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LOCALIZATION OF THE COEFFICIENT THEOREM

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Abstract

Let f be holomorphic and univalent in $D = \{|z| < 1\}$ and set $K(z) = z/(1-z)^2$. We prove $|f^{(n)}(z)/f'(z)| \le K^{(n)}(|z|)/K'(|z|)$ at each $z \in D$ and for each $n \ge 2$. This inequality at $z = 0$ is just the coefficient theorem of de Branges, the very solution of the Bieberbach conjecture. The equality condition is given in detail. In the specified case where $f(D)$ is convex we have again a similar and sharp result. We also consider $|f^{(n)}(z)/f'(z)|$ for f univalent in a hyperbolic domain Ω with the Poincaré density $P_\Omega(z)$ and the radius of univalency $\rho_\Omega(z)$ at $z \in \Omega$. We obtain the estimate $(\rho_\Omega(z)/P_\Omega(z))^{n-1}|f^{(n)}(z)/f'(z)| \le n!4^{n-1}$ at $z \in \Omega$ for $n \ge 2$, together with the detailed equality condition on f, Ω, and z.

1. Introduction

Let $\mathscr{U}$ be the family of functions holomorphic and univalent in $D = \{z; |z| < 1\}$. Writing $f_\gamma(z) = \bar{\gamma} f(\gamma z)$ for $f \in \mathscr{U}$ and for $\gamma \in \partial D \equiv \{z; |z| = 1\}$, we know that important members of $\mathscr{U}$ are K_γ, the γ-rotations of the Koebe function $K(z) = z/(1-z)^2$. The coefficient theorem proved by L. de Branges [B] then reads as follows. For each $f \in \mathscr{U}$ and for each $n \ge 2$, the inequality

$$\left|\frac{f^{(n)}(0)}{f'(0)}\right| \le n!n \tag{1.1}$$

holds. If the equality holds in (1.1) for an $n \ge 2$, then $f = f'(0)K_\gamma + f(0)$ for some $\gamma \in \partial D$. Conversely the equality holds in (1.1) for all $n \ge 2$ and for all $f = AK_\gamma + B$, where $A \ne 0, B$, and $\gamma \in \partial D$ are complex constants.

By induction we have

$$K_\gamma^{(n)}(z) \equiv (K_\gamma)^{(n)}(z) = \frac{\gamma^{n-1} n!(n + \gamma z)}{(1-\gamma z)^{n+2}} \quad (n \ge 1, \gamma \in \partial D), \tag{1.2}$$

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so that (1.1) is precisely

$$\left|\frac{f^{(n)}(0)}{f'(0)}\right| \le \frac{K^{(n)}(0)}{K'(0)}. \tag{1.3}$$

We may therefore call the following a localization of the coefficient theorem.

THEOREM A. *For $f \in \mathscr{U}$ the estimate*

$$\left|\frac{f^{(n)}(z)}{f'(z)}\right| \le \frac{K^{(n)}(|z|)}{K'(|z|)} = \frac{n!(n+|z|)}{(1-|z|)^{n-1}(1+|z|)} \tag{1.4}$$

holds for each $n \ge 2$ and at each $z \in D$. If the equality holds in (1.4) *at a point z and for an $n \ge 2$, then*

$$f(w) \equiv AK_\beta(w) + B, \tag{1.5}$$

where $A \neq 0, B$, and $\beta \in \partial D$ are all complex constants. Conversely for f of (1.5) *the equality holds in* (1.4) *for all $n \ge 2$ and at all points of the radius*

$$\Lambda(\beta) \equiv \{\bar{\beta}t; 0 \le t < 1\}.$$

Furthermore, the inequality (1.4) *is strict for all $n \ge 2$ and at all points of $D \backslash \Lambda(\beta)$.*

Let $\mathscr{S}$ be the family of $f \in \mathscr{U}$ with $f(0) = f'(0) - 1 = 0$. Supposing (1.1) the proof of which was unknown at that time, Z. J. Jakubowski [J, p. 67] proved that

$$\left|\frac{f^{(n)}(z)}{f'(z)}\right| \le \frac{n!(n+|z|)}{(1-|z|)^{n-1}(1+|z|)} \tag{1.4J}$$

for $f \in \mathscr{S}, z \in D$, and $n \ge 2$, so that (1.4) is essentially due to him. However, Jakubowski never gave any equality condition for (1.4J) even for $f \in \mathscr{S}$. Under the condition that $f \in \mathscr{S}$, the equality condition for (1.4J) is the same as in Theorem A except for the restriction that $A = 1$ and $B = 0$ in (1.5). Actually, in Section 2 we shall propose Theorem 1 which may be called the first generalization of the coefficient theorem and which is a generalized form of Theorem A, in terms of the radius of univalency. In particular, the proof of (1.4) is different from Jakubowski's.

