''_\mu.$$

So that

$$F' = \bigcup_{j=1}^{q} \bigcup_k F^{\iota}_{j,k} + \bigcup_\mu F''_\mu.$$

Since F' is decomposed into $\{F^{\iota}_{j,k}\}$ and $\{F''_\mu\}$ by ring cuts, so its characteristic not change.

Hence

$$\rho(F') = \sum_{j=1}^{q} \sum_{k} \rho(F'_{j,k}) + \sum_{\mu} \rho(F''_{\mu}).$$

By (4)

$$\tilde{n} - N(F') \geq \rho(F') = \sum_{j=1}^{q} \sum_{k} \rho(F'_{j,k}) + \sum_{\mu} \rho^{+}(F''_{\mu}) - N(F''), \tag{5}$$

where $N(F'')$ is the number of simply connected F''_{μ}.

Since

$$\sum_{j=1}^{q} \sum_{k} \rho(F'_{j,k}) = \sum_{j=1}^{q} \sum_{k} \rho^{+}(F'_{j,k}) - \sum_{j=1}^{q} n_j, \tag{6}$$

so by (5), (6) we have

$$\begin{aligned} \sum_{j=1}^{q} n_j &= \sum_{j=1}^{q} \sum_{k} \rho^{+}(F'_{j,k}) - \sum_{j=1}^{q} \sum_{k} \rho(F'_{j,k}) \\ &\geq \sum_{j=1}^{q} \sum_{k} \rho^{+}(F'_{j,k}) + \sum_{\mu} \rho^{+}(F''_{\mu}) - N(F'') + N(F') - \tilde{n}. \end{aligned} \tag{7}$$

We see easily that $N(F') - N(F'') \geq 0$, hence

$$\tilde{n} + \sum_{j=1}^{q} n_j \geq \sum_{\mu} \rho^{+}(F''_{\mu}). \tag{8}$$

Put $F_0 = V - \bigcup_{v=1}^{q} D_v$, then F''_{μ} is a covering surface of F_0, by Lemma 2,

$$\rho^{+}(F''_{\mu}) \geq (q-2)S''_{\mu} - \frac{2^5 \pi^2}{\delta^3} L''_{\mu}, \tag{9}$$

where $S''_{\mu} = |F''_{\mu}|/|F_0|$ and L''_{μ} is the length of the relative boundary of F''_{μ} with respect to F_0.

By (8), (9)

$$\begin{aligned} \tilde{n} + \sum_{j=1}^{q} n_j &\geq \sum_{\mu} (q-2) S''_{\mu} - \sum_{\mu} \frac{2^5 \pi^2}{\delta^3} L''_{\mu} \\ &= (q-2) S'' - \frac{2^5 \pi^2}{\delta^3} L'', \end{aligned} \tag{10}$$

where $S'' = \sum_{\mu} S''_{\mu} = |F''|/|F_0|$, $L'' = \sum_{\mu} L''_{\mu}$.

By Lemma 1, $|S - S''| < \max\{2/\delta, \pi^2/|F_0|\} L''$.

Since $|F_0| < |V| = \pi$, then $\pi^2/|F_0| \geq 1$, so that $|S - S''| < (2/\delta)(\pi^2/|F_0|) L''$.

Since $(\delta/2)^2 q \leq |F_0|$, $L'' \leq L$, we have

$$S'' > S - \frac{2^3\pi^2}{\delta^3 q} L. \tag{11}$$

Hence by (10), (11)

$$\tilde{n} + \sum_{j=1}^{q} n_j \geq (q-2)S - \frac{C}{\delta^3} L,$$

where $C = 40\pi^2$.

Lemma 4. *Under the same condition as in Lemma 3, let D_j $(j = 1, 2, \ldots, q)$ be $q (q \geq 3)$ disjoint disks with radius $\delta/3$, and $n_j^{l)}$ be the number of simply connected islands in $F(D_j)$, which consist of not more than l sheets, then*

$$(l+1)\tilde{n} + l \sum_{j=1}^{q} n_j^{l)} \geq (l+1)(q-2)S - \frac{C}{\delta^5} L,$$

where $l \geq 3$ is a positive integer.

