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ON SECTIONAL GENUS OF QUASI-POLARIZED MANIFOLDS WITH NON-NEGATIVE KODAIRA DIMENSION, II

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Abstract

Let (X, L) be a quasi-polarized manifold over the complex number field with $\dim X = n$ and $\kappa(X) \geq 0$. If $n = 2, \kappa(X) \geq 0$, and $h^0(L) = \dim H^0(L) \geq 2$, then in our previous paper we studied a lower bound for sectional genus $g(L)$. In this paper, we mainly consider the case in which $n = 3$, $\kappa(X) \geq 0$, and $h^0(L) \geq 3$, and we obtain a lower bound for $g(L)$ which is a generalization of the result of our previous paper.

0. Introduction

Let X be a smooth projective manifold over the complex number field $\boldsymbol{C}$ with $\dim X = n$ and let L be a Cartier divisor on X. Then (X, L) is called a polarized (resp. quasi-polarized) manifold if L is ample (resp. nef and big). The sectional genus is defined by the following formula:

$$g(L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1},$$

where K_X is the canonical divisor of X.

A classification of (X, L) with small value of sectional genus was obtained by several authors. On the other hand, Fujita proved the following Theorem (see Theorem (2.13.1) in [Fj0]).

THEOREM 0.1. *Let (X, L) be a polarized manifold. Then for any fixed n and $g(L)$ there are only finitely many deformation type of (X, L) unless (X, L) is a scroll over a smooth curve.*

(For a definition of the deformation type of (X, L), see §13 of Chapter II in [Fj0].) By this theorem, Fujita proposed the following Conjecture; which is interesting but difficult.

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CONJECTURE. *Let (X, L) be a quasi-polarized manifold. Then $g(L) \geq q(X)$, where $q(X) = \dim H^1(\mathcal{O}_X)$ is the irregularity of X.*

If $\dim X = 2$ and $h^0(L) > 0$, then we can easily prove the above Conjecture. In [Fk4], we proved that the above Conjecture is true if $\dim X = 3$ and $h^0(L) \geq 2$. In [Fk3], we improved the above inequality if the Kodaira dimension $\kappa(X)$ of X is non-negative and $h^0(L) \geq 2$, that is,

THEOREM 0.2. *Let (X, L) be a quasi-polarized surface with $\kappa(X) \geq 0$ and $h^0(L) \geq 2$. Then we get $g(L) \geq 2q(X) - 1$ unless the rational map defined by $|L|$ is of special type.*

(In detail, see Theorem 3.1 in [Fk3].)

In this paper, we consider the 3-dimensional version of Theorem 0.2, that is, we improve a lower bound for $g(L)$ if $\dim X = 3$, $h^0(L) \geq 3$, and $\kappa(X) \geq 0$. The main result, which is a generalization of Theorem 0.2, is the following:

THEOREM 2.2. *Let (X, L) be a quasi-polarized 3-fold with $\kappa(X) \geq 0$ and $h^0(L) \geq 3$. We use Notation 2.1. Then (X, L) satisfies one of the following:*

(1) $g(L) \geq 2q(X) - 1$.

(2) $\dim W = 2$, *M_3 is not big, $g(L) \geq q(W_r) + 2g(F_\psi) \geq q(X) + g(F_\psi)$, and (W_r, A_r) is a scroll over a curve with $q(W_r) \geq 2$, where we take W_r as a minimal resolution of $\widetilde{W}$.*

(3) $\dim W = 1$, *M_3 is not big, $g(B_\alpha) \geq 3$, and $g(L) \geq g(B_\alpha) + 2q(F_\alpha) + 1 \geq q(X) + q(F_\alpha) + 1$.*

Here we should mention that at present we do not know whether there exists an example of the cases (2) and (3) in Theorem 2.2 or not.

The main theorem seems to enables us to study (X, L) with $\dim X = 3$, $\kappa(X) \geq 0$ and $h^0(L) \geq 3$ in detail. If $n = 2$, then in [Fk5] we obtained some results about a lower bound of $K_X L$ by using the result of [Fk3]. So if $n = 3$, we expect that we can get a result about a lower bound of $K_X L^2$ by using the result of this paper. We use the customary notations in algebraic geometry.

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1. Preliminaries

DEFINITION 1.1. Let X be a smooth projective variety with $\dim X > \dim Y \geq 1$. Then a morphism $f : X \to Y$ is a fiber space if f is surjective with connected fibers. Let L be a Cartier divisor on X. Then (f, X, Y, L) is called a quasi-polarized (resp. polarized) fiber space if $f : X \to Y$ is a fiber space and L is nef and big (resp. ample).

DEFINITION 1.2. Let X and Y be projective varieties with $\dim X = n$ and $\dim Y = m$, and let L be a line bundle on X. Then we say that (X, L) is a scroll over Y if there exists a fiber space $\pi : X \to Y$ such that any fiber of π is isomorphic to $\boldsymbol{P}^{n-m}$ and $L|_F = \mathcal{O}_{\boldsymbol{P}^{n-m}}(1)$, where $1 \le m < n$. A quasi-polarized fiber space (f, X, Y, L) is called a scroll if $(F, L_F) \cong (\boldsymbol{P}^{n-m}, \mathcal{O}_{\boldsymbol{P}^{n-m}}(1))$ for any fiber F of f, where $\dim X = n > m = \dim Y \ge 1$.

DEFINITION 1.3. (1) Let (X_1, L_1) and (X_2, L_2) be quasi-polarized manifolds where X_i may have singularities for $i = 1, 2$. Then (X_1, L_1) and (X_2, L_2) are said to be birationally equivalent if there is another variety G with birational morphisms $g_i : G \to X_i$ $(i = 1, 2)$ such that $g_1^* L_1 = g_2^* L_2$.