For each function

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

of $\mathscr{S}$ we know that

$$|h'(z)| \le K'(|z|) = \frac{1+|z|}{(1-|z|)^3} \tag{1.6}$$

for all $z \in D$ [G, p. 65]. Applying (1.4) and (1.6) to h we now have

$$|h^{(n)}(z)| \le K^{(n)}(|z|) = \frac{n!(n+|z|)}{(1-|z|)^{n+2}} \tag{1.7}$$

for all $n \ge 2$ and all $z \in D$, a known result in [L, Satz] and [M, (12)], where (1.1) is again supposed; see also [G, pp. 74 and 103], This is also an immediate consequence of $|a_k| \le k, k \ge 2$ for h because

$$|h^{(n)}(z)| \le \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)k|z|^{k-n} = K^{(n)}(|z|).$$

However, the proofs in [L] and [M] are not short. The equality condition is incompletely given in the cited three literatures, so that the following might be noteworthy. If the equality holds in (1.7) for an $n \ge 2$ and at a point $z \in D$, then $h = K_\beta$ for a $\beta \in \partial D$. Conversely, for $h = K_\beta, \beta \in \partial D$, the equality holds in (1.7) for all $n \ge 2$ and at all points of $\Lambda(\beta)$, whereas the inequality (1.7) is strict for all $n \ge 2$ and at all points of $D \backslash \Lambda(\beta)$.

To consider a convex version of Theorem A we recall the function $L(z) = z/(1-z)$ of $\mathscr{S}$ for which

$$\frac{L^{(n)}(z)}{L'(z)} = \frac{n!}{(1-z)^{n-1}} \quad (n \ge 2);$$

note that $L(D)$ is a half-plane, so that this is convex.

THEOREM B. *Suppose that the image $f(D)$ of D by $f \in \mathscr{U}$ is convex. Then*

$$\left|\frac{f^{(n)}(z)}{f'(z)}\right| \le \frac{L^{(n)}(|z|)}{L'(|z|)} = \frac{n!}{(1-|z|)^{n-1}} \tag{1.8}$$

for each $n \ge 2$ and at each $z \in D$. If the equality holds in (1.8) *at a point z and for an $n \ge 2$, then*

$$f(w) \equiv AL_\beta(w) + B, \tag{1.9}$$

where $A \ne 0, B$, and $\beta \in \partial D$ are all complex constants. Conversely for f of (1.9) *the equality holds in* (1.8) *for all $n \ge 2$ and at all points of $\Lambda(\beta)$. Furthermore, the inequality* (1.8) *is strict for all $n \ge 2$ and at all points of $D \backslash \Lambda(\beta)$.*

The inequality (1.8) at $z = 0$ is familiar [G, p. 117].

Jakubowski [J, p. 68] proved that

$$\left|\frac{f^{(n)}(z)}{f'(z)}\right| \le \frac{n!}{(1-|z|)^{n-1}} \tag{1.8J}$$

for $f \in \mathscr{S}$ with convex $f(D)$ again without detailed equality condition as ours. Actually, in Section 3 we shall prove Theorem 2, a generalized form of Theorem

B, in terms of the radius of convexity. In particular, the proof of (1.8) is different from Jakubowski's.

Suppose that $h(D)$ is convex for $h \in \mathscr{S}$. Then

$$|h'(z)| \leq L'(|z|) \tag{1.10}$$

for all $n \geq 2$ and all $z \in D$ [G, p. 118]. Applying (1.8) and (1.10) to $h \in \mathscr{S}$ with convex $h(D)$, we have

$$|h^{(n)}(z)| \leq L^{(n)}(|z|) \tag{1.11}$$

for all $n \geq 2$ and all $z \in D$; this is a known result [G, p. 118] and also is a trivial consequence of the coefficient theorem [G, p. 117] in the convex case. The equality conditions like for (1.7) can easily be obtained.

In Section 4 we shall consider the inequalities containing $f', f'', \ldots, f^{(n)}$, $n \geq 2$, at the same time; the equality conditions in Theorems 3 and 4 there are different from those in Theorems 1 and 2. One can regard Theorem 3 as the second localization of the coefficient theorem.

In Section 5 we shall prove Theorem 5, a version of Theorem A in a hyperbolic domain with the Poincaré density. Theorem 5 is sharp yet is not an extension of Theorem A.

2. Radius of univalency

Suppose that $f'(z) \neq 0$ at a point $z \in D$ for f holomorphic in D. Then there exists $\rho(z, f) > 0$, the greatest r such that $0 < r \leq 1$ and f is univalent in

$$\left\{w; \left|\frac{w - z}{1 - \bar{z}w}\right| < r\right\} \tag{2.1}$$

which is the non-Euclidean disk of center z and the non-Euclidean radius arctanh r, and also is the disk of

$$\text{center } \mathscr{Z}(z, r) \equiv \frac{z(1 - r^2)}{1 - r^2|z|^2} \in D \text{ and radius } \mathscr{R}(z, r) \equiv \frac{r(1 - |z|^2)}{1 - r^2|z|^2} \leq 1 - |\mathscr{Z}(z, r)|.$$

We call $\rho(z, f)$ the radius of univalency of f at z.

