G. QIU
KODAI MATH. J.
23 (2000), 1–11

UNIQUENESS OF ENTIRE FUNCTIONS THAT SHARE SOME SMALL FUNCTIONS

GANGDI QIU

Abstract

In this paper we obtain a unicity theorem of an entire function and its derivative that share two small functions IM. So we generalize and improve some results given by Rubel–Yang, Mues–Steinmetz and J. H. Zheng etc.

1. Introduction and main results

In this paper, we use the same signs as given in Nevanlinna theory of meromorphic functions (see [1]). By $S(r,f)$ we denote any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$, possibly outside a set of r with finite linear measure. Let f and g be two meromorphic functions. Then the meromorphic function α is said a small function of f if and only if $T(r,\alpha) = S(r,f)$. We say that f and g share a value a IM(CM) if $f - a$ and $g - a$ have the same zeros ignoring multiplicities (with the same multiplicity). When a is a small function of f and g, a is said a common small function of f and g IM(CM). In addition, we introduce the following denotations:

$S(m,n)(b) = \{z | z$ is a common zero of $f - b$ and $f' - b$ with multiplicities m and n respectively$\}$. $\bar{N}(m,n)(r, 1/(f-b))$ denotes the counting function of f with respect to the set $S(m,n)(b)$.

On the problems of uniqueness of an entire function and its derivative that share some values, Rubel–Yang (see [2]) proved that if the entire function f and f' share two distinct finite values CM then $f \equiv f'$. Mues–Steinmetz (see [3]) improved this result to the case when f and f' share two values IM. In 1992, J. H. Zheng and S. P. Wang (see [4]) generalized this result to the f and f' which share two small functions CM. In this paper, we generalize and improve above results to obtain the following result:

THEOREM 1. *Let f be a nonconstant entire function, a and b two distinct small functions of f with $a \not\equiv \infty$ and $b \not\equiv \infty$. If f and f' share a and b IM, then $f \equiv f'$.*

Supported by Fujian Provincal Science Foundation.
Keywords: Entire function, Meromorphic function, Small function, Uniqueness.
Received August 24, 1998; revised July 5, 1999

2. Some lemmas

LEMMA 1. *Let f be a nonconstant entire function, a_1 and a_2 two distinct small functions of f with $a_1 \not\equiv \infty$ and $a_2 \not\equiv \infty$. Set*

$$\Delta(f) = \begin{vmatrix} f - a_1 & a_1 - a_2 \\ f' - a_1' & a_1' - a_2' \end{vmatrix} = \begin{vmatrix} f - a_2 & a_1 - a_2 \\ f' - a_2' & a_1' - a_2' \end{vmatrix}. \tag{1}$$

Then

$$\text{(i)} \quad \Delta(f) \not\equiv 0, \tag{2}$$

$$\text{(ii)} \quad m\left[r, \frac{\Delta(f)}{f - a_i}\right] = S(r,f), \quad (i = 1,2) \tag{3}$$

$$\text{(iii)} \quad m\left[r, \frac{\Delta(f)}{(f - a_1)(f - a_2)}\right] = S(r,f), \tag{4}$$

$$\text{(iv)} \quad m\left[r, \frac{\Delta(f)(f - \beta)}{(f - a_1)(f - a_2)}\right] = S(r,f), \tag{5}$$

where β is an arbitrary small function of f.

$$\text{(v)} \quad \sum_{i=1}^{2} N\left(r, \frac{1}{f - a_i}\right) - N\left(r, \frac{1}{\Delta(f)}\right) \leq \sum_{i=1}^{2} \overline{N}\left(r, \frac{1}{f - a_i}\right) + S(r,f). \tag{6}$$

Proof. Suppose that $\Delta(f) \equiv 0$, then from (1) we have

$$\frac{f' - a_1'}{f - a_1} = \frac{a_1' - a_2'}{a_1 - a_2}.$$

By integrating for above two side we get

$$f = a_1 + c(a_1 - a_2), \quad (c \neq 0 \text{ is a constant})$$

which contradicts the fact that a_1 and a_2 are small functions of f. Hence

$$\Delta(f) \not\equiv 0.$$

Again by (1) we have

$$m\left(r, \frac{\Delta(f)}{f - a_i}\right) \leq m(r, a_1' - a_2') + m(r, a_1 - a_2) + m\left(r, \frac{f' - a_i'}{f - a_i}\right) + \log 2$$
$$= S(r,f), \quad (i = 1,2)$$

i.e., (3) holds.