(2) Let (f_1, X_1, Y, L_1) and (f_2, X_2, Y, L_2) be quasi-polarized fiber spaces, where X_i may have singularities for $i = 1, 2$. Then (f_1, X_1, Y, L_1) and (f_2, X_2, Y, L_2) are said to be birationally equivalent if there is another variety G with birational morphisms $g_i : G \to X_i$ $(i = 1, 2)$ such that $g_1^* L_1 = g_2^* L_2$ and $f_1 \circ g_1 = f_2 \circ g_2$.

THEOREM 1.4. *Let (X, L) be a quasi-polarized 3-fold. Then there exists a quasi-polarized variety (X', L') which is birationally equivalent to (X, L) and satisfies one of the following conditions*:

(1) *$K_{X'} + 2L'$ is nef for the canonical $\boldsymbol{Q}$-bundle $K_{X'}$*;

(2) $\Delta(L') = 0$;

(3) *(X', L') is a scroll over a curve,*

where X' is a normal projective variety with only $\boldsymbol{Q}$-factorial terminal singularities.

Proof. See Theorem 4.2 in [Fj1]. □

Remark 1.5. Theorem 1.4 is true for $\dim X = n$ if the Flip Conjecture (see [KMM]) is true for $\dim X = n$.

THEOREM 1.6. *Let (f, X, C, L) be a quasi-polarized fiber space with $\dim X = 3$ and $\dim C = 1$. Then there exists a quasi-polarized fiber space (f', X', C, L') which is birationally equivalent to (f, X, C, L) such that (f', X', C, L') satisfies one of the following conditions*:

(1) *$K_{X'/C} + 2L'$ is nef*;

(2) *(f', X', C, L') is a scroll,*

where X' is a normal projective variety with only $\boldsymbol{Q}$-factorial terminal singularities and $K_{X'/C} = K_{X'} - (f')^* K_C$.

Proof. See Lemma 0.2 and Theorem 1.3 in [Fk2]. Here we show the outline of proof. We can prove the following by a method similar to the proof of Theorem 4.2 in [Fj1]:

There exists a quasi-polarized fiber space (f', X', C, L') which is birationally equivalent to (f, X, C, L) such that (f', X', C, L') satisfies one of the following conditions:

(1) $K_{X'/C} + 2L'$ is f'-nef;

(2) (f', X', C, L') is a scroll,

where X' is a normal projective variety with only $\boldsymbol{Q}$-factorial terminal singularities and $K_{X'/C} = K_{X'} - (f')^* K_C$. Next we can prove that $K_{X'/C} + 2L'$ is nef if $K_{X'/C} + 2L'$ is f'-nef by the same argument as in the proof of Theorem 1.1.2 in [Fk1]. □

THEOREM 1.7. *Let (X, L) be a quasi-polarized manifold with* $\dim X \geq 2$. *Assume that L is spanned by global sections. Then* $g(L) \geq q(X)$.

Proof. See Theorem 7.2.10 in [BS]. □

THEOREM 1.8. *Let (X, L) be a quasi-polarized manifold with* $\dim X \geq 2$ *and* $\kappa(X) \geq 0$ *such that L is spanned by global sections. Then* $g(L) \geq 2q(X) - 1$.

Proof. See Corollary 3.2 and Corollary 3.3 in [Fk3]. □

DEFINITION 1.9. Let (X, L) be a quasi-polarized manifold with $h^0(L) \geq 2$. Let X' be a smooth projective manifold and let $\mu : X' \to X$ be a birational morphism such that $\mathrm{Bs}|M'| = \phi$, where $|M'|$ is the movable part of $|\mu^*(L)|$. Then we define $\dim \varphi_{|L|}(X)$ as $\dim \varphi_{|M'|}(X')$.

DEFINITION 1.10. Let D_1 and D_2 be divisors on a smooth projective manifold X. We denote $D_1 \geq D_2$ if $D_1 - D_2$ is linearly equivalent to an effective divisor on X.

DEFINITION 1.11 (See [Fk0] and [Fk3]). (1) Let (X, L) be a quasi-polarized surface. Then (X, L) is called L-minimal if $LE > 0$ for any (-1)-curve E on X.

(2) For any quasi-polarized surface (X, L), there is a quasi-polarized surface (X_1, L_1) and a birational morphism $\mu : X \to X_1$ such that $L = \mu^*(L_1)$ and (X_1, L_1) is L_1-minimal. Then we call (X_1, L_1) an L-minimalization of (X, L).

(3) Let (f, X, C, L) be a quasi-polarized fiber space with $\dim X = 2$ and $\dim C = 1$. Then (f, X, C, L) is said to be relatively L-minimal if $LE > 0$ for any (-1)-curve E on X which is contained in a fiber of f.

(4) For any quasi-polarized fiber space (f, X, C, L) with $\dim X = 2$ and $\dim C = 1$, there exist a quasi-polarized fiber space (f_1, X_1, C, L_1) and a birational morphism $\mu : X \to X_1$ such that $f = f_1 \circ \mu$, $L = \mu^*(L_1)$, and (f_1, X_1, C, L_1) is relatively L_1-minimal. Then we call (f_1, X_1, C, L_1) a relative L-minimalization of (f, X, C, L).

