Z. JIAO AND S. WANG
KODAI MATH. J.
23 (2000), 88–99

A CLASS OF TWISTED BRAIDED GROUPS*

Zhengming Jiao and Shuanhong Wang

1. Introduction

As a dual concept of quasitriangular Hopf algebra, the coquasitriangular Hopf algebra was introduced by Larson and Towber [1] in 1991. In 1993, some properties of coquasitriangular Hopf algebra were studied by Doi [2]. In special, a class of coquasitriangular Hopf algebra construction $(A^\sigma, \tilde{\sigma})$ was discovered. Recently Doi and Takeuchi [3] investigated a class of A^σ type Hopf algebra $B \bowtie_\tau H$ which is a special case of Majid's double crossproduct $B \bowtie H$ [4], [5]. So Doi's coquasitriangular Hopf algebra construction $(A^\sigma, \tilde{\sigma})$ is an important significant coquasitriangular Hopf algebra. While Majid has introduced the braided group over coquasitriangular Hopf algebra in monoidal categories [6]. It is known that every coquasitriangular Hopf algebra A can be converted by a process of transmutation into a braided group $\underline{A}$.

In this paper, we work with an invertible bilinear σ on A where A is a Hopf algebra. In section 2, we introduce the concept of T-Hopf algebra where $T : A \otimes A \to A \otimes A$ is a bilinear map. Then we show that A_σ with the invertible 2-cocycle σ [2], [3], [7] is a T_σ-Hopf algebra where $T_\sigma : A \otimes A \to A \otimes A$ is defined by

$$a \otimes b \mapsto \sum \sigma(a_1 S(a_5), b_1 S(b_5)) b_3 \otimes a_3 \sigma^{-1}(b_2 S(b_4), a_2 S(a_4))$$

which is different from that of [2].

Next in section 3, we give some necessary lemmas. In section 4, we show that if A is a commutative Hopf algebra, then A_σ can be obtained as a braided group $\underline{A}^\sigma$. We work over a fixed field k and follow Sweedler's book [10] for terminology on coalgebras, bialgebras and Hopf algebras. Let C be a coalgebra, the sigma notation

$$\Delta(c) = \sum c_1 \otimes c_2$$

for all $c \in C$ will be used frequently later. The antipode of a Hopf algebra will be denoted by S. We mainly refer to [2] and [3].

---

* This work was supported by the NSF of P.R. China and the Foundation of CSC.
Received March 16, 1999; revised July 7, 1999.

2. The T_σ-Hopf algebras A_σ

In this section we will give the definition of the T_σ-Hopf algebra and construct a T_σ-Hopf algebra A_σ.

Let A be a bialgebra and let σ be an invertible bilinear form on A, here "invertible" means σ has an inverse in the dual algebra $(A \otimes A)^*$. We say that σ is a 2-cocycle if [2], [3], [7]:

$$\langle 1\rangle \qquad \sum \sigma(x_1, y_1)\sigma(x_2y_2, z) = \sum \sigma(y_1, z_1)\sigma(x, y_2z_2)$$

Let σ^{-1} is the convolution inverse of σ, then we easily have [1, Theorem 1.6(a)]:

$$\langle 2\rangle \qquad \sum \sigma^{-1}(x_1y_1, z)\sigma^{-1}(x_2, y_2) = \sum \sigma^{-1}(x, y_1z_1)\sigma^{-1}(y_2, z_2)$$

We next define a coquasitriangular Hopf algebra [2] to be a pair (A, σ), where A is a Hopf algebra over k and σ is a bilinear form on A satisfying the followings:

(BR1) $\sigma(xy, z) = \sum \sigma(x, z_1)\sigma(y, z_2)$;
(BR2) $\sigma(x, yz) = \sum \sigma(x_1, z)\sigma(x_2, y)$:
(BR3) $\sigma(1, x) = \sigma(x, 1) = \varepsilon(x)$;
(BR4) $\sum \sigma(x_1, y_1)x_2y_2 = \sum y_1x_1\sigma(x_2, y_2)$

for all $x, y, z \in A$.

The σ satisfying condition (BR3) is called a normal bilinear form on A.

It is not hard to see that the following is true.

LEMMA 2.1. *Assume that σ is an invertible 2-cocycle bilinear form on A, then we have the followings*:

(1) $\sigma(xy, z) = \sum \sigma^{-1}(x_1, y_1)\sigma(y_2, z_1)\sigma(x_2, y_3z_2)$;
(2) $\sigma(x, yz) = \sum \sigma^{-1}(y_1, z_1)\sigma(x_1, y_2)\sigma(x_2y_3, z_2)$;
(3) $\sigma^{-1}(xy, z) = \sum \sigma^{-1}(x_1, y_1z_1)\sigma^{-1}(y_2, z_2)\sigma(x_2, y_3)$;
(4) $\sigma^{-1}(x, yz) = \sum \sigma^{-1}(x_1y_1, z_1)\sigma^{-1}(x_2, y_2)\sigma(y_3, z_2)$

for all $x, y, z \in A$.

LEMMA 2.2 ([2, Theorem 1.6(b), (c)]). *Let A be a Hopf algebra with a normal invertible 2-cocycle σ, then*:

(i) *Define $A^\sigma = A$ as a coalgebra and*

$$x \cdot y = \sum \sigma(x_1, y_1)x_2y_2\sigma^{-1}(x_3, y_3);$$

$$S^\sigma(x) = \sum \sigma(x_1, S(x_2))S(x_3)\sigma^{-1}(S(x_4), x_5).$$

Then A^σ is a Hopf algebra.