Note that

$$\frac{1}{(f - a_1)(f - a_2)} = \frac{1}{a_1 - a_2}\left[\frac{1}{f - a_1} - \frac{1}{f - a_2}\right], \tag{7}$$

and

$$\frac{\Delta(f)(f-\beta)}{(f-a_1)(f-a_2)} = \frac{\Delta(f)}{f-a_2} + \frac{(a_2-\beta)\Delta(f)}{(f-a_1)(f-a_2)}. \tag{8}$$

So (3) and (7) imply (4). (5) follows from (3), (4) and (8).

Next, it is easy to see from (1) if any zero of $f - a_i$ $(i = 1, 2)$ with multiplicity p is not the pole of a_1 and a_2, as well as is not the zero of $(a_1 - a_2)$, then it must be a zero of $\Delta(f)$ with multiplicity at least $(p-1)$. Thus (6) holds.

This completes the proof of Lemma 1.

LEMMA 2. *Let f be a nonconstant entire function, a and b two distinct small functions of f with $a \not\equiv \infty$ and $b \not\equiv \infty$. Again let*

$$c_k = a + k(a-b), \quad (k \text{ is a positive integer}). \tag{9}$$

Then

$$(n+1)T(r,f') \le \overline{N}\left(r,\frac{1}{f'-a}\right) + \overline{N}\left(r,\frac{1}{f'-b}\right) + \sum_{k=1}^{n}\overline{N}\left(r,\frac{1}{f'-c_k}\right) + S(r,f). \tag{10}$$

Proof. It is easy to see from (9) that $c_k \not\equiv a$ and $c_k \not\equiv b$ $(k = 1, 2, \ldots, n)$, and they are distinct small function of f. Let

$$F = \frac{f'-a}{b-a}, \tag{11}$$

then

$$T(r,F) = T(r,f') + S(r,f). \tag{12}$$

By the second fundamental theorem

$$\begin{aligned}(n+1)T(r,F) &< \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-1}\right) \\ &\quad + \sum_{k=1}^{n}\overline{N}\left(r,\frac{1}{F+k}\right) + S(r,f) \\ &\le \overline{N}\left(r,\frac{1}{f'-a}\right) + \overline{N}\left(r,\frac{1}{f'-b}\right) + \sum_{k=1}^{n}\overline{N}\left(r,\frac{1}{f'-c_k}\right) + S(r,f).\end{aligned}$$

This and (12) imply (10).

This completes the proof of Lemma 2.

LEMMA 3. *Let f be a nonconstant entire functıon, a and b two distinct samll functions of f with $a \not\equiv \infty$* and *$b \not\equiv \infty$. If f and f' share a and b* IM, *and*

$$T(r,f) = T(r,f') + S(r,f), \tag{13}$$

then $f \equiv f'$.

Proof. Assume that $f \not\equiv f'$. From the fact that f and f' share a and b IM we know that

$$\begin{aligned}\bar{N}\left(r,\frac{1}{f-a}\right) + \bar{N}\left(r,\frac{1}{f-b}\right) &\le N\left(r,\frac{1}{f-f'}\right) \le T(r,f-f') + O(1)\\ &\le m(r,f) + m\left(r,1-\frac{f'}{f}\right) + S(r,f)\\ &\le T(r,f) + S(r,f). \end{aligned} \tag{14}$$

Now by the second fudamental theorem

$$T(r,f) \le \bar{N}\left(r,\frac{1}{f-a}\right) + \bar{N}\left(r,\frac{1}{f-b}\right) + S(r,f). \tag{15}$$

Combining (14) and (15) we have

$$T(r,f) = \bar{N}\left(r,\frac{1}{f-a}\right) + \bar{N}\left(r,\frac{1}{f-b}\right) + S(r,f). \tag{16}$$

Set

$$\varphi = \frac{\Delta(f)(f-f')}{(f-a)(f-b)}, \tag{17}$$

and

$$\chi = \frac{\Delta(f')(f-f')}{(f'-a)(f'-b)}, \tag{18}$$

where $\Delta(f)$ and $\Delta(f')$ are defined by (1), $a_1 = a$ and $a_2 = b$.