LEMMA 1.12. *Let (X, L) be an L-minimal quasi-polarized surface with* $\kappa(X) \geq 0$. *Then $K_X + L$ is nef.*

Proof (See Lemma 2.4 in [Fk3]). If $K_X + L$ is not nef, then there is a (-1)-curve E on X such that $(K_X + L)E < 0$ since $\kappa(X) \geq 0$. Because $K_X E = -1$, we have $LE = 0$. But this contradicts assumption. □

LEMMA 1.13. *Let (f, X, C, L) be a relatively L-minimal quasi-polarized fiber space with* $\dim X = 2$ *and* $\kappa(X) \geq 0$. *Then* $K_{X/C} + L$ *is nef, where* $K_{X/C} = K_X - f^*K_C$: *relative canonical divisor of f.*

Proof (See Lemma 2.5 in [Fk3]). Let D be an irreducible reduced curve on X such that $f(D)$ is not a point. Let $\mu : X \to X'$ be a relatively minimal model of $f : X \to C$ and $D' = \mu(D)$. Then $K_{X/C}D \geq K_{X'/C}D'$. On the other hand $K_{X'/C}$ is nef because $\kappa(X) \geq 0$. Hence $(K_{X/C} + L)D \geq 0$. Next we prove that $K_{X/C} + L$ is f-nef. If $K_{X/C} + L$ is not f-nef, then there is a (-1)-curve E on X such that $f(E)$ is a point and $(K_{X/C} + L)E < 0$ because $\kappa(X) \geq 0$. Since $K_{X/C}E = -1$, we have $LE = 0$. But this contradicts the assumption. □

LEMMA 1.14 (G. Xiao). *Let (f, X, C) be a fiber space with* $\dim X = 2$, $\kappa(X) \geq 0$ *and* $g(C) = 0$. *Then* $q(X) \leq (1/2)(g(F) + 1)$, *where F is a general fiber of f.*

Proof. See [X]. □

LEMMA 1.15. *Let (X, L) be an L-minimal quasi-polarized surface. Assume that $|L|$ is spanned. Then* $g(L) \geq 2q(X) - 1$ *unless (X, L) is a scroll over a smooth curve.*

Proof. If $\kappa(X) \geq 0$, then this is proved by Corollary 3.2 in [Fk3]. So we assume $\kappa(X) = -\infty$. If $q(X) \leq 1$, then this is true. Hence we may assume that $q(X) \geq 2$. Then $K_X^2 \leq 8(1 - q(X))$. So we obtain

$$\begin{aligned}(K_X + L)^2 &= K_X^2 + 2(K_X + L)L - L^2 \\ &\leq 8(1 - q(X)) + 2(2g(L) - 2) - L^2 \\ &= 4(g(L) - 2q(X) + 1) - L^2.\end{aligned}$$

If $K_X + L$ is nef, then $(K_X + L)^2 \geq 0$. Therefore $g(L) \geq 2q(X)$. If $K_X + L$ is not nef, then (X, L) is a scroll over a smooth curve since (X, L) is L-minimal and $q(X) \geq 2$ (see the proof of Theorem 3.1 in [Fk0]). □

2. Main result

Before we prove the main theorem, we fix the notation which are used later.

NOTATION 2.1. (1) Let (X, L) be a quasi-polarized manifold with $\dim X = n$, $\kappa(X) \geq 0$, and $h^0(L) \geq 2$. We assume that the Flip Conjecture is true. Then by Theorem 1.4 and Remark 1.5, there exist a quasi-polarized variety (V_1, L_1), a smooth projective variety Y, and birational morphisms $\eta : Y \to X$ and $\eta_1 : Y \to V_1$ such that $\eta^*L = \eta_1^*L_1$ and $K_{V_1} + 2L_1$ is nef, where V_1 is a normal projective variety with only $\boldsymbol{Q}$-factorial terminal singularities. Let $\theta : V_2 \to V_1$ be a resolution of V_1, that is, θ is a birational morphism with $V_2 \backslash \theta^{-1}(\operatorname{Sing} V_1) \cong V_1 \backslash \operatorname{Sing} V_1$, where $\operatorname{Sing} V_1$ denotes the singular locus of V_1. Let $L_2 = \theta^*(L_1)$. Let $|M_2|$ be the movable part of $|L_2|$ and let Z_2 be the fixed part of $|L_2|$. We put $V_{2,0} = V_2$, $L_{2,0} = L_2$, and $M_{2,0} = M_2$. By Hironaka Theory, there exist a sequence of blowing ups: $\mu_k : V_{2,k} \to V_{2,k-1}$ along a smooth center B_{k-1}, and a non-negative integer t such that $\mathrm{Bs}|M_{2,t}| = \emptyset$ and $\mathrm{Bs}|M_{2,t-1}| \neq \emptyset$, where $M_{2,k}$ is the movable part of $|\mu_k^*(M_{2,k-1})|$. Let $\mu = \mu_1 \circ \cdots \circ \mu_t$, $V_3 = V_{2,t}$, $L_3 = \mu^*(L_2)$, and $M_3 = M_{2,t}$. Let E_k be the μ_k-exceptional effective divisor. Then there is a morphism $\varphi_{|M_3|} : V_3 \to \boldsymbol{P}^N$ defined by $|M_3|$. Let $W = \varphi_{|M_3|}(V_3)$. Then, by taking Stein factorization, there exist a normal projective variety $\widetilde{W}$, a morphism $\tilde{\varphi} : V_3 \to \widetilde{W}$, and a finite morphism $\varepsilon : \widetilde{W} \to W$ such that $\varphi_{|M_3|} = \varepsilon \circ \tilde{\varphi}$.