(ii) *If A is commutative as an algebra, then $(A^\sigma, \tilde{\sigma})$ is a symmetric coquasi-triangular Hopf algebra, where $\tilde{\sigma}$ is defined by*

$$\tilde{\sigma}(x, y) = \sum \sigma(y_1, x_1)\sigma^{-1}(x_2, y_2).$$

DEFINITION 2.3. Let (H, m_H, Δ_H) be an algebra and a coalgebra (not necessary bialgebra). Let $T : H \otimes H \to H \otimes H$ be a linear map such that $(H \otimes H, \cdot_T)$ is an algebra, where

$$(a \otimes b) \cdot_T (c \otimes d) = (m_H \otimes m_H)(I \otimes T \otimes I)(a \otimes b \otimes c \otimes d).$$

H is called a twist T-bialgebra if Δ_H is a algebra homomorphism. Furthermore, if there is an antipode for H, then H is called a twist T-Hopf algebra.

EXAMPLE 2.4. Let H be any Hopf algebra, and $\tau : H \otimes H \to H \otimes H$ is the classical twist, then H is a twist τ-Hopf algebra.

A left H-comodule coalgebra C means $(C, \Delta, \varepsilon, \rho)$ where (C, Δ, ε) is a coalgebra and $\rho : C \to H \otimes C, \rho(c) = \sum c^{(1)} \otimes c^{(2)}$, for all $c \in C$ is the comodule structure map such that

i) $\sum c^{(1)} \varepsilon(c^{(2)}) = \varepsilon(c)$;

ii) $\sum c^{(1)} \otimes c^{(2)}{}_1 \otimes c^{(2)}{}_2 = \sum c_1^{(1)} c_2^{(1)} \otimes c_1^{(2)} \otimes c_2^{(2)}$.

In [9], Molnar has affirmed that (H, Co_H) forms a left H-comodule coalgebra [9, (2.5)(a)], where Co_H is a coadjoint action on H and is assigned to $\mathrm{Co}_H(h) = \sum h_1 S(h_3) \otimes h_2$. Now we can obtain a class of twist T-Hopf algebra.

PROPOSITION 2.5. *Let A be a Hopf algebra with a normal invertible 2-cocycle σ and*

$$\sigma(xyz, mnt) = \sigma(yxz, nmt) = \sigma(xzy, mtn) \tag{2.5.1}$$

for all $x, y, z, m, n, t \in A$, then there is a T_σ-bialgebra A_σ defined by

$$\Delta_\sigma(a) = \sum \sigma^{-1}(a_1 S(a_3), a_4 S(a_6))(a_2 \otimes a_5);$$

$$a \cdot_\sigma b = \sum \sigma(a_1 S(a_3), b_1 S(b_3))(a_2 b_2);$$

$$T_\sigma : A \otimes A \to A \otimes A,$$

$$T_\sigma(a \otimes b) = \sum \sigma^{-1}(b_2 S(b_4), a_2 S(a_4))(b_3 \otimes a_3)\sigma(a_1 S(a_5), b_1 S(b_5)).$$

Furthermore, if $S^2 = I$ then A_σ is a T_σ-Hopf algebra, and it's antipode is S.

Proof. 1. Coassociative law.

$$\begin{aligned}
&(I \otimes \Delta_\sigma)\Delta_\sigma(a)\\
&= \sum \sigma^{-1}(a_1 S(a_3), a_4 S(a_{11}))\sigma^{-1}(a_5 S(a_7), a_8 S(a_{10}))(a_2 \otimes a_6 \otimes a_9)\\
&= \sum \sigma^{-1}(a_1 S(a_3), a_4 S(a_8) a_9 S(a_{13}))\sigma^{-1}(a_5 S(a_7), a_{10} S(a_{12}))(a_2 \otimes a_6 \otimes a_{11})\\
&= \sum \sigma^{-1}(a_2 S(a_4), a_7 S(a_9))\sigma^{-1}(a_1 S(a_5) a_6 S(a_{10}), a_{11} S(a_{13}))\\
&\quad (a_3 \otimes a_8 \otimes a_{12}) \qquad \text{(by } \langle 2 \rangle\text{)}
\end{aligned}$$

$$= \sum \sigma^{-1}(a_2 S(a_4), a_5 S(a_7)) \sigma^{-1}(a_1 S(a_8), a_9 S(a_{11}))(a_3 \otimes a_6 \otimes a_{10})$$
$$= (\Delta_\sigma \otimes I)\Delta_\sigma(a).$$

2. Associative law.

$$(a \cdot_\sigma b) \cdot_\sigma c$$
$$= \sum \sigma(a_1 S(a_5), b_1 S(b_5)) \sigma(a_2 b_2 S(a_4 b_4), c_1 S(c_3))(a_3 b_3 c_2)$$
$$= \sum \sigma(a_1 S(a_5), b_1 S(b_5)) \sigma(a_2 S(a_4) b_2 S(b_4), c_1 S(c_3))(a_3 b_3 c_2) \quad \text{(by (2.5.1))}$$
$$= \sum \sigma(a_1 S(a_3), b_2 S(b_4) c_2 S(c_4)) \sigma(b_1 S(b_5), c_1 S(c_5))(a_2 b_3 c_3) \quad \text{(by } \langle 1 \rangle)$$
$$= \sum \sigma(a_1 S(a_3), b_2 c_2 S(b_4 c_4)) \sigma(b_1 S(b_5), c_1 S(c_5))(a_2 b_3 c_3) \quad \text{(by (2.5.1))}$$
$$= a \cdot_\sigma (b \cdot_\sigma c).$$