From (2) we know that $\Delta(f) \not\equiv 0$ and $\Delta(f') \not\equiv 0$. Therefore, it follows that $\varphi \not\equiv 0$ and $\chi \not\equiv 0$. It is easy to see from (6) that $N(r,\varphi) = S(r,f)$ and $N(r,\chi) = S(r,f)$. Again by (5) we get

$$m(r,\varphi) \le m\left(r,\frac{\Delta(f)\cdot f}{(f-a)(f-b)}\right) + m\left(r,1-\frac{f'}{f}\right) = S(r,f).$$

Thus

$$T(r,\varphi) = S(r,f). \tag{19}$$

Next, for any positive integer k, by (3), (4) and (5) we have

$$
\begin{aligned}
m(r,\chi) &\le m\left[r,\frac{\Delta(f')(f'-c_k)\left(\frac{f-c_k}{f'-c_k}-1\right)}{(f'-a)(f'-b)}\right]\\
&\le m\left(r,\frac{\Delta(f')}{f'-b}\right)+m\left(r,\frac{\Delta(f')(a-c_k)}{(f'-a)(f'-b)}\right)+m\left(r,\frac{f-c_k}{f'-c_k}-1\right)+\log 2\\
&\le m\left(r,\frac{f-c_k}{f'-c_k}\right)+S(r,f)\\
&= m\left(r,\frac{f'-c_k}{f-c_k}\right)+N\left(r,\frac{f'-c_k}{f-c_k}\right)-N\left(r,\frac{f-c_k}{f'-c_k}\right)+S(r,f)\\
&\le m\left(r,\frac{f'-c_k'}{f-c_k}\right)+m\left(r,\frac{c_k'-c_k}{f-c_k}\right)+N\left(r,\frac{1}{f-c_k}\right)+N(r,f'-c_k)\\
&\quad -N\left(r,\frac{1}{f'-c_k}\right)-N(r,f-c_k)+S(r,f)\\
&\le m\left(r,\frac{1}{f-c_k}\right)+N\left(r,\frac{1}{f-c_k}\right)-N\left(r,\frac{1}{f'-c_k}\right)+S(r,f)\\
&\le T(r,f)-N\left(r,\frac{1}{f'-c_k}\right)+S(r,f).
\end{aligned}
$$

Thus

$$
T(r,\chi)\le T(r,f)-N\left(r,\frac{1}{f'-c_k}\right)+S(r,f). \tag{20}
$$

On the other hand, combining (10), (13) and (14) we get

$$
\begin{aligned}
2T(r,f') &\le \overline{N}\left(r,\frac{1}{f'-a}\right)+\overline{N}\left(r,\frac{1}{f'-b}\right)+\overline{N}\left(r,\frac{1}{f'-c_k}\right)+S(r,f)\\
&\le \overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{f-b}\right)+N\left(r,\frac{1}{f'-c_k}\right)+S(r,f)\\
&= T(r,f)+N\left(r,\frac{1}{f'-c_k}\right)+S(r,f)\\
&= T(r,f')+N\left(r,\frac{1}{f'-c_k}\right)+S(r,f),
\end{aligned}
$$

which results in

$$
N\left(r,\frac{1}{f'-c_k}\right)=T(r,f')+S(r,f). \tag{21}
$$

Combining (13), (20) and (21) we deduce that

$$T(r,\chi) = S(r,f). \tag{22}$$

Again let

$$Hm,n = m\varphi - n\chi. \quad (m \text{ and } n \text{ are positive integers}). \tag{23}$$

