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A NEW INTERPRETATION OF THE BÄCKLUND TRANSFORMATION OF THE SINE-GORDON EQUATION

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Abstract

Using special geodesic orthogonal coordinates on surfaces of constant Gauss curvature -1 in R^3, a new interpretation of the Bäcklund transformation of the sine-Gordon equation is given by elementary geometric procedure.

1. Introduction

Let $\psi = \psi(u, v)$ be a solution of the sine-Gordon equation

$$\psi_{uu} - \psi_{vv} = -\sin\psi\cos\psi. \tag{1}$$

Then for any constant τ (with $\sin 2\tau \neq 0$), the following system of equations

$$\begin{aligned} \sin\tau(\alpha_u - \psi_v) &= \cos\tau\cos\alpha\cos\psi + \sin\alpha\sin\psi, \\ \sin\tau(\alpha_v - \psi_u) &= -\cos\tau\sin\alpha\sin\psi - \cos\alpha\cos\psi, \end{aligned} \tag{2}$$

is solvable with integrability condition being given by (1). It is easily seen that every solution $\alpha = \alpha(u, v)$ of (2) satisfies

$$\alpha_{uu} - \alpha_{vv} = \sin\alpha\cos\alpha,$$

and hence the solutions of (2) produce new-solutions of the sine-Gordon equation. This fact is just the so-called classical Bäcklund transformation.

It is well-known that there is a correspondence between nontrivial solutions of the sine-Gordon equation (1) and the surfaces of constant Gauss curvature -1 in R^3 (see e.g., [4] and [7]). In particular, solution $\psi(u, v)$ of (1) with $\sin 2\psi \neq 0$ corresponds to a local surface of constant Gauss curvature -1 in R^3 with the first and second fundamental forms being given by

$$\begin{aligned} \mathrm{I} &= \sin^2\psi\, du^2 + \cos^2\psi\, dv^2, \\ \mathrm{II} &= \sin\psi\cos\psi(du^2 - dv^2). \end{aligned} \tag{3}$$

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In this note, we emphasize the existence of special geodesic orthogonal coordinates on surfaces of constant Gauss curvature -1 in R^3 and applying these coordinates we then get our main result which provides a new interpretation for the Bäcklund transformation of the sine-Gordon equation.

THEOREM. *Let $\psi = \psi(u, v)$ be a solution of* (1) *with* $\sin 2\psi \neq 0$. *ψ determines a surface Σ of constant Gauss curvature -1 in R^3 with the first and second fundamental forms being given by* (3) *in coordinates (u, v). Let (x, y) be geodesic orthogonal coordinates on Σ such that all the x-curves are geodesics and all the y-curves are of constant geodesic curvature* 1. *Then the angle $\alpha = \alpha(u, v)$ that y-curves makes with the u-curves, is also a solution of the sine-Gordon equation* (1).

Remark 1. The classical Bäcklund transformation (2) is geometrically derived from the pseudosphere line congruence and the fact that on surfaces of constant Gauss curvature -1 in R^3 there exist the so-called Tschebyscheff coordinates (see [1] and [4] for the details). Our theorem depends heavily on the special geodesic orthogonal coordinates as stated in the theorem. An advantage of our proof for the theorem is the fact that the new solution $\alpha(u, v)$ is achieved by solving two correlated initial value ODE problems.

Remark 2. For more information about the geometric sine-Gordon equation and the similar sinh-Gordon or sinh-Laplace equations, one may consult [3], [5], [7] and [8].

2. Proof of the theorem

Let $\psi = \psi(u, v)$ be a solution of (1) with $\sin 2\psi \neq 0$, i.e., without loss of generality, $0 < \psi < \pi/2$. Then ψ corresponds to a local surface Σ of constant Gauss curvature -1 in R^3 with the first and second fundamental forms being given by (3) in coordinates (u, v) and the integrability condition being given by (1).

For any fixed point $p_0 \in \Sigma$ with coordinate (u_0, v_0) and any constant α_0, let $V_0 = \cos\alpha_0 \vec{e}_u + \sin\alpha_0 \vec{e}_v \in T_{p_0}\Sigma$ be a unit tangent vector. Then there exists a unique curve Γ passing through p_0 along the direction V_0 and possessing constant geodesic curvature $k_g \equiv 1$.

We can choose a geodesic orthogonal coordinates (x, y) on Σ such that Γ is a y-curve with $x = 0$ and p_0 corresponding to $y = 0$, and with respect to (x, y) the first fundamental form of Σ takes the form (compare with, e.g., pp. 80–81 of [6])

$$I = dx^2 + e^{2x}\, dy^2. \tag{4}$$

Suppose, in the coordinates (u, v), Γ is parametrized by $u = u_1(y)$, $v = v_1(y)$ with y being the arc length of Γ, and let $\alpha_1(y)$ be the angle that Γ makes with the u-curves. Then $u = u_1(y)$, $v = v_1(y)$ and $\alpha = \alpha_1(y)$ satisfy the following equations (cf. p. 254 of [2]):

$$(5)\qquad \begin{cases} \dfrac{\partial u}{\partial y} = \dfrac{\cos\alpha}{\sin\psi}, \quad \dfrac{\partial v}{\partial y} = \dfrac{\sin\alpha}{\cos\psi}, \\ \dfrac{\partial \alpha}{\partial y} = 1 + \dfrac{\cos\alpha}{\sin\psi}\psi_v + \dfrac{\sin\alpha}{\cos\psi}\psi_u, \\ u(0) = u_0, \quad v(0) = v_0, \quad \alpha(0) = \alpha_0. \end{cases}$$