3. Δ_σ is an algebra homomorphism.

$$\Delta_\sigma(a \cdot_\sigma b)$$
$$= \sum \sigma(a_1 S(a_8), b_1 S(b_8)) \sigma^{-1}(a_2 b_2 S(a_4 b_4), a_5 b_5 S(a_7 b_7))(a_3 b_3 \otimes a_6 b_6)$$
$$= \sum \sigma(a_1 S(a_5) a_6 S(a_{10}), b_1 S(b_5) b_6 S(b_{10}))$$
$$\sigma^{-1}(a_2 S(a_4) b_2 S(b_4), a_7 S(a_9) b_7 S(b_9))(a_3 b_3 \otimes a_8 b_8) \quad \text{(by (2.5.1))}$$
$$= \sum \sigma^{-1}((a_1 S(a_5))_1, (a_6 S(a_{10}))_1) \sigma((a_6 S(a_{10}))_2, (b_1 S(b_5) b_6 S(b_{10}))_1)$$
$$\sigma((a_1 S(a_5))_2, (a_6 S(a_{10}))_3 (b_1 S(b_5) b_6 S(b_{10}))_2) \sigma((a_2 S(a_4))_2, (b_2 S(b_4))_3)$$
$$\sigma^{-1}((a_2 S(a_4))_1, (b_2 S(b_4))_1 (a_7 S(a_9) b_7 S(b_9))_1)$$
$$\sigma^{-1}((b_2 S(b_4))_2, (a_7 S(a_9) b_7 S(b_9))_2)(a_3 b_3 \otimes a_8 b_8) \quad \text{(by (1), (3))}$$
$$= \sum \sigma^{-1}(a_1 S(a_9), (a_{10} S(a_{14}))_1) \sigma((a_{10} S(a_{14}))_2, (b_1 S(b_9) b_{10} S(b_{14}))_1)$$
$$\sigma(a_2 S(a_8), (b_2 S(b_8))(a_{10} S(a_{14}))_3 (b_{10} S(b_{14}))_2) \sigma(a_4 S(a_6), (b_4 S(b_6))_2)$$
$$\sigma^{-1}(a_3 S(a_7), (b_3 S(b_7))(a_{11} S(a_{13}))_1 (b_{11} S(b_{13}))_1)$$
$$\sigma^{-1}((b_4 S(b_6))_1, (a_{11} S(a_{13}))_2 (b_{11} S(b_{13}))_2)(a_5 b_5 \otimes a_{12} b_{12}) \quad \text{(by (2.5.1))}$$
$$= \sum \sigma^{-1}(a_1 S(a_9), a_{10} S(a_{18})) \sigma(a_{11} S(a_{17}), b_1 S(b_9 b_{10} S(b_{18}))$$
$$\sigma(a_2 S(a_8), (b_2 S(b_8))(a_{12} S(a_{16}))(b_{11} S(b_{17}))) \sigma(a_4 S(a_6), (b_4 S(b_6))_2)$$
$$\sigma^{-1}(a_3 S(a_7), (b_3 S(b_7))(a_{13} S(a_{15}))(b_{12} S(b_{16})))$$
$$\sigma^{-1}((b_4 S(b_6))_1, (a_{14} S(a_{16}))(b_{13} S(b_{15})))(a_5 b_5 \otimes a_{15} b_{14})$$