(2) Assume that $2 \leq \dim W \leq n-1$ and $h^0(L) \geq n$. Then any general member S_3 of $|M_3|$ is irreducible by Bertini Theorem. Then $h^0(\mu^*(L_2)|_{S_3}) \geq n-1$ and $(\varphi_{|M_3|})_{S_3} : S_3 \to \varphi_{|M_3|}(S_3)$ is the morphism defined by $|M_3|_{S_3}$. Let A be a hyperplane bundle $\mathcal{O}_W(1)$. Then $S_3 = \varphi_{|M_3|}^*(A)$. Let W_r be a resolution $r : W_r \to \widetilde{W}$, and let $\tilde{A} = \varepsilon^*A$ and $A_r = r^*\tilde{A}$. Then there exist a smooth projective variety $\tilde{V}_3$, a birational morphism $\nu : \tilde{V}_3 \to V_3$, and a fiber space $\psi : \tilde{V}_3 \to W_r$ such that $\tilde{\varphi} \circ \nu = r \circ \psi$. Let F_ψ be a general fiber of ψ and $\tilde{L}_3 = \nu^*L_3$.

(3) Assume that $\dim W = 1$. Let $\rho : X_\alpha \to X$ be a birational morphism such that the movable part of $|\rho^*L|$ is base point free. Let $L_\alpha = \rho^*L$ and let M_α be the movable part of $|L_\alpha|$. Let $\varphi_\alpha : X_\alpha \to \boldsymbol{P}^N$ be the morphism defined by $|M_\alpha|$. Then $W = \varphi_\alpha(X_\alpha)$. By taking Stein factorization, there exists a smooth projective curve B_α, a finite morphism $\omega : B_\alpha \to W$, and a fiber space $\psi_\alpha : X_\alpha \to B_\alpha$ such that $\varphi_\alpha = \omega \circ \psi_\alpha$. Let F_α be a general fiber of ψ_α.

THEOREM 2.2. *Let (X, L) be a quasi-polarized 3-fold with $\kappa(X) \geq 0$ and $h^0(L) \geq 3$. We use Notation 2.1. Then (X, L) satisfies one of the following*:

(1) $g(L) \geq 2q(X) - 1$.

(2) $\dim W = 2$, *M_3 is not big, $g(L) \geq q(W_r) + 2g(F_\psi) \geq q(X) + g(F_\psi)$, and (W_r, A_r) is a scroll over a curve with $q(W_r) \geq 2$, where we take W_r as a minimal resolution of $\widetilde{W}$.*

(3) $\dim W = 1$, *M_3 is not big, $g(B_\alpha) \geq 3$, and $g(L) \geq g(B_\alpha) + 2q(F_\alpha) + 1 \geq q(X) + q(F_\alpha) + 1$.*

Proof. We use Notation 2.1.

(I) The case in which M_3 is big.

Then we consider a quasi-polarized 3-fold (V_3, M_3). By Theorem 1.4, there exist a smooth projective variety U, a quasi-polarized 3-fold (V_3', M_3') and birational morphisms $\sigma_1 : U \to V_3$ and $\sigma_2 : U \to V_3'$ such that $\sigma_1^*M_3 = \sigma_2^*M_3'$ and $K_{V_3'} + 2M_3'$ is nef, where V_3' is a normal projective variety with $\dim V_3' = 3$ with only $\boldsymbol{Q}$-factorial terminal singularities. Then

$$
\begin{aligned}
g(L) = g(L_3) &= 1 + \frac{1}{2}(K_{V_3} + 2L_3)L_3^2 \\
&\geq 1 + \frac{1}{2}(K_{V_3} + 2M_3)L_3^2 \\
&= 1 + \frac{1}{2}\sigma_1^*(K_{V_3} + 2M_3)(\sigma_1^* L_3)^2 \\
&= 1 + \frac{1}{2}(K_U + 2\sigma_1^* M_3)(\sigma_1^* L_3)^2
\end{aligned}
$$

because L_3 is nef. Since $\sigma_1^* M_3 = \sigma_2^* M_3'$ and V_3' has only $\boldsymbol{Q}$-factorial terminal singularities, we obtain

$$
\begin{aligned}
g(L) &\geq 1 + \frac{1}{2}(K_U + 2\sigma_1^* M_3)(\sigma_1^* L_3)^2 \\
&= 1 + \frac{1}{2}(K_U + 2\sigma_2^* M_3')(\sigma_1^* L_3)^2 \\
&\geq 1 + \frac{1}{2}\sigma_2^*(K_{V_3'} + 2M_3')(\sigma_1^* L_3)^2
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\sigma_2^*(K_{V_3'} + 2M_3')(\sigma_1^*(L_3))^2 \\
&\quad = \sigma_2^*(K_{V_3'} + 2M_3')(\sigma_1^*(M_3))^2 \\
&\qquad + \sigma_2^*(K_{V_3'} + 2M_3')((\sigma_1^* L_3) + (\sigma_1^*(M_3)))(\sigma_1^* Z_3) \\
&\quad \geq \sigma_2^*(K_{V_3'} + 2M_3')(\sigma_1^*(M_3))^2 \\
&\quad = \sigma_2^*(K_{V_3'} + 2M_3')(\sigma_2^*(M_3'))^2 \\
&\quad = (K_{V_3'} + 2M_3')(M_3')^2
\end{aligned}
$$

because $\sigma_1^* L_3$ and $\sigma_1^* M_3$ are nef. Therefore $g(L) \geq 1 + (1/2)(K_{V_3'} + 2M_3')(M_3')^2 = g(M_3')$. Since $\mathrm{Bs}|M_3'| = \phi$ and V_3' has only $\boldsymbol{Q}$-factorial terminal singularities, any general section S_3' of $|M_3'|$ is smooth by Bertini's Theorem. Hence $g(M_3') = g(M_3'|_{S_3'}) \geq 2q(S_3') - 1$ by Theorem 1.8 since $\kappa(S_3') \geq 0$.