For any $p \in \Gamma$ with coordinate $(u_1(y), v_1(y))$, let C_y be the geodesic which meets Γ at p orthogonally, i.e., C_y is an x-curve with $y = constant$ and C_y is parametrized by arc length x. Let $\theta(x, y)$ be the angle that C_y makes with the u-curves. Then according to Liouville's formula (cf. p. 253, Proposition 4 of [2]), the coordinates $(u(x, y), v(x, y))$ of C_y and $\theta(x, y)$ satisfy the following equations

$$(6)\qquad \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\cos\theta}{\sin\psi}, \quad \dfrac{\partial v}{\partial x} = \dfrac{\sin\theta}{\cos\psi}, \\ \dfrac{\partial \theta}{\partial x} = \dfrac{\cos\theta}{\sin\psi}\psi_v + \dfrac{\sin\theta}{\cos\psi}\psi_u, \\ u(0, y) = u_1(y), \quad v(0, y) = v_1(y), \quad \theta(0, y) = \alpha_1(y) - \pi/2. \end{cases}$$

Thus we get functions $u = u(x, y)$, $v = v(x, y)$ and $\theta = \theta(x, y)$ which are defined on Σ and satisfy, for all the x-curves,

$$(7)\qquad \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\cos\theta}{\sin\psi}, \quad \dfrac{\partial v}{\partial x} = \dfrac{\sin\theta}{\cos\psi}, \\ \dfrac{\partial \theta}{\partial x} = \dfrac{\cos\theta}{\sin\psi}\psi_v + \dfrac{\sin\theta}{\cos\psi}\psi_u, \end{cases}$$

and for all the y-curves that are parametrized by arc length $e^x y$ with $x = constant$,

$$(8)\qquad \begin{cases} \dfrac{\partial u}{\partial y} = -e^x \dfrac{\sin\theta}{\sin\psi}, \quad \dfrac{\partial v}{\partial y} = e^x \dfrac{\cos\theta}{\cos\psi}, \\ \dfrac{\partial \theta}{\partial y} = e^x\left(1 - \dfrac{\sin\theta}{\sin\psi}\psi_v + \dfrac{\cos\theta}{\cos\psi}\psi_u\right). \end{cases}$$

Hence, for the coordinates transformation

$$(9)\qquad \begin{cases} u = u(x, y), \\ v = v(x, y), \end{cases}$$

we have

$$\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\cos\theta}{\sin\psi} & -e^x \dfrac{\sin\theta}{\sin\psi} \\ \dfrac{\sin\theta}{\cos\psi} & e^x \dfrac{\cos\theta}{\cos\psi} \end{pmatrix}. \tag{10}$$

Then for the inverse coordinates transformation of (9)

$$\begin{cases} x = x(u, v), \\ y = y(u, v), \end{cases} \tag{11}$$

we get from (10)

$$\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta\sin\psi & \sin\theta\cos\psi \\ -e^{-x}\sin\theta\sin\psi & e^{-x}\cos\theta\cos\psi \end{pmatrix}. \tag{12}$$

Now, from (6), (7), (11) and (12) we see that

$$\theta = \theta(x, y) = \theta(x(u, v), y(u, v)) \equiv: \theta(u, v),$$

and

$$\begin{aligned} \frac{\partial\theta}{\partial u} &= \frac{\partial\theta}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial\theta}{\partial y}\frac{\partial y}{\partial u} \\ &= \left(\frac{\cos\theta}{\sin\psi}\psi_v + \frac{\sin\theta}{\cos\psi}\psi_u\right)\cos\theta\sin\psi \\ &\quad + e^x\left(1 - \frac{\sin\theta}{\sin\psi}\psi_v + \frac{\cos\theta}{\cos\psi}\psi_u\right)(-e^{-x}\sin\theta\sin\psi) \\ &= \psi_v - \sin\theta\sin\psi, \\ \frac{\partial\theta}{\partial v} &= \frac{\partial\theta}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial\theta}{\partial y}\frac{\partial y}{\partial v} \\ &= \left(\frac{\cos\theta}{\sin\psi}\psi_v + \frac{\sin\theta}{\cos\psi}\psi_u\right)\sin\theta\cos\psi \\ &\quad + e^x\left(1 - \frac{\sin\theta}{\sin\psi}\psi_v + \frac{\cos\theta}{\cos\psi}\psi_u\right)e^{-x}\cos\theta\cos\psi \\ &= \psi_u + \cos\theta\cos\psi, \end{aligned}$$

that is,

$$\begin{cases} \dfrac{\partial\theta}{\partial u} = \psi_v - \sin\theta\sin\psi, \\ \dfrac{\partial\theta}{\partial v} = \psi_u + \cos\theta\cos\psi. \end{cases} \tag{13}$$

Hence

$$\theta_{uu} - \theta_{vv} = \sin\theta\cos\theta,$$

and $\alpha(u,v) = \theta(u,v) + \pi/2$ is a new solution of the sine-Gordon equation (1), which is the angle that y-curves makes with the u-curves on Σ. This completes the proof of the theorem.

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