$$=\sum\sigma^{-1}(a_1S(a_5),a_6S(a_{12}))\sigma(a_7S(a_{11}),b_1S(b_7)b_8S(b_{12}))\sigma(a_2S(a_4)),b_3S(b_5))$$
$$\sigma^{-1}(b_2S(b_6),a_8S(a_{10})b_9S(b_{11}))(a_3b_4\otimes a_9b_{10})$$
$$=\sum\sigma^{-1}(a_1S(a_5),a_6S(a_{12}))\sigma^{-1}((b_1S(b_7))_1,(b_8S(b_{12}))_1)\sigma((a_7S(a_{11}))_1,$$
$$(b_1S(b_7))_2)\sigma((a_7S(a_{11}))_2(b_1S(b_7))_3,(b_8S(b_{12}))_2)\sigma(a_2S(a_4),b_3S(b_5))$$
$$\sigma^{-1}((b_2S(b_6))_1(a_8S(a_{10}))_1,(b_9S(b_{11}))_1)\sigma^{-1}((b_2S(b_6))_2,(a_8S(a_{10}))_2)$$
$$\sigma((a_8S(a_{10}))_3,(b_9S(b_{11}))_2)(a_3b_4\otimes a_9b_{10}) \qquad \text{(by (2),(4))}$$
$$=\sum\sigma^{-1}(a_1S(a_5),a_6S(a_{18}))\sigma^{-1}((b_1S(b_7))_1,(b_8S(b_{12}))_1)\sigma(a_7S(a_{17}),(b_1S(b_7))_2)$$
$$\sigma((a_8S(a_{16}))(b_1S(b_7))_3,(b_8S(b_{12}))_2)\sigma(a_2S(a_4),b_3S(b_5))$$
$$\sigma^{-1}((a_9S(a_{15}))(b_2S(b_6))_1,(b_9S(b_{11}))_1)\sigma^{-1}((b_2S(b_6))_2,a_{10}S(a_{14}))$$
$$\sigma(a_{11}S(a_{13}),(b_9S(b_{11}))_2)(a_3b_4\otimes a_{12}b_{10}) \qquad \text{(by (2.5.1))}$$
$$=\sum\sigma^{-1}(a_1S(a_5),a_6S(a_{18}))\sigma^{-1}(b_1S(b_{13}),b_{14}S(b_{22}))$$
$$(a_7S(a_{17}),b_2S(b_{12}))\sigma((a_8S(a_{16}))(b_3S(b_{11})),b_{15}S(b_{21}))\sigma(a_2S(a_4),b_6S(b_8))$$
$$\sigma^{-1}((a_9S(a_{15}))(b_4S(b_{10})),b_{16}S(b_{20}))\sigma^{-1}(b_5S(b_9),a_{10}S(a_{14}))$$
$$\sigma(a_{11}S(a_{13}),b_{17}S(b_{19}))(a_3b_7\otimes a_{12}b_{18})$$
$$=\sum\sigma^{-1}(a_1S(a_5),a_6S(a_{14}))\sigma^{-1}(b_1S(b_9),b_{10}S(b_{14}))$$
$$\sigma(a_7S(a_{13}),b_2S(b_8))\sigma(a_2S(a_4),b_4S(b_6))$$
$$\sigma^{-1}(b_3S(b_7),a_8S(a_{12}))\sigma(a_9S(a_{11}),b_{11}S(b_{13}))(a_3b_5\otimes a_{10}b_{12})$$
$$=\sum\sigma^{-1}(a_1S(a_5),a_6S(a_{12}))\sigma^{-1}(b_1S(b_9),(b_{10}S(b_{12}))$$
$$\sigma(a_7S(a_{11}),b_2S(b_8))\sigma^{-1}(a_2S(a_4),b_4S(b_6))$$
$$\sigma^{-1}(b_3S(b_7),a_8S(a_{10}))(a_3b_5\otimes a_9\cdot_\sigma b_{11})$$
$$=\sum\sigma^{-1}(a_1S(a_3),a_4S(a_{10}))\sigma^{-1}(b_1S(b_7),b_8S(b_{10}))$$
$$\sigma(a_5S(a_9),b_2S(b_6))\sigma^{-1}(b_3S(b_5),a_6S(a_8))(a_2\cdot_\sigma b_4\otimes a_7\cdot_\sigma b_9)$$
$$=\sum\sigma^{-1}(a_1S(a_3),a_4S(a_6))\sigma^{-1}(b_1S(b_3),b_4S(b_6))[(a_2\otimes a_5)\cdot_{T_\sigma}(b_2\otimes b_5)]$$
$$=\Delta_\sigma(a)\cdot_{T_\sigma}\Delta_\sigma(b).$$

If $S^2=I$, then we have

$$\begin{aligned}&\cdot_\sigma(I\otimes S)\Delta_\sigma(a)\\ &\quad=\sum\sigma^{-1}(a_1S(a_3),a_4S(a_6))(a_2\cdot_\sigma S(a_5))\\ &\quad=\sum\sigma^{-1}(a_1S(a_5),a_6S(a_{10})\sigma(a_2S(a_4),a_7S(a_9))(a_3S(a_8))\\ &\quad=\varepsilon(a).\end{aligned}$$

In a similar manner, we can show that

$$\cdot_\sigma(S \otimes I)\Delta_\sigma(a) = \varepsilon(a).$$

This completes the proof of Proposition 2.5.

Remark 2.6. Let A be any Hopf algebra with trivial normal invertible 2-cocycle $\sigma = \varepsilon_{A\otimes A}$, then $A = A_\sigma$ as Hopf algebra and T_σ becomes a classical twist.

We have the following important result.

COROLLARY 2.7. *Let A be a commutative Hopf algebra with a normal invertible 2-cocycle σ, then A_σ is T_σ-Hopf algebra.*

Proof. Invoking of [8, 1.5.12], we see that $S^2 = I$. It is obvious that the Corollary 2.7 is true.

3. Some lemmas

In [2], we can see that if σ is a normal invertible 2-cocycle map on A, then (A, σ) is not necessary a coquasitriangular Hopf algebra. In the same manner as [2], we can show that [2, Theorem 1.3] is true for a normal invertible 2-cocycle σ. List it following as a Lemma:

LEMMA 3.1. *Let A be a Hopf algebra with a normal invertible 2-cocycle σ on A. We set*

$$\lambda(a) = \sum \sigma(a_1, S(a_2))$$

for all $a \in A$. Then λ is convolution invertible with

$$\lambda^{-1}(a) = \sum \sigma^{-1}(S(a_1), a_2).$$

LEMMA 3.2. *Let A be a Hopf algebra with a normal invertible 2-cocycle σ, then we have*

$$\lambda^{-1}(ab) = \sum \lambda^{-1}(b_2)\lambda^{-1}(a_2)\sigma(a_3, b_3)\sigma(S(b_1), S(a_1))$$

for all $a, b \in A$.