Claim 2.3. $q(S_3') = h^1(\mathcal{O}_{V_3'}) = q(X)$.

Proof. Let $\lambda : V_3'' \to V_3'$ be a resolution of V_3' and $M_3'' = \lambda^*(M_3')$. Then $\mathrm{Bs}|M_3''| = \phi$. Since S_3' is a general member of $|M_3'|$, we may assume that $S_3'' = \lambda^*(S_3')$ is a smooth projective surface which is a member of $|M_3''|$. Then S_3'' is birationally equivalent to S_3' and so $q(S_3'') = q(S_3')$. Since S_3'' is nef and big on V_3'', we obtain that $q(S_3'') = q(V_3'')$ by Kawamata-Viehweg Vanishing Theorem. Because V_3' has only rational singularities, we have $h^1(\mathcal{O}_{V_3'}) = q(V_3'') = q(X)$. This completes the proof of Claim 2.3. $\square$

Therefore by this Claim, we get $g(L) \geq 2q(X) - 1$.

(II) The case in which M_3 is not big.
Then $\dim W \leq 2$.

(II-1) The case in which $\dim W = 2$.
First we prove the following Claim.

CLAIM 2.4. $q(S_3) \geq q(V_3) = q(X)$.

Proof. Let $\Lambda \subseteq |M_3|$ be a linear pencil such that $\mathrm{Bs}\,\Lambda \neq \emptyset$ and a general member of Λ is smooth and irreducible. (We can take this Λ since $\dim W = 2$.) Then we make a fiber space defined by Λ. Let $\varphi_\Lambda : V_3 \to \boldsymbol{P}^1$ be the rational map defined by Λ. Let $\tau : V_4 \to V_3$ be an elimination of indeterminacy of φ_Λ. Then there exists a morphism $h : V_4 \to \boldsymbol{P}^1$. We remark that h has connected fibers by the choice of Λ. Hence $q(F_h) \geq q(V_4) = q(V_3)$, where F_h is a general fiber of h. On the other hand, since S_3 is a general member of $|M_3|$, we may assume that F_h is birationally equivalent to S_3. Hence $q(S_3) = q(F_h) \geq q(V_3) = q(X)$. This completes the proof of Claim 2.4. □

Since $K_{V_1} + 2L_1$ is nef, we obtain

$$g(L) = g(L_3) = 1 + \frac{1}{2}(\mu^* \circ \theta^*(K_{V_1} + 2L_1))(\mu^* \circ \theta^*(L_1))^2$$

$$\geq 1 + \frac{1}{2}(\mu^* \circ \theta^*(K_{V_1} + 2L_1))(\mu^* \circ \theta^*(L_1))S_3.$$

On the other hand, let E be an effective θ-exceptional divisor on V_2, then $\mu^*(E)(\mu^* \circ \theta^*(L_1))S_3 = 0$ since $\theta(E)$ is 0-dimensional. Hence

$$g(L) \geq 1 + \frac{1}{2}\mu^*(K_{V_2} + 2\theta^* L_1)(\mu^* \circ \theta^*(L_1))S_3.$$

By Claim 2.4 in [Fk4], we have

$$\begin{aligned}
&\mu^*(K_{V_2} + 2\theta^* L_1)(\mu^* \circ \theta^*(L_1))S_3 \\
&\quad \geq (K_{V_3} + S_3 + (\mu^* \circ \theta^*(L_1)))(\mu^* \circ \theta^*(L_1))S_3 \\
&\quad = (K_{S_3} + L_3|_{S_3})(L_3|_{S_3}).
\end{aligned}$$

Therefore $g(L) \geq g(L_3|_{S_3})$.

Let $\widetilde{S_3}$ be a general member of $|\nu^* M_3|$ such that $\nu(\widetilde{S_3}) = S_3$. We consider $\widetilde{\psi_3} = \psi|_{\widetilde{S_3}} : \widetilde{S_3} \to A_r$. Then $(\widetilde{\psi_3}, \widetilde{S_3}, A_r, \widetilde{L_3}|_{\widetilde{S_3}})$ is a quasi-polarized fiber space. We put $N_3 = \widetilde{L_3}|_{\widetilde{S_3}}$. We remark that $g(N_3) = g(L_3|_{S_3})$. So we obtain $g(L) \geq g(N_3)$. By construction, $\nu^* M_3|_{\widetilde{S_3}} \sim \sum_{i=1}^{a} \widetilde{F_{3,i}}$, where $\widetilde{F_{3,i}}$ is a general fiber of $\widetilde{\psi_3}$ and a is a positive integer. Hence $N_3 - \sum_{i=1}^{a} \widetilde{F_{3,i}} \geq 0$.

(II-1-1) The case in which $g(A_r) = 0$.
Let $\widetilde{F_3}$ be a general fiber of $\widetilde{\psi_3}$.

CLAIM 2.5. $g(N_3) \geq g(\widetilde{F_3})$.