Proof. Let n_j be the number of simply connected islands in $F(D_j)$, and $n_j^{(l}$ be that of such ones, which consist of more than l sheets, then

$$n_j = n_j^{l)} + n_j^{(l}, \quad S(D_j) \geq n_j^{l)} + (l+1) n_j^{(l},$$

so that

$$S(D_j) \geq (l+1)(n_j^{l)} + n_j^{(l}) - l n_j^{l)} = (l+1) n_j - l n_j^{l)}. \tag{12}$$

Since $|D_j| \geq \delta^2/9$, from Lemma 1, we have

$$S + \frac{18\pi^2}{\delta^3} L > S(D_j),$$

hence by (12), $S + (18\pi^2/\delta^3) L > (l+1) n_j - l n_j^{l)}$, so that

$$qS + l \sum_{j=1}^{q} n_j^{l)} + \frac{18\pi^2}{\delta^3} qL > (l+1) \sum_{j=1}^{q} n_j. \tag{13}$$

Note that $q(\delta/2)^2 \leq \pi$, by Lemma 3 and (13) we have

$$(l+1)\tilde{n} + l \sum_{j=1}^{q} n_j^{l)} \geq (l+1)(q-2)S - \frac{C}{\delta^5} L.$$

3. A fundamental inequality of algebroid functions

Theorem 1. *Let $w = w(z)$ be a ν-valued algebroid function in $|z| < R$ $(0 < R < \infty)$, F be the Riemann surface, generated by $w = w(z)$ on the w-sphere V,* *and $D_1, D_2, \ldots, D_q$ be $q(q \geq 3)$ disjoint disks with radius $\delta/3$ on V, $d(D_i, D_j) \geq \delta \in (0, 1/2)$ $(i \neq j)$. Suppose that $n_j^{l)}$ is the number of simply connected islands in $F(D_j)$, which consist of not more than l sheets, then for any $r \in (0, R)$*

$$(q-2)S(r) \leq \sum_{j=1}^{q} n_j^{l)} + n(R, \tilde{R}_z) + \frac{C}{\delta^{25}} \frac{R}{R-r},$$

where $l \geq 3$ is a positive integer, $n(R, \tilde{R}_z)$ is as in section 1.

Proof. We take off $D_1, D_2, \ldots, D_q$ from the w-sphere V and let F_0 be the remaining surface, then $\rho(F_0) = q - 2$.

Let $\tilde{D}_r = \{|\tilde{z}| < r\} (r \in (0, R))$ be the part of $\tilde{R}_z$ in $|z| < r$, then by M. Hurwitz formula [8]

$$\rho(\tilde{D}_r) = n(r, \tilde{R}_z) - \nu.$$

From Lemma 4, we easily obtain

$$(l+1)n(r, \tilde{R}_z) - (l+1)\nu + l \sum_{j=1}^{q} n_j^{l)} \geq (l+1)(q-2)S(r) - \frac{C}{\delta^5} L(r), \tag{14}$$

where

$$S(r) = \frac{1}{\pi} \iint_{|\tilde{z}| \leq r} \left(\frac{|w'(z)|}{1 + |w(z)|^2} \right)^2 d\omega,$$

$$L(r) = \int_{|\tilde{z}|=r} \frac{|w'(re^{i\varphi})|}{1 + |w(re^{i\varphi})|^2} r \, d\varphi.$$

By Schwarz's inequality,

$$L^2(r) \leq 2\pi^2 \nu r \frac{dS(r)}{dr}. \tag{15}$$

Put $\sum_{j=1}^{q} n_j^{l)} + n(r, \tilde{R}_z) \overset{\text{def}}{=} N$, from (14), we have

$$N \geq (q-2)S(r) - \frac{C}{\delta^5} L(r).$$

If $(q-2)S(r') - N > 0$ for all $r' \in (r, R)$, then from (15), we have

$$((q-2)S(r') - N)^2 \leq \frac{C}{\delta^{25}} L^2(r') \leq \frac{2\pi^2 \nu RC}{\delta^{25}} \frac{dS(r')}{dr'},$$

so that

$$R - r = \int_r^R dr' \leq \frac{C}{\delta^{25}} R \int_r^R \frac{dS(r')}{[(q-2)S(r') - N]^2} \leq \frac{C}{\delta^{25}} \frac{R}{(q-2)S(r) - N}.$$

From this, we have

$$(16)\qquad (q-2)S(r) \le N + \frac{C}{\delta^{25}}\frac{R}{R-r} = \sum_{j=1}^{q} n_j^{l)} + n(r, \tilde{R}_z) + \frac{C}{\delta^{25}}\frac{R}{R-r}.$$

If $(q-2)S(r') - N \le 0$ for some $r' \in (r, R)$, then $(q-2)S(r') \le N$, so that (16) holds in general.