Proof. We compute as following:

$$\begin{aligned}\lambda^{-1}(ab) &= \sum \sigma^{-1}(S(b_1)S(a_1), a_2b_2)\\ &= \sum \sigma^{-1}(S(b_2), S(a_3)a_4b_3)\sigma^{-1}(S(a_2), a_5b_4)\sigma(S(b_1), S(a_1)) \qquad \text{(by (3))}\\ &= \sum \sigma^{-1}(S(b_2), b_3)\sigma^{-1}(S(a_2), a_3b_4)\sigma(S(b_1), S(a_1))\end{aligned}$$

$$= \sum \sigma^{-1}(S(b_2), b_3)\sigma^{-1}(S(a_3)a_4, b_4)$$

$$\sigma^{-1}(S(a_2), a_5)\sigma(a_6, b_5)\sigma(S(b_1), S(a_1)) \qquad \text{(by (4))}$$

$$= \sum \sigma^{-1}(S(b_2), b_3)\sigma^{-1}(S(a_2), a_3)\sigma(a_4, b_4)\sigma(S(b_1), S(a_1))$$

$$= \sum \lambda^{-1}(b_2)\lambda^{-1}(a_2)\sigma(a_3, b_3)\sigma(S(b_1), S(a_1)).$$

Let A be Hopf algebra, $\mathcal{M}$ denote the category of left A-comodule. We have

LEMMA 3.3 ([6, Theorem 4.1]). *Let (A, σ) be a coquasitriangular Hopf* algebra. *Then there is a braided group $\underline{A}$ in the category $\mathcal{M}$. As a coalgebra, $\underline{A}$ coincides with A. The algebra structure and the antipode are transmuted to*

$$a \underline{\cdot} b = \sum \sigma(b_1 S(b_3), S(a_2))a_1 b_2,$$

$$\underline{S}(b) = \sum \sigma(S(b_4)S^2(b_2), b_1)S(b_3).$$

By [2], we can obtain that

LEMMA 3.4. *Let (A, σ) be a coquasitriangular Hopf algebra, then we have*

$$a \underline{\cdot} b = \sum \sigma(b_3, a_2)\sigma(b_1, S(a_3))a_1 b_2,$$

$$\underline{S}(b) = \sum \sigma(S(b_6)b_3, b_1)\lambda^{-1}(b_2)\lambda(b_4)S(b_5).$$

LEMMA 3.5. *Let A be a Hopf algebra with a normal invertible 2-cocycle σ. Then there is an isomorphism $\Psi : A^\sigma \to A_\sigma$ defined by*

$$\Psi(a) = \sum \sigma^{-1}(a_1 S(a_3), a_4)a_2$$

and Ψ is an A^σ-comodule homomorphism, where all comodule structure maps are always coadjoint actions [9].

Proof. Firstly, we have

$$\Psi(a) = \sum \sigma^{-1}(a_1 S(a_3), a_4)a_2$$

$$\sum \sigma^{-1}(a_1, S(a_6)a_7)\sigma^{-1}(S(a_5), a_8)\sigma(a_2, S(a_4))a_3 \qquad \text{(by (3))}$$

$$= \sum \sigma(a_1, S(a_3))a_2\lambda^{-1}(a_4).$$

It is easy to show that Ψ is invertible with

$$\Psi^{-1}(a) = \sum \sigma^{-1}(a_1), S(a_4))a_2\lambda(a_3).$$

To check that Ψ is an A^σ-comodule homomorphism, we calculate that

$$
\begin{aligned}
&(I\otimes\Psi)Co_{A^\sigma}(a)\\
&\quad=\sum a_1\cdot S^\sigma(a_3)\otimes\Psi(a_2)\\
&\quad=\sum \sigma(a_3,S(a_4))\sigma^{-1}(S(a_6),a_7)(a_1S(a_5)\otimes\Psi(a_2))\\
&\quad=\sum \sigma(a_1,S(a_9))\sigma^{-1}(a_3,S(a_7))\sigma(a_5,S(a_6))\sigma^{-1}(S(a_{10}),a_{11})(a_2S(a_8)\otimes\Psi(a_4))\\
&\quad=\sum \sigma(a_1,S(a_8))\sigma^{-1}(a_3,S(a_6))\lambda(a_5)\lambda^{-1}(a_9)(a_2S(a_7)\otimes\Psi(a_4))\\
&\quad=\sum \sigma(a_1,S(a_{11}))\sigma^{-1}(a_3,S(a_9))\lambda(a_8)\lambda^{-1}(a_{12})\\
&\qquad\sigma(a_4,S(a_6))\lambda^{-1}(a_7)(a_2S(a_{10})\otimes a_5)\\
&\quad=\sum \sigma(a_1,S(a_9))\sigma^{-1}(a_3,S(a_7))\sigma(a_4,S(a_6))\lambda^{-1}(a_{10})(a_2S(a_8)\otimes a_5)\\
&\quad=\sum \sigma(a_1,S(a_5))\lambda^{-1}(a_6)(a_2S(a_6)\otimes a_3)\\
&\quad=Co_{A^\sigma}\Psi(a).
\end{aligned}
$$

Lemma 3.6. *Let A be a Hopf algebra with a normal invertible 2-cocycle σ. Then Ψ (same as in Lemma 3.5) is a coalgebra homomorphism.*