If there exist m_0 and n_0 such that $Hm_0,n_0 \equiv 0$, i.e., $m_0\varphi \equiv n_0\chi$. Then from (17) and (18) we have

$$m_0\frac{\Delta(f)}{(f-a)(f-b)} = n_0\frac{\Delta(f')}{(f'-a)(f'-b)}.$$

By (1)

$$\left(\frac{f-b}{f-a}\right)^{m_0} = D\left(\frac{f'-b}{f'-a}\right)^{n_0}, \quad (D \neq 0 \text{ is a constant}).$$

According to the condition (13) we know that $m_0 = n_0$. Hence

$$\frac{f-b}{f-a} = D_1\left(\frac{f'-b}{f'-a}\right), \quad (D_1 \neq 0 \text{ is a constant}). \tag{24}$$

Since $f \not\equiv f'$, thus $D_1 \neq 1$. So from (24) we have

$$f[(D_1-1)f' + a - D_1 b] = (D_1 a - b)f' + (1-D_1)ab.$$

By Clunie's lemma (see [5]) we have

$$m[r,(D_1-1)f' + a - D_1 b] = S(r,f),$$

which results in $m(r,f') = S(r,f)$, i.e., $T(r,f') = S(r,f)$. This is impossible to satisfy.

Hence, $Hm,n = m\varphi - n\chi \not\equiv 0$ for all positive integers m and n. Now let $z_0 \in S(n,m)(a) \cup S(n,m)(b)$, i.e, z_0 be a common zero of $f-a$ (or $f-b$) and $f'-a$ (or $f'-b$) with multiplicities n and m respectively. From (17) and (18) it follows that $Hm,n(z_0) = 0$. So

$$\begin{aligned} &\bar{N}(n,m)\left(r,\frac{1}{f-a}\right) + \bar{N}(n,m)\left(r,\frac{1}{f-b}\right) \\ &\leq N\left(r,\frac{1}{Hm,n}\right) + S(r,f) \leq T(r,Hm,n) + S(r,f) \\ &\leq T(r,\varphi) + T(r,\chi) + S(r,f) = S(r,f), \end{aligned}$$

for all positive integers m and n. Again by (16) we have

$$\begin{aligned} T(r,f) &= \bar{N}\left(r,\frac{1}{f-a}\right) + \bar{N}\left(r,\frac{1}{f-b}\right) + S(r,f) \\ &= \sum_{m,n}\left[\bar{N}(m,n)\left(r,\frac{1}{f-a}\right) + \bar{N}(m,n)\left(r,\frac{1}{f-b}\right)\right] + S(r,f) \end{aligned}$$

$$= \sum_{m\geq 4, n\geq 4}\left[\overline{N}(m,n)\left(r,\frac{1}{f-a}\right)+\overline{N}(m,n)\left(r,\frac{1}{f-b}\right)\right]+S(r,f)$$
$$\leq \frac{1}{4}\left[N\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{f-b}\right)\right]+S(r,f)$$
$$\leq \frac{1}{2}T(r,f)+S(r,f).$$

It is impossible for this to hold, thus $f\equiv f'$.

This completes the proof of Lemma 3.

3. The proof of Theorem 1

Assume that $f\not\equiv f'$. Let φ and χ be defined by (17) and (18) respectively. From (17) we have

$$\varphi(f-a)(f-b)=\Delta(f)(f-f'),$$

we rewrite this in the following form

$$[\varphi-(a'-b')]f^2=b_1f+b_2f'+b_3ff'+b_4f'^2+b_5, \tag{25}$$

where $b_1=ab'-ba'+(a+b)\varphi$, $b_2=ba'-ab'$, $b_3=b+b'-a-a'$, $b_4=a-b$, $b_5=-ab\varphi$ are all small functions of f. We discuss the following two cases:

(I) Suppose that $\varphi-(a'-b')\not\equiv 0$. By (25) we have

$$2m(r,f)\leq m\left(r,\frac{1}{\varphi-a'+b'}\right)+m\left[r,f\left(b_1+b_2\frac{f'}{f}+b_3f'+b_4f'\cdot\frac{f'}{f}\right)\right]+m(r,b_5)$$
$$\leq m(r,f)+m\left[r,f'\left(b_3+b_4\cdot\frac{f'}{f}\right)\right]+S(r,f)$$
$$\leq m(r,f)+m(r,f')+S(r,f),$$

which results in

$$m(r,f)\leq m(r,f')+S(r,f),$$

i.e.,

$$T(r,f)\leq T(r,f')+S(r,f).$$

Noting that f is an entire function, we have obviously

$$T(r,f')\leq T(r,f)+S(r,f).$$

Hence

$$T(r,f)=T(r,f')+S(r,f).$$

So by Lemma 3, this contradicts the assumption of Theorem 1.