This completes the proof of Theorem 1.

Applying Theorem 1, we have the following

COROLLARY 1. *Let $w = w(z)$ be a v-valued algebroid function in $|z| < R$, and $a_1, a_2, \dots, a_q (q \ge 3)$ be q disjoint points on w-sphere V, where the spherical distances between any two of them satisfy $d(a_i, a_j) \ge \delta \in (0, 1/2)$ $(i \ne j)$, then for any $r \in (0, R)$, we have*

$$(q-2)S(r) \le \sum_{j=1}^{q} \bar{n}^{l)}(R, a_j) + n(R, \tilde{R}_z) + \frac{C}{\delta^{25}}\frac{R}{R-r}.$$

where $l \ge 3$ is a positive integer, $\bar{n}^{l)}(R, a_j)$ is as in section 1.

4. The sequence of filling disks for algebroid functions

LEMMA 5. *Let $w = w(z)$ be a v-valued algebroid function of order ρ $(0 < \rho < \infty)$ defined by (1) in $|z| < \infty$, and $l \ge 3$ be a positive integer. For arbitrarily constants $\varepsilon \in (0, \rho)$ and $R > 1$, there exists $a_0 \in (1, 2)$ such that for any $a \in (1, a_0)$ the following assertion is true:*

Set $r_n = a^n$ and $m = [2\pi r_{n-1}/(r_n - r_{n-1})] = [2\pi/(a-1)]$, where $[x]$ is the integral part of x. For integers p, q with $p > 0$ and $0 \le q < m$, let $\theta_q = 2\pi(q+1)/m$, let $\Omega_{p,q}$ be the domain $\{a^{p-1} \le |z| < a^{p+2}\} \cap \{|\arg z - \theta_q| \le 2\pi/m\}$ and let $\bar{n}^{l)}(\Omega_{p,q}, w = \alpha)$ be the number of distinct zeros with multiplicity $\le l$ of $w(z) - \alpha$ in $\tilde{\Omega}_{p,q}$. Then there exist at least a pair of integers p_0, q_0, with $a^{p_0} > R$, and $(lv - 1)$ domains enclosed by spherical circles of radius $\delta = a^{-p_0\rho/26}$ on the Riemann sphere such that $\bar{n}^{l)}(\Omega_{p_0,q_0}, w = \alpha) \ge a^{p_0(\rho-\varepsilon)}$ for any complex value α not in the $(lv - 1)$ domains.

Proof. Suppose that the conclusion is false. Then, for some $\varepsilon \in (0, \rho)$ and $R > 1$, and any given sequence $\{a_i\}$ *with* $a_i > 1$ *and* $a_i \to 1$ $(i \to \infty)$, there exists at least a point $a \in (1, a_i)$ for each i, such that to any integers $p > P \overset{\text{def}}{=} [\log R/\log a]$ and $q \in \{0, 1, 2, \dots, m-1\}$, there exist accordingly lv complex numbers $\{\alpha_j = \alpha_j(p, q)\}_{j=1}^{lv}$ such that

$$(17)\qquad \bar{n}^{l)}(\Omega_{p,q}, w = \alpha_j) < a^{p(\rho-\varepsilon)},$$

where the spherical distance $d(\alpha_j, \alpha_k) \ge \delta = a^{-p\rho/26} (j \ne k)$.

Taking $r > R$ arbitrarily, set $T = [\log r/\log a]$ $(a^T \le r < a^{T+1})$. For any positive integers M and N, put

$$b = a^{\frac{1}{M}}, r_{p,t} = b^{Mp+t}, \quad t = 0, 1, 2, \ldots, M-1,$$

$$L_{p,t} = \{r_{p,t} \leq |z| < r_{p,t+1}\}, \quad \theta_{q,J} = \frac{2\pi q}{m} + \frac{2\pi j}{Nm},$$

$$\Delta_{q,J} = \{z; |z| < a^T, \theta_{q,J} \leq \arg z < \theta_{q,J+1}\}.$$

Since

$$\{a^{-1} \leq |z| < a^T\} = \bigcup_{t=0}^{M-1} \bigcup_{p=-1}^{T-1} L_{p,t},$$