Proof. We compute

$$
\begin{aligned}
\Delta_\sigma\Psi(a)&=\sum \sigma^{-1}(a_1S(a_8),a_9)\sigma^{-1}(a_2S(a_4),a_5S(a_7))(a_3\otimes a_6)\\
&=\sum \sigma^{-1}(a_1S(a_5)a_6S(a_{10}),a_{11})\sigma^{-1}(a_2S(a_4),a_7S(a_9))(a_3\otimes a_8)\\
&=\sum \sigma^{-1}(a_1S(a_3),a_4S(a_8)a_9)\sigma^{-1}(a_5S(a_7),a_{10})(a_2\otimes a_6)\\
&=\sum \sigma^{-1}(a_1S(a_3),a_4)\sigma^{-1}(a_5S(a_7),a_8)(a_2\otimes a_6)\\
&=(\Psi\otimes\Psi)\Delta(a).
\end{aligned}
$$

4. Braided group $\underline{A}^\sigma$

Throughout this section, A is always a commutative Hopf algebra with a normal invertible 2-cocycle σ. Lemma 2.2 says that $(A^\sigma,\tilde{\sigma})$ is a coquasitriangular Hopf algebra. In this section we will show that the braided group $\underline{A}^\sigma\cong A_\sigma$.

Lemma 4.1. *Let A be a commutative Hopf algebra with a normal invertible 2-cocycle σ. Then $\Psi:\underline{A}^\sigma\to A_\sigma$ (defined as in Lemma 3.5) is an algebra homomorphism.*

Proof. For all $a, b \in A$, we have

$$
\begin{aligned}
\Psi(a \cdot b) &= \sum \tilde{\sigma}(b_1 \cdot S^\sigma(b_3), S^\sigma(a_2))\Psi(a_1 \cdot b_2)\\
&= \sum \tilde{\sigma}(b_1 \cdot S^\sigma(b_{10}), S^\sigma(a_9))\sigma(a_1, b_2)\sigma^{-1}(a_8, b_9)\sigma(a_2b_3, S(a_4b_5))\\
&\quad \lambda^{-1}(b_7)\lambda^{-1}(a_6)\sigma(a_7, b_8)\sigma(S(b_6), S(a_5))a_3b_4\\
&= \sum \tilde{\sigma}(b_1 \cdot S^\sigma(b_{11}), S^\sigma(a_9))\sigma(a_2b_3, S(b_7))\sigma(a_3b_4S(b_6), S(a_5))\\
&\quad \sigma(a_7, b_9)\sigma(a_1, b_2)\sigma^{-1}(a_8, b_{10})\lambda^{-1}(a_6)\lambda^{-1}(b_8)a_4b_5 \qquad \text{(by } \langle 1\rangle\text{)}\\
&= \sum \tilde{\sigma}(b_1, S^\sigma(a_{10}))\tilde{\sigma}(S^\sigma(b_{11}), S^\sigma(a_9))\sigma(a_2b_3, S(b_7))\sigma(a_3b_4S(b_6), S(a_5))\\
&\quad \sigma(a_7, b_9)\sigma(a_1, b_2)\sigma^{-1}(a_8, b_{10})\lambda^{-1}(a_6)\lambda^{-1}(b_8)a_4b_5 \qquad \text{(by (BR1))}\\
&= \sum \tilde{\sigma}(b_1, S^\sigma(a_8))\tilde{\sigma}(b_9, a_7)\sigma(a_2b_3, S(b_7))\sigma(a_3b_4S(b_6), S(a_5))\\
&\quad \sigma(a_1, b_2)\lambda^{-1}(a_6)\lambda^{-1}(b_8)a_4b_5 \qquad \text{(by definition of } S^\sigma\text{)}\\
&= \sum \tilde{\sigma}(b_1, S(a_{10}))\sigma(a_8, S(a_9))\tilde{\sigma}(b_9, a_7)\sigma^{-1}(S(a_{11}), a_{12})\sigma(a_2b_3, S(b_7))\\
&\quad \sigma(a_3b_4S(b_6), S(a_5))\sigma(a_1, b_2)\lambda^{-1}(a_6)\lambda^{-1}(b_8)a_4b_5 \qquad \text{(by definition of } S^\sigma\text{)}\\
&= \sum \tilde{\sigma}(b_1, S(a_9))\tilde{\sigma}(b_9, a_7)\lambda(a_8)\lambda^{-1}(a_{10})\sigma(a_2b_3, S(b_7))\\
&\quad \sigma(a_3b_4S(b_6), S(a_5))\sigma(a_1, b_2)\lambda^{-1}(a_6)\lambda^{-1}(b_8)a_4b_5\\
&= \sum \tilde{\sigma}(b_1, S(a_{11}))\tilde{\sigma}(b_{11}, a_9)\sigma^{-1}(a_8, b_{10})\sigma(a_7, b_9)\lambda(a_{10})\lambda^{-1}(a_{12})\\
&\quad \sigma(a_2b_3, S(b_7))\sigma(a_3b_4S(b_6), S(a_5))\sigma(a_1, b_2)\lambda^{-1}(a_6)\lambda^{-1}(b_8)a_4b_5\\
&= \sum \tilde{\sigma}(b_1, S(a_6))\tilde{\sigma}(b_5, a_4)\sigma^{-1}(a_3, b_4)\\
&\quad \lambda(a_5)\lambda^{-1}(a_7)\sigma(a_1, b_2)\Psi(a_2b_3) \qquad \text{(by definition of } \Psi\text{)}\\
&= \sum \sigma(S(a_7), b_1)\sigma^{-1}(b_2, S(a_6))\tilde{\sigma}(b_6, a_4)\\
&\quad \sigma^{-1}(a_3, b_5)\lambda(a_5)\lambda^{-1}(a_8)\sigma(a_1, b_3)\Psi(a_2b_4) \qquad \text{(by definition of } \tilde{\sigma}\text{)}\\
&= \sum \sigma(S(a_8), b_1)\sigma^{-1}(b_2, S(a_7))\sigma(a_4, b_6)\sigma^{-1}(b_7, a_5)\\
&\quad \sigma^{-1}(a_3, b_5)\lambda(a_6)\lambda^{-1}(a_9)\sigma(a_1, b_3)\Psi(a_2b_4) \qquad \text{(by definition of } \tilde{\sigma}\text{)}\\
&= \sum \sigma(S(a_6), b_1)\sigma^{-1}(b_2, S(a_5))\sigma^{-1}(b_5, a_3)\lambda(a_4)\lambda^{-1}(a_7)\sigma(a_1, b_3)\Psi(a_2b_4)\\
&= \sum \sigma(S(a_{12}), b_1)\sigma^{-1}(b_2, S(a_{11}))\sigma^{-1}(b_{10}, a_9)\sigma(a_1, b_3)\\
&\quad \lambda(a_{10})\lambda^{-1}(a_{13})\sigma(b_4a_2, S(a_6))\sigma(b_5a_3S(a_5), S(b_7))\sigma(b_9, a_8)\\
&\quad \lambda^{-1}(b_8)\lambda^{-1}(a_7)b_6a_4 \qquad \text{(by definition of } \Psi \text{ and the commutativety)}
\end{aligned}
$$