Proof (See also Theorem 3.1 in [Fk3]). Let $\beta : \widetilde{S_3} \to T$ be an N_3-minimalization of $(\widetilde{S_3}, N_3)$ and $N_T = \beta_*(N_3)$. Then $N_3 = \beta^*(N_T)$ and $K_T + N_T$ is nef by Lemma 1.12. Let $\beta_j : T_j \to T_{j+1}$ be a blowing up at a point of T_{j+1}, $\beta = \beta_{t-1} \circ \cdots \circ \beta_0$, $T_0 = \widetilde{S_3}$, and $T_t = T$. Let $M_{T,j} = (\beta_{j-1})_*(M_{T,j-1})$ for $j = 1, \dots, t$, $M_{T,0} = \sum_{i=1}^{a} \widetilde{F_{3,i}}$, and $M_T = M_{T,t}$. Then M_T is nef. Let $M_{T,j} = (\beta_j)^*(M_{T,j+1}) - n_j E_j$, where E_j is the (-1)-curve of β_j for $j = 0, \dots, t-1$. Then we remark that $n_j \geq 0$ for any j. Then

$$(K_{\widetilde{S_3}} + N_3)(N_3 - M_{T,0}) = (K_T + N_T)(N_T - M_{T,t}) - \sum_{j=0}^{t-1} n_j.$$

Since $(M_{T,0})^2 = (\sum_{i=1}^{a} \widetilde{F_{3,i}})^2 = 0$, we obtain that $M_T^2 = \sum_{j=0}^{t-1} n_j^2$. Because $n_j \geq 0$, we obtain that $\sum_{j=0}^{t-1} n_j \leq \sum_{j=0}^{t-1} n_j^2$. Therefore

$$\begin{aligned}(K_{\widetilde{S_3}} + N_3)N_3 &= (K_{\widetilde{S_3}} + N_3)M_{T,0} + (K_T + N_T)(N_T - M_{T,t}) - \sum_{j=0}^{t-1} n_j \\ &\geq (K_{\widetilde{S_3}} + N_3)M_{T,0} - \sum_{j=0}^{t-1} n_j^2 \\ &= K_{\widetilde{S_3}} M_{T,0} + N_T M_T - M_T^2.\end{aligned}$$

Since $N_T - M_T \geq 0$ and M_T is nef, we obtain that $N_T M_T - M_T^2 \geq 0$. Hence $(K_{\widetilde{S_3}} + N_3)N_3 \geq K_{\widetilde{S_3}} M_{T,0} \geq 2g(\widetilde{F_3}) - 2$ since $a \geq 1$. Hence $g(N_3) \geq g(\widetilde{F_3})$. This completes the proof of Claim 2.5. □

Hence $g(N_3) \geq g(\widetilde{F_3})$. On the other hand, by Lemma 1.14 $g(\widetilde{F_3}) \geq 2q(\widetilde{S_3}) - 1$. Therefore $g(L) \geq g(N_3) \geq g(\widetilde{F_3}) \geq 2q(\widetilde{S_3}) - 1 = 2q(S_3) - 1 \geq 2q(X) - 1$ by Claim 2.4.

(II-1-2) The case in which $g(A_r) \geq 1$.

Then $a \geq 2$. Indeed, if $a = 1$, then $h^0(v^* M_3|_{\widetilde{S_3}}) = 1$ and $h^0(v^* M_3) = 2$. But this is a contradiction because $h^0(L_3) = h^0(L_2) = h^0(L_1) = h^0(L) \geq 3$.

CLAIM 2.6.

$$g(N_3) \geq \begin{cases} 2q(\widetilde{S_3}) - 1, & \text{if } N_3 \widetilde{F_3} \geq 2, \\ g(A_r) + 2g(\widetilde{F_3}), & \text{if } N_3 \widetilde{F_3} = 1, \end{cases}$$

where $\widetilde{F_3}$ is a general fiber of $\widetilde{\psi_3}$.

Proof. By taking a relative N_3-minimalization of $(\widetilde{\psi_3}, \widetilde{S_3}, A_r, N_3)$, we may assume that $(\widetilde{\psi_3}, \widetilde{S_3}, A_r, N_3)$ is relatively N_3-minimal (see Definition 1.11). Hence by Lemma 1.13, $K_{\widetilde{S_3}/A_r} + N_3$ is nef. Since $N_3 - \sum_{i=1}^{a} \widetilde{F_{3,i}} \geq 0$, we obtain

$$\begin{aligned}(K_{\widetilde{S_3}/A_r}+N_3)N_3 &\geq \sum_{i=1}^{a} K_{\widetilde{S_3}/A_r}\widetilde{F_{3,i}} + N_3\left(\sum_{i=1}^{a}\widetilde{F_{3,i}}\right)\\ &= \sum_{i=1}^{a}(2g(\widetilde{F_{3,i}})-2)+\sum_{i=1}^{a}N_3\widetilde{F_{3,i}}\\ &= 2a(g(\widetilde{F_3})-1)+\sum_{i=1}^{a}N_3\widetilde{F_{3,i}},\end{aligned}$$

where $\widetilde{F_3}$ is a general fiber of $\widetilde{\psi_3}$.
On the other hand

$$g(N_3)=g(A_r)+\frac{1}{2}(K_{\widetilde{S_3}/A_r}+N_3)N_3+(N_3\widetilde{F_3}-1)(g(A_r)-1).$$

Hence

$$g(N_3)\geq g(A_r)+a(g(\widetilde{F_3})-1)+\frac{1}{2}\sum_{i=1}^{a}N_3\widetilde{F_{3,i}}+(N_3\widetilde{F_3}-1)(g(A_r)-1).$$

If $N_3\widetilde{F_3}\geq 2$, then $\sum_{i=1}^{a}N_3\widetilde{F_{3,i}}\geq 2a\geq 4$ and we obtain

$$\begin{aligned}g(N_3)&\geq 2g(A_r)-1+2g(\widetilde{F_3})\\ &=2(g(A_r)+g(\widetilde{F_3}))-1\\ &\geq 2q(\widetilde{S_3})-1\end{aligned}$$

since $a\geq 2$.