$$\{|z| < a^T\} = \bigcup_{J=0}^{N-1} \bigcup_{q=0}^{m-1} \Delta_{q,J},$$

then there certainly exist a pair of t_0, j_0, which depend on T, without loss of generality, we may assume that $t_0 = 0$, $j_0 = 0$, such that

$$\sum_{p=-1}^{T-1} n(L_{p,0}, \tilde{R}_z) \leq \frac{1}{M} n(a^T, \tilde{R}_z), \tag{18}$$

$$\sum_{q=0}^{m-1} n(\Delta_{q,0}, \tilde{R}_z) \leq \frac{1}{N} n(a^T, \tilde{R}_z), \tag{19}$$

where $n(L_{p,0}, \tilde{R}_z)$ and $n(\Delta_{q,0}, \tilde{R}_z)$ are the number of branch points of $\tilde{R}_z$ in $\tilde{L}_{p,0}$ and $\tilde{\Delta}_{q,0}$ respectively.

Put

$$\Omega_{p,q}^0 = \left\{\frac{b^{Mp} + b^{Mp+1}}{2} \leq |z| < \frac{b^{Mp+M} + b^{Mp+M+1}}{2}\right\}$$
$$\cap \left\{\frac{\theta_{q,0} + \theta_{q,1}}{2} \leq \arg z < \frac{\theta_{q+1,0} + \theta_{q+1,1}}{2}\right\},$$

$$\bar{\Omega}_{p,q} = \{b^{Mp} \leq |z| < b^{Mp+M+1}\} \cap \{\theta_{q,0} \leq \arg z < \theta_{q+1,1}\}.$$

Then

$$\Omega_{p,q}^0 \subset \bar{\Omega}_{p,q} \subset \Omega_{p,q}.$$

Since $\{\bar{\Omega}_{p,q}\}$ overlap $\bigcup_{p=-1}^{T-1} L_{p,0}$ and $\bigcup_{q=0}^{m-1} \Delta_{q,0}$ twice at most, from (18), (19) we have

$$\sum_{p=1}^{T-2} \sum_{q=0}^{m-1} n(\bar{\Omega}_{p,q}, \tilde{R}_z) \leq \left(1 + \frac{1}{M} + \frac{1}{N}\right) n(a^T, \tilde{R}_z), \tag{20}$$

where $n(\bar{\Omega}_{p,q}, \tilde{R}_z)$ is the number of branch points of $\tilde{R}_z$ in $\tilde{\bar{\Omega}}_{p,q}$.

Obviously, $\bar{\Omega}_{p,q}$ can be mapped conformally to unit disk $|\zeta| < 1$ such that the center of $\Omega_{p,q}^0$ corresponds to $\zeta = 0$ and the image of $\Omega_{p,q}^0$ is contained in the disk $|\zeta| < \kappa < 1$, where $\kappa > 0$ is a constant, independent of p and q. Hence by Corollary 1 and (17), (20), we have

$$\begin{aligned}(lv-2)S(a^{T-1},w) &\leq (lv-2)\sum_{p=P+1}^{T-2}\sum_{q=0}^{m-1}(S(\Omega^0_{p,q},w)+(lv-2)S(a^{P+2},w)\\ &\leq \sum_{p=P+1}^{T-2}\sum_{q=0}^{m-1}\left(\sum_{j=1}^{lv}\bar{n}^{l)}(\bar{\Omega}_{p,q},w=\alpha_j)+n(\bar{\Omega}_{p,q},\tilde{R}_z)+\frac{C}{\delta^{25}}\frac{1}{1-\kappa}\right)\\ &\quad +(lv-2)S(a^{P+2},w)\\ &\leq lvTma^{(T-1)(\rho-\varepsilon)}+\left(1+\frac{1}{M}+\frac{1}{N}\right)n(a^T,\tilde{R}_z)\\ &\quad +Tm\frac{C}{1-\kappa}(a^{T\rho/26})^{25}+(lv-2)S(a^{P+2},w),\end{aligned}$$

where $S(\Omega^0_{p_0,q_0},w)=1/\pi\iint_{\tilde{\Omega}^0_{p_0,q_0}}(|w'(z)|/(1+|w(z)|^2)^2\,d\omega$, $\bar{n}^{l)}(\bar{\Omega}_{p,q},w=\alpha_j)$ is the number of distinct zeros with multiplicity $\leq l$ of $w(z)-\alpha_j$ in $\bar{\tilde{\Omega}}_{p,q}$. Taking $T(=[\log r/\log a])$ sufficiently large, then r sufficiently large too, and $r\in[a^T,a^{T+1})$, thus we have

$$(lv-2)S(ra^{-2},w)\leq r^{\rho-(\varepsilon/2)}+\left(1+\frac{1}{M}+\frac{1}{N}\right)n(r,\tilde{R}_z)+Cr^{(25/26)\rho}+C.$$