$$= \sum \sigma(S(a_{10}), b_1)\sigma^{-1}(b_2, S(a_9))\sigma(a_1, b_3)\lambda(a_8)\lambda^{-1}(a_{11})$$
$$\sigma(b_4a_2, S(a_6))\sigma(b_5a_3S(a_5), S(b_7))\lambda^{-1}(b_8)\lambda^{-1}(a_7)b_6a_4$$
$$= \sum \sigma(S(a_8), b_1)\sigma^{-1}(b_2, S(a_7))\sigma(a_1, b_3)\lambda^{-1}(a_9)$$
$$\sigma(b_4a_2, S(a_6))\sigma(b_5a_3S(a_5), S(b_7))\lambda^{-1}(b_8)b_6a_4$$
$$= \sum \sigma(S(a_8), b_1)\sigma^{-1}(b_2, S(a_7))\sigma(a_1, b_4S(a_5))\lambda^{-1}(a_9)\sigma(b_3, S(a_6))$$
$$\sigma(b_5a_2S(a_4), S(b_7))\lambda^{-1}(b_8)b_6a_3 \qquad \text{(by } \langle 1\rangle \text{ and the commutativety)}$$
$$= \sum \sigma(S(a_6), b_1)\sigma(a_1, b_2S(a_5))\lambda^{-1}(a_7)\sigma(b_3a_2S(a_4), S(b_5))\lambda^{-1}(b_6)b_4a_3$$
$$= \sum \sigma(S(a_6), b_1)\sigma(a_1, S(a_5)b_2)\lambda^{-1}(a_7)$$
$$\sigma(a_2S(a_4)b_3, S(b_5))\lambda^{-1}(b_6)a_3b_4 \qquad \text{(by the commutativety)}$$
$$= \sum \sigma(a_1, S(a_7))\sigma(a_2S(a_6), b_1)\lambda^{-1}(a_8)$$
$$\sigma(a_3S(a_5)b_2, S(b_4))\lambda^{-1}(b_5)a_4b_3 \qquad \text{(by } \langle 1\rangle)$$
$$= \sum \sigma(a_1, S(a_5))\sigma((a_2S(a_4))_1, b_1)\lambda^{-1}(a_6)$$
$$\sigma((a_2S(a_4))_2b_2, S(b_4))\lambda^{-1}(b_5)a_3b_3$$
$$= \sum \sigma(a_1, S(a_5))\sigma(a_2S(a_4), b_2S(b_4))$$
$$\lambda^{-1}(a_6)\sigma(b_1, S(b_5))\lambda^{-1}(b_6)a_3b_3 \qquad \text{(by } \langle 1\rangle)$$
$$= \sum \sigma(a_1, S(a_3))\lambda^{-1}(a_4)(a_2 \cdot_\sigma b_2)\sigma(b_1, S(b_3))\lambda^{-1}(b_4)$$
$$= \Psi(a) \cdot_\sigma \Psi(b).$$

Note that the braided group $\underline{A}$ has the structure of a T-Hopf algebra relative to the braiding T in the category $\mathscr{M}$, where T is defined by

$$T : A \otimes A \to A \otimes A;$$
$$T(a \otimes b) = \sum (b_2 \otimes a_2)\sigma(a_1S(a_3), b_1S(b_3)).$$

Therefore, we can show:

Lemma 4.2. *Let A be a commutative Hopf algebra with a normal invertible 2-cocycle σ. Then $\Psi : \underline{A}^\sigma \to A_\sigma$ (defined as in Lemma 3.5) preserves the twist map.*