(II) Suppose that $\varphi-(a'-b')\equiv 0$, i.e., $\varphi\equiv a'-b'$. We again divide the following three cases:

(II.1) Suppose that $a'\not\equiv a$ and $b'\not\equiv b$. Since f and f' share a and b IM, so the zeros of $f-a$ and $f-b$ with multiplicities larger than one are the zeros of $a'-a$ and $b'-b$ respectively. It follows that

$$\sum_{p\geq 2,m\geq 1}\overline{N}(p,m)\left(r,\frac{1}{f-a}\right)+\overline{N}(p,m)\left(r,\frac{1}{f-b}\right)\Bigg]=S(r,f). \tag{26}$$

Now let $z_1\in S(1,p)(a)$, i.e, z_1 be a simple zero of $f-a$ and a zero of $f'-a$ with multiplicity p. When $p\geq 2$, we get by (17)

$$\varphi(z_1)=a'(z_1)-a(z_1)=a'(z_1)-b'(z_1),$$

which results in $a(z_1)-b'(z_1)=0$. If $a-b'\equiv 0$, from (17) we get

$$\frac{f'-a'}{f-a}=\frac{a'-b}{a-b}.$$

This implies that $T(r,f)=T(r,f')+S(r,f)$.

By Lemma 3 this is a contradiction again.

Thus $a-b'\not\equiv 0$. Hence

$$\sum_{p\geq 2}\overline{N}(1,p)\left(r,\frac{1}{f-a}\right)=S(r,f).$$

Similary, we have

$$\sum_{p\geq 2}\overline{N}(r,p)\left(r,\frac{1}{f-b}\right)=S(r,f).$$

Therefore

$$\begin{aligned}T(r,f)&=\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{f-b}\right)+S(r,f)\\&=\overline{N}(1,1)\left(r,\frac{1}{f-a}\right)+\overline{N}(1,1)\left(r,\frac{1}{f-b}\right)+S(r,f).\end{aligned} \tag{27}$$

Setting $H=\varphi-\chi$, we suppose that $H\equiv 0$. It is easy to see from (17) and (18) that

$$T(r,f)=T(r,f')+S(r,f).$$

This is also a contradiction.

Thus $H\not\equiv 0$. By (17) and (18) we know that $H(z_2)=0$ for any $z_2\in S(1,1)(a)\cup S(1,1)(b)$. Combining this, (20) and (27) we get

$$\begin{aligned}T(r,f)&\leq N\left(r,\frac{1}{H}\right)+S(r,f)\leq T(r,\chi)+S(r,f)\\&\leq T(r,f)-N\left(r,\frac{1}{f'-c_k}\right)+S(r,f),\end{aligned}$$

which results in

$$N\left(r,\frac{1}{f'-c_k}\right)=S(r,f),\quad (k\in N^+).$$

This is also impossible.

(II.2) Suppose that either $a'\equiv a$ and $b'\not\equiv b$ or $a'\not\equiv a$ and $b'\equiv b$. Without loss of generality, we can assume that $a'\equiv a$ and $b'\not\equiv b$. According to the discussion in (II.1) we know that

$$\overline{N}\left(r,\frac{1}{f-b}\right)+S(r,f)=\overline{N}(1,1)\left(r,\frac{1}{f-b}\right)+S(r,f). \tag{28}$$

Since the zeros of $f-a$ are all the zeros of $f'-a=f'-a'$, it follows that

$$\overline{N}(1,1)\left(r,\frac{1}{f-a}\right)=S(r,f).$$

It is easy to see from (17) that the counting function corresponding to the zeros of $f-a$ and $f'-a$ with multiplicities all larger than one equals to $S(r,f)$. This derives that

$$\begin{aligned} T(r,f)&=\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{f-b}\right)+S(r,f)\\ &=\overline{N}(2,1)\left(r,\frac{1}{f-a}\right)+\overline{N}(1,1)\left(r,\frac{1}{f-b}\right)+S(r,f). \end{aligned} \tag{29}$$