If $N_3\widetilde{F_3}=1$, then there exists a section $\widetilde{C_3}$ of $\widetilde{\psi_3}$ such that $N_3-\widetilde{C_3}-\sum_{i=1}^{a}\widetilde{F_{3,i}}\geq 0$. Since $N_3-\widetilde{C_3}-\sum_{i=1}^{a}\widetilde{F_{3,i}}$ is contained in fibers of $\widetilde{\psi_3}$, we obtain $N_3\widetilde{C_3}\geq(\widetilde{C_3}+\sum_{i=1}^{a}\widetilde{F_{3,i}})\widetilde{C_3}$. Hence

$$\begin{aligned}(K_{\widetilde{S_3}/A_r}+N_3)N_3&\geq\sum_{i=1}^{a}(K_{\widetilde{S_3}/A_r}+N_3)\widetilde{F_{3,i}}+(K_{\widetilde{S_3}/A_r}+N_3)\widetilde{C_3}\\ &\geq\sum_{i=1}^{a}(K_{\widetilde{S_3}/A_r}+N_3)\widetilde{F_{3,i}}+(K_{\widetilde{S_3}/A_r}\widetilde{C_3}+(\widetilde{C_3})^2)+\sum_{i=1}^{a}\widetilde{F_{3,i}}\widetilde{C_3}\\ &=\sum_{i=1}^{a}(K_{\widetilde{S_3}/A_r}+N_3)\widetilde{F_{3,i}}+\sum_{i=1}^{a}\widetilde{F_{3,i}}\widetilde{C_3}\\ &=(2ag(\widetilde{F_3})-a)+a.\end{aligned}$$

Hence $(K_{\widetilde{S_3}/A_r}+N_3)N_3\geq 2a(g(\widetilde{F_3}))\geq 4g(\widetilde{F_3})$. This completes the proof of Claim 2.6. □

If $N_3\widetilde{F_3}\geq 2$, then $g(L)\geq g(N_3)\geq 2q(\widetilde{S_3})-1=2q(S_3)-1\geq 2q(X)-1$ by Claim 2.4. So we consider the case in which $N_3\widetilde{F_3}=1$.

If (W_r, A_r) is not a scroll over a curve, then $g(A_r) \geq 2q(W_r) - 1$ by Lemma 1.15. (We remark that (W_r, A_r) is A_r-minimal because W_r is a minimal resolution of $\widetilde{W}$.) On the other hand $g(\widetilde{F_3}) = q(F_\psi)$. Hence

$$\begin{aligned} g(N_3) &\geq g(A_r) + 2g(\widetilde{F_3}) \\ &\geq 2q(W_r) + 2g(F_\psi) - 1 \\ &\geq 2q(X) - 1. \end{aligned}$$

Hence (W_r, A_r) is a scroll over a curve if $g(L) < 2q(X) - 1$.

If $g(A_r) = 1$, then $q(W_r) \leq 1$ and we obtain

$$\begin{aligned} g(L) &\geq 1 + 2g(\widetilde{F_3}) \\ &\geq 2q(W_r) - 1 + 2g(F_\psi) \\ &\geq 2q(X) - 1. \end{aligned}$$

Hence $g(A_r) \geq 2$ if $g(L) < 2q(X) - 1$.

(II-2) The case in which $\dim W = 1$.

Here we use the notation in Notation 2.1 (3). We remark that $M_\alpha \sim \sum_{i=1}^{b} F_{\alpha,i}$ for some positive integer b and a general fiber $F_{\alpha,i}$ of φ_α.

(II-2-1) The case in which $g(B_\alpha) = 0$.

Then $b \geq 2$. Indeed if $b = 1$, then $h^0(L_\alpha) = h^0(M_\alpha) \leq 2$. This is a contradiction since $h^0(L_\alpha) = h^0(L) \geq 3$.

By the same argument as the proof of the Case (2) of Theorem 2.1 in [Fk4],

$$\begin{aligned} g(L) &= g(L_\alpha) \\ &= 1 + \frac{1}{2}(K_{X_\alpha} + 2L_\alpha)(L_\alpha)^2 \\ &\geq 1 + \frac{1}{2}(K_{X_\alpha} + 2L_\alpha)L_\alpha M_\alpha \\ &\geq 1 + \frac{1}{2}(K_{X_\alpha} + M_\alpha + L_\alpha)L_\alpha M_\alpha \\ &= 1 + \frac{b}{2}(K_{F_\alpha} + L_\alpha|_{F_\alpha})(L_\alpha|_{F_\alpha}), \end{aligned}$$

where F_α is a general fiber of φ_α.

Since $\kappa(F_\alpha) \geq 0$ and $b \geq 2$, we have

$$\begin{aligned} g(L) &\geq 1 + 2g(L_\alpha|_{F_\alpha}) - 2 \\ &= 2g(L_\alpha|_{F_\alpha}) - 1. \end{aligned}$$

On the other hand, $q(F_\alpha) \geq q(V_\alpha) = q(X)$ and $g(L_\alpha|_{F_\alpha}) \geq q(F_\alpha)$ since $h^0(L_\alpha|_{F_\alpha}) \geq 1$. Therefore

$$\begin{aligned} g(L) &\geq 2g(L_\alpha|_{F_\alpha}) - 1 \\ &\geq 2q(F_\alpha) - 1 \\ &\geq 2q(X) - 1. \end{aligned}$$

(II-2-2) The case in which $g(B_\alpha) \geq 1$.
Then we remark that $b \geq 3$. Indeed if $b \leq 2$, then $h^0(L) = h^0(L_\alpha) = h^0(M_\alpha) \leq 2$ since $g(B_\alpha) \geq 1$. But this is a contradiction. We consider the quasi-polarized fiber space $(\psi_\alpha, X_\alpha, B_\alpha, L_\alpha)$. By the same argument as the proof of the Case (1) of Theorem 2.1 in [Fk4], we can prove

$$\begin{aligned} g(L) &= g(L_\alpha) \\ &= g(B_\alpha) + \frac{1}{2}(K_{X_\alpha/B_\alpha} + 2L_\alpha)(L_\alpha)^2 + (g(B_\alpha) - 1)(L_\alpha^2 F_\alpha - 1) \\ &\geq g(B_\alpha) + \frac{1}{2}(K_{X_\alpha/B_\alpha} + 2L_\alpha)L_\alpha M_\alpha \\ &\geq g(B_\alpha) + \frac{b}{2}(K_{X_\alpha/B_\alpha} + 2L_\alpha)L_\alpha F_\alpha, \end{aligned}$$

where F_α is a general fiber of φ_α.