Proof. For all $a, b \in A$, we have

$$(\Psi \otimes \Psi)T(a \otimes b)$$
$$= \sum (\Psi \otimes \Psi)(b_2 \otimes a_2)\tilde{\sigma}(a_1 \cdot S^\sigma(a_3), b_1 \cdot S^\sigma(b_3))$$

$$=\sum(\Psi\otimes\Psi)(b_2\otimes a_2)\tilde{\sigma}(a_1\cdot S(a_5),b_1\cdot S(b_5))$$
$$\sigma(a_3,S(a_4))\sigma^{-1}(S(a_6),a_7)\sigma(b_3,S(b_4))\sigma^{-1}(S(b_6),b_7)\quad\text{(by definition of }S^{\sigma})$$
$$=\sum(\Psi\otimes\Psi)(b_4\otimes a_4)\tilde{\sigma}(a_2S(a_8),b_2S(b_8))$$
$$\sigma(a_1,S(a_9))\sigma^{-1}(a_3,S(a_7))\sigma(b_1,S(b_9))\sigma^{-1}(b_3,S(b_7))$$
$$\sigma(a_5,S(a_6))\sigma^{-1}(S(a_{10}),a_{11})\sigma(b_5,S(b_6))\sigma^{-1}(S(b_{10}),b_{11})\quad\text{(by definition of }\cdot)$$
$$=\sum(b_5\otimes a_5)\sigma(b_4,S(b_6))\lambda^{-1}(b_7)\sigma(a_4,S(a_6))\lambda^{-1}(a_7)$$
$$\tilde{\sigma}(a_2S(a_{11}),b_2S(b_{11}))\sigma(a_1,S(a_{12}))\sigma^{-1}(a_3,S(a_{10}))\sigma(b_1,S(b_{12}))\sigma^{-1}(b_3,S(b_{10}))$$
$$\sigma(a_8,S(a_9))\sigma^{-1}(S(a_{13}),a_{14})\sigma(b_8,S(b_9))\sigma^{-1}(S(b_{13}),b_{14})\quad\text{(by definition of }\Psi)$$
$$=\sum(b_5\otimes a_5)\sigma(b_4,S(b_6))\sigma(a_4,S(a_6))\tilde{\sigma}(a_2S(a_8),b_2S(b_8))$$
$$\sigma(a_1,S(a_9))\sigma^{-1}(a_3,S(a_7))\sigma(b_1,S(b_9))\sigma^{-1}(b_3,S(b_7))$$
$$\sigma^{-1}(S(a_{10}),a_{11})\sigma^{-1}(S(b_{10}),b_{11})\quad\text{(by the invertibility of }\lambda)$$
$$=\sum(b_3\otimes a_3)\tilde{\sigma}(a_2S(a_4),b_2S(b_4))\sigma(a_1,S(a_5))\sigma(b_1,S(b_5))\lambda^{-1}(a_6)\lambda^{-1}(b_6)$$
$$=\sum(b_4\otimes a_4)\sigma(a_2S(a_6),b_2S(b_6))\sigma^{-1}(b_3S(b_5),a_3S(a_5))$$
$$\sigma(a_1,S(a_7))\sigma(b_1,S(b_7))\lambda^{-1}(a_8)\lambda^{-1}(b_8)\quad\text{(by definition of }\tilde{\sigma})$$
$$=T_{\sigma}(\Psi\otimes\Psi)(a\otimes b)$$

as required. This finishes our proof.

Note that both $\underline{A}^{\sigma}$ and A_{σ} are Hopf algebra, by [7], the bialgebra map Ψ between $\underline{A}^{\sigma}$ and A_{σ} is automatically a Hopf algebra map. Therefore, we have established the main result in this paper.

THEOREM 4.3. *Let A be a commutative Hopf algebra with a normal invertible 2-cocycle σ, A_{σ} be T_{σ}-Hopf algebra and $\underline{A}^{\sigma}$ be the braided group of $(A^{\sigma},\tilde{\sigma})$. Then $\underline{A}^{\sigma}\cong A_{\sigma}$ as T-Hopf algebras.*

REFERENCES

[1] R. LARSON AND J. TOWBER, Two dual classes of bialgebras related to the concepts of "quantum groups" and "quantum Lie algebras", Comm. Algebra, **19** (1991), 3295–3345.

[2] Y DOI, Braided bialgebras and quadratic bialgebras, Comm. Algebra, **21** (1993), 1731–1749.

[3] Y DOI AND M. TAKEUCHI, Multiplication alteration by two-cocycles (the quantum view), Comm. Algebra, **22** (1994), 5715–5732.

[4] S. MAJID, Quasitriangular Hopf algebras and Yang-Baxter equations, Internat. J. Modern Phys. A, **5** (1990), 1–91.

[5] D. E. RADFORD, Minimal quasitriangular Hopf algebra, J. Algebra, **157** (1993), 185–315.

[6] S. MAJID, Braided groups, J. Pure Appl. Algebra, **86** (1993), 187–221.
[7] S. MAJID, Foundations of Quantum Group Theory, Cambridge University Press, Cambridge, 1995.
[8] S. MONTGOMERY, Hopf Algebra and their Actions on Rings, CBMS Regional Conf. Ser. in Math., **82**, American Mathematical Society, Providence, 1993.
[9] R. K. MOLNAR, Semi-direct products of Hopf algebra, J. Algebra, **47** (1977), 29–51.
[10] M. E. SWEEDLER, Hopf Algebras, Benjamin, New York, 1969.

DEPARTMENT OF MATHEMATICS
HENAN NORMAL UNIVERSITY
XINXIANG, HENAN 453002
P.R. CHINA