Dividing this inequality by r and integrating it, then

$$(lv-2)T(ra^{-2},w)\leq r^{\rho-(\varepsilon/2)}+\left(1+\frac{1}{M}+\frac{1}{N}\right)N(r,\tilde{R}_z)+Cr^{(25/26)\rho}+C\log r.$$

From (2), we have

$$\begin{aligned}(lv-2)T(ra^{-2},w)&\leq r^{\rho-(\varepsilon/2)}+\left(1+\frac{1}{M}+\frac{1}{N}\right)(2v-2)T(r,w)\\ &\quad +O(1)+Cr^{(25/26)\rho}+C\log r.\end{aligned}\tag{21}$$

Suppose that $\rho(r)$ is a precise order of $T(r,w)$. Put $U(r)=r^{\rho(r)}$, then $\lim_{r\to\infty}\rho(r)=\rho$ and

$$\lim_{r\to\infty}\frac{U(ra^{-2})}{U(r)}=a^{-2\rho},\quad \overline{\lim_{r\to\infty}}\frac{T(r,w)}{U(r)}=1.$$

Dividing (21) by $U(r)$ and letting $r\to\infty$, we obtain

$$(lv-2)a^{-2\rho}\leq\left(1+\frac{1}{M}+\frac{1}{N}\right)(2v-2).$$

Since $a \in (1, a_i)$, thus

$$lv - 2 \leq \left(1 + \frac{1}{M} + \frac{1}{N}\right) a_i^{2\rho}(2v - 2)$$

Letting i, M and $N \to \infty$ respectively, we have

$$lv \leq 2v.$$

This is contrary to the condition $l \geq 3$, and Lemma 5 is proved.

THEOREM 2. *Let $w = w(z)$ be a v-valued algebroid function of order ρ $(0 < \rho < \infty)$ defined by* (1) *in $|z| < \infty$, and $l \geq 3$ be a positive integer. Then there exists a sequence of filling disks for $w(z)$*

$$\Gamma_n : \{|z - z_n| < r_n \sigma_n\}, \quad n = 1, 2, \ldots$$

$$z_n = r_n e^{i\theta_n}, \quad \lim_{n \to \infty} r_n = \infty, \quad \lim_{n \to \infty} \sigma_n = 0 (\sigma_n > 0),$$

such that for any complex value α,

$$\bar{n}^{l)}(\Gamma_n, w = \alpha) \geq r_n^{\rho - \varepsilon_n},$$

with some possible exceptions for α enclosed in $lv - 1$ spherical circles with radius $r_n^{-\rho/26}$ and $\varepsilon_n \to 0 (n \to \infty)$.

Proof. Let $\varepsilon_n = \rho/2^n$, $R_n = 2^n$. By Lemma 5, we have

$$a_n \in \left(1, 1 + \frac{1}{n}\right), \quad m_n = \left[\frac{2\pi}{a_n - 1}\right], \quad p_n, \quad q_n, \quad \theta_{q_n} = \frac{2\pi(q_n + 1)}{m_n}$$

and

$$\Omega_{p_n, q_n} = \{a_n^{p_n - 1} \leq |z| \leq a_n^{p_n + 2}\} \cap \left\{|\arg z - \theta_{q_n}| \leq \frac{2\pi}{m_n}\right\} (n = 1, 2, \ldots).$$

Let $\theta_n = \theta_{q_n}$, $z_n = a_n^{p_n} e^{i\theta_n}$, then $r_n = |z_n| = a_n^{p_n} > R_n = 2^n \to \infty (n \to \infty)$.
Take $\sigma_n = 4(a_n - 1) \in (0, 4/n)$, then $\sigma_n \to 0 (n \to \infty)$.
Let $\Gamma_n = \{|z - z_n| < \sigma_n r_n\}$, then $\Gamma_n \supset \Omega_{p_n, q_n}$.
Hence for any complex value α, we have

$$\bar{n}^{l)}(\Gamma_n, w = \alpha) \geq \bar{n}^{l)}(\Omega_{p_n, q_n}, w = \alpha) \geq r_n^{\rho - \varepsilon_n},$$

with some possible exceptions for α enclosed in $lv - 1$ spherical circles with radius $\delta = r_n^{-\rho/26}$ and $\varepsilon_n \to 0 (n \to \infty)$.
This proves Theorem 2.