Since $b \geq 3$ and $\kappa(F_\alpha) \geq 0$, we obtain that

$$\begin{aligned} g(L) &\geq g(B_\alpha) + \frac{b}{2}(K_{F_\alpha} + 2L_\alpha|_{F_\alpha})L_\alpha|_{F_\alpha} \\ &\geq g(B_\alpha) + 3g(L_\alpha|_{F_\alpha}) - 3 + \frac{3}{2}(L_\alpha|_{F_\alpha})^2. \end{aligned}$$

Since $g(L) \in \boldsymbol{Z}$, we obtain that

$$\begin{aligned} g(L) &\geq g(B_\alpha) + 3g(L_\alpha|_{F_\alpha}) - 3 + 2 \\ &= g(B_\alpha) + 2g(L_\alpha|_{F_\alpha}) + g(L_\alpha|_{F_\alpha}) - 1. \end{aligned}$$

Because $\kappa(F_\alpha) \geq 0$, we have $g(L_\alpha|_{F_\alpha}) \geq 2$. Moreover $g(L_\alpha|_{F_\alpha}) \geq q(F_\alpha)$ since $h^0(L_\alpha|_{F_\alpha}) > 0$. Hence

$$\begin{aligned} g(L) &\geq g(B_\alpha) + 2g(L_\alpha|_{F_\alpha}) + 1 \\ &\geq g(B_\alpha) + 2q(F_\alpha) + 1 \\ &\geq q(X) + q(F_\alpha) + 1. \end{aligned}$$

If $g(B_\alpha) = 1$ or 2, then

$$\begin{aligned} g(L) &\geq 2g(B_\alpha) - 1 + 2g(L_\alpha|_{F_\alpha}) \\ &\geq 2q(V_\alpha) - 1 \\ &= 2q(X) - 1. \end{aligned}$$

This completes the proof of Theorem 2.2. □

3. Conjecture

Before we propose the Conjecture, we give the notations used later.

NOTATION 3.1. Let (X, L) be a polarized n-fold with $h^0(L) \geq 2$. Let $|M|$ be the movable part of $|L|$, and let Z be the fixed part of $|L|$. We put $X_0 = X$, $L_0 = L$, and $M_0 = M$. By Hironaka Theory, there exist a sequence of blowing ups: $\mu_k : V_{2,k} \to V_{2,k-1}$ along a smooth center B_{k-1}, and a non-negative integer t such that $\mathrm{Bs}|M_{2,t}| = \emptyset$ and $\mathrm{Bs}|M_{2,t-1}| \neq \emptyset$, where $M_{2,k}$ is the movable part of $|\mu_k^*(M_{2,k-1})|$. Let $\mu = \mu_1 \circ \cdots \circ \mu_t$, $X' = X_t$, and $M' = M_t$. Let E_k be the μ_k-exceptional effective divisor and $Z' = \mu^*L - M'$. Then there is a morphism $\varphi_{|M'|} : X' \to \boldsymbol{P}^N$ defined by $|M'|$. Let $W = \varphi_{|M'|}(X')$. Then there exist a normal projective variety $\widetilde{W}$, a morphism $\tilde{\varphi} : X' \to \widetilde{W}$, and a finite morphism $\varepsilon : \widetilde{W} \to W$ such that $\varphi_{|M'|} = \varepsilon \circ \tilde{\varphi}$. Let $r : W_r \to \widetilde{W}$ be a resolution of $\widetilde{W}$. Then there exist a smooth projective variety X'', a birational morphism $\theta' : X'' \to X'$, and a fiber space $f'' : X'' \to W_r$ such that $\tilde{\varphi} \circ \theta' = r \circ f''$. By definition there exists an ample and spanned line bundle A on W such that $M' = \varphi_{|M'|}^*(A)$. Let $\tilde{A} = \varepsilon^*(A)$ and $A_r = r^*(\tilde{A})$. We remark that A_r is nef, big, and spanned. Let $L'' = (\theta')^*(L')$ and let F'' be a general fiber of f''.

CONJECTURE 3.2. *Let (X, L) be a quasi-polarized manifold with $\dim X = n$, $\kappa(X) \geq 0$, and $h^0(L) \geq n$. We use Notation 3.1. Then (X, L) satisfies one of the following*:

(1) $g(L) \geq 2q(X) - 1$.

(2) $2 \leq m = \dim W \leq n - 1$, *$M'$ is not big*, $g(L) \geq q(W_r) + 2q(F'') + (n - m - 1) \geq q(X) + q(F'') + (n - m - 1)$, *and (W_r, A_r) is birationally equivalent to a scroll over a curve with* $q(W_r) \geq n - m + 1$.

(3) $\dim W = 1$, *M' is not big*, $g(\widetilde{W}) \geq n$, *and* $g(L) \geq g(W_r) + 2q(F'') + (n - 2) \geq q(X) + q(F'') + (n - 2)$.

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