Set

$$G=2\frac{f''-b'}{f'-b}-2\frac{f'-b'}{f-b}+\frac{a'-b'}{a-b}. \tag{30}$$

It is easy to see from (28) that $T(r,G)=S(r,f)$. When $G\equiv 0$, by (30) we have

$$T(r,f)=T(r,f')+S(r,f).$$

This is also a contradiction. When $G\not\equiv 0$, combining (17), we get

$$\overline{N}(2,1)\left(r,\frac{1}{f-a}\right)\le N\left(r,\frac{1}{G}\right)+S(r,f)=S(r,f).$$

Hence

$$\begin{aligned} T(r,f)&=\overline{N}(1,1)\left(r,\frac{1}{f-b}\right)+S(r,f)\\ &\le N\left(r,\frac{1}{\varphi-\chi}\right)+S(r,f)\\ &\le T(r,\varphi)+T(r,\chi)+S(r,f)\\ &\le T(r,f)-N\left(r,\frac{1}{f'-c_k}\right)+S(r,f), \end{aligned}$$

i.e.,

$$N\left(r, \frac{1}{f' - c_k}\right) = S(r, f), \quad (k \in N^+).$$

This is also impossible.

(II.3) Suppose that $a' \equiv a$ and $b' \equiv b$. By the discussion in (II.2) we know that

$$T(r, f) = \bar{N}(2, 1)\left(r, \frac{1}{f - a}\right) + \bar{N}(2, 1)\left(r, \frac{1}{f - b}\right) + S(r, f). \tag{31}$$

Now let z^* be a simple zero of $f' - a$ and a zero of $f - a$ with multiplicity two but not a pole of a and b, also not a zero of $a - b$. Set

$$G_1 = 2\frac{f'' - b''}{f' - b'} - \frac{f' - b'}{f - b} - 2\frac{a' - b'}{a - b}. \tag{32}$$

If $G_1 \equiv 0$, by (32)

$$(f' - b)^2 = D_2(a - b)^2(f - b). \quad (D_2 \neq 0 \text{ is a constant}).$$

This implies that z^* must be a zero of $a - b - (1/D_2)$. Since $\varphi = a' - b' = a - b \not\equiv 0$, so $a - b - (1/D_2) \not\equiv 0$, which results in

$$\bar{N}(2, 1)\left(r, \frac{1}{f - a}\right) = S(r, f).$$

If $G_1 \not\equiv 0$, from (17) and (32) it follows that $G_1(z^*) = 0$, also we have that

$$\bar{N}(2, 1)\left(r, \frac{1}{f - a}\right) \leq N\left(r, \frac{1}{G_1}\right) + S(r, f) \leq N(r, G_1) + S(r, f) = S(r, f).$$

Thus, from (31) we get

$$T(r, f) = N(2, 1)\left(r, \frac{1}{f - b}\right) + S(r, f) \leq \frac{1}{2}T(r, f) + S(r, f).$$

This is also a contradiction.

According to above all discussion we obtain that $f \equiv f'$.

This completes the proof of Theorem 1.

References

[1] W K. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.

[2] L. A. Rubel and C. C. Yang, Values shared by an entire function and its derivative, Lecture Notes in Math., **599**, Springer, Berlin, 1977, 101–103.

[3] E. Mues and N. Steinmetz, Meromorphe Funktionen, die mit ihrer Ableitung werte teilen, Manuscripta Math., **29** (1979), 195–206.

[4] J. H. ZHENG AND S. P WANG, On the unicity of meromorphic functions and their derivatives, Adv. in Math. (China), **21** (1992), 334–341.

[5] J. CLUNIE, On integral and meromorphic functions, J. London Math. Soc., **37** (1962), 17–27

DEPARTMENT OF MATHEMATICS
NINGDE TEACHERS COLLEGE
NINGDE, FUJIAN 352100
P.R. CHINA