5. The Borel direction of algebroid functions

THEOREM 3. *Suppose that $w = w(z)$ is a v-valued algebroid function of order ρ $(0 < \rho < \infty)$ defined by* (1) *in $|z| < \infty$, $l \geq 3$ is a positive integer,* *then there exists a direction* $B: \arg z = \theta_0 (0 \leq \theta_0 < 2\pi)$ *such that for any given* $\delta (0 < \delta < \pi/2)$

$$\overline{\lim_{r\to\infty}} \frac{\log^+ \bar{n}^{l)}(r, \Delta(\theta_0, \delta), \alpha)}{\log r} = \rho$$

for any value of α, *with* $lv - 1$ *possible exceptions.*

Proof. By Theorem 2, there exists a sequence of filling disks of $w(z)$

$$\Gamma_n : \{|z - z_n| < |z_n|\sigma_n\}, \quad |z_n| = r_n, \quad \sigma_n \to 0 (n \to \infty)$$

such that for any complex value α,

$$\bar{n}^{l)}(\Gamma_n, w = \alpha) \geq |z_n|^{\rho - \varepsilon_n},$$

with some possible exceptions for α enclosed in $lv - 1$ spherical circles with radius $r_n^{-\rho/26}$ on the Riemann sphere, where $\varepsilon_n \to 0 (n \to \infty)$.

Let θ_0 be a cluster point of $\{\arg z_n\}$, then the direction $B: \arg z = \theta_0$ has the properties of Theorem 3. Otherwise, then there exists a positive number $\delta_0 (0 < \delta_0 < \pi/2)$ and lv exceptional values $a_j (1 \leq j \leq lv)$ such that

$$\overline{\lim_{n\to\infty}} \frac{\log \bar{n}^{l)}(r, \Delta(\theta_0, \delta_0), a_j)}{\log r} < \rho \quad (j = 1, 2, \ldots, lv). \tag{22}$$

On the other hand, since θ_0 is a cluster point of $\{\arg z_n\}$, then there is a subsequence of $\{\arg z_n\}$ which converges to θ_0. We may assume without loss of generality that $\lim_{n\to\infty} \arg z_n = \theta_0 (0 \leq \theta_0 < 2\pi)$. Hence for sufficiently large n we have

$$\Gamma_n \subset \{z; |\arg z - \theta_0| < \delta_0\}.$$

Let $\varepsilon_0 = \min_{1 \leq i \neq j \leq lv} \{d(a_i, a_j)\}$, then $\varepsilon_0 > 0$. Note that $r_n^{-\rho/26} \to 0$ $(n \to \infty)$, then

$$r_n^{-\rho/26} < \frac{\varepsilon_0}{2}$$

when n is large enough.

Because $\{\Gamma_n\}$ are a sequence of filling disks of $w(z)$, then there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ and an $a_{j_0} \in \{a_j\}_{j=1}^{vl}$ such that

$$\bar{n}^{l)}(\Gamma_{n_k}, w = a_{j_0}) \geq r_{n_k}^{\rho - \varepsilon_{n_k}}.$$

Hence

$$\begin{aligned} &\overline{\lim_{r\to\infty}} \frac{\log \bar{n}^{l)}(r, \Delta(\theta_0, \delta_0), a_{j_0})}{\log r} \\ &\quad \geq \overline{\lim_{k\to\infty}} \frac{\log \bar{n}^{l)}(\Gamma_{n_k}, w = a_{j_0})}{\log 2r_{n_k}} \end{aligned}$$

$$\geq \varlimsup_{k\to\infty} \frac{\log r_{n_k}^{\rho-\varepsilon_{n_k}}}{\log 2r_{n_k}}$$

$$= \rho.$$

This is contrary to (22) and Theorem 3 is proved.

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