A generalization of Theorem A is the following.

THEOREM 1. *Let f be holomorphic in D and suppose that $f'(z) \neq 0$ at a point $z \in D$, so that $\rho = \rho(z, f) > 0$. Then*

$$\left|\frac{f^{(n)}(z)}{f'(z)}\right| \leq \mathscr{R}(z, \rho)^{1-n} \frac{K^{(n)}(\rho|z|)}{K'(\rho|z|)} = \frac{n!(\rho|z| + 1)^{n-2}(\rho|z| + n)}{\rho^{n-1}(1 - |z|^2)^{n-1}} \tag{2.2}$$

for each $n \geq 2$. If the equality holds in (2.2) *for an $n \geq 2$, then $\rho(z, f) = 1$, so that $f \in \mathscr{U}$. Furthermore, f is of the form* (1.5). *Conversely for f of* (1.5) *the equality holds* (*in* (2.2), *i.e.,*) *in* (1.4) *for all $n \geq 2$ and at all points of $\Lambda(\beta)$, whereas the inequality* (1.4) *is strict for all $n \geq 2$ and at all points of $D \backslash \Lambda(\beta)$.*

We shall make use of the identity

$$\sum_{k=0}^{m}(k+1)\binom{m}{k}P^{m-k}Q^{k}=(P+Q)^{m-1}(P+(m+1)Q) \tag{2.3}$$

for complex numbers P, Q and for a natural number m. Actually, it follows from

$$k\binom{m}{k}=m\binom{m-1}{k-1}$$

for $1 \le k \le m$ that

$$\sum_{k=1}^{m}k\binom{m}{k}P^{m-k}Q^{k}=mQ(P+Q)^{m-1}.$$

Proof of Theorem 1. Since the function

$$g(w)=\frac{f\left(\frac{\rho w+z}{1+\bar{z}\rho w}\right)-f(z)}{\rho(1-|z|^2)f'(z)}=\sum_{k=1}^{\infty}b_k w^k \tag{2.4}$$

of $w \in D$ is in $\mathcal{S}$, since

$$f(\zeta)=\rho(1-|z|^2)f'(z)g(w)+f(z)$$

for

$$\zeta=\frac{\rho w+z}{1+\bar{z}\rho w} \quad \text{with } d\zeta=\frac{\rho(1-|z|^2)}{(1+\bar{z}\rho w)^2}dw, \quad w\in D,$$

and since

$$\frac{(1+\bar{z}\rho w)^{n-1}}{w^{n+1}}=\sum_{k=0}^{n-1}\binom{n-1}{k}(\bar{z}\rho)^{n-1-k}w^{-k-2}, \quad w\neq 0,$$

for $n \ge 1$, it follows, after short computation, that

$$\begin{aligned} \frac{f^{(n)}(z)}{n!} &= \frac{1}{2\pi i}\int_{|(\zeta-z)/(1-\bar{z}\zeta)|=\rho/2}\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta \\ &= \frac{f'(z)}{\rho^{n-1}(1-|z|^2)^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}(\bar{z}\rho)^{n-1-k}\frac{1}{2\pi i}\int_{|w|=1/2}\frac{g(w)}{w^{k+2}}dw \\ &= \frac{f'(z)}{\rho^{n-1}(1-|z|^2)^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}(\bar{z}\rho)^{n-1-k}b_{k+1}. \end{aligned} \tag{2.5}$$

Since $|b_{k+1}| \le k+1$ for all $k \ge 1$ (with $b_1 = 1$), and since (2.3) for $m =$

$n-1, P=\rho|z|$, and $Q=1$ holds, it finally follows from (2.5) that

$$\frac{|f^{(n)}(z)|}{n!} \leq \frac{|f'(z)|(\rho|z|+1)^{n-2}(\rho|z|+n)}{\rho^{n-1}(1-|z|^2)^{n-1}}, \tag{2.6}$$

or (2.2).

If the equality holds in (2.2) for an $n \geq 2$, then there exists a $\beta \in \partial D$ with $g = K_\beta$. If $\rho < 1$, then f has $(\rho\bar{\beta}+z)/(1+\bar{z}\rho\bar{\beta}) \in D$ as a pole. This contradiction shows that $\rho = 1$, so that $f \in \mathscr{U}$. We thus have

$$\frac{f\left(\frac{w+z}{1+\bar{z}w}\right) - f(z)}{(1-|z|^2)f'(z)} = K_\beta(w) = \sum_{k=1}^{\infty} k\beta^{k-1}w^k.$$

Furthermore, with the aid of (2.3) for $m=n-1, P=\bar{z}$ and $Q=\beta$, (2.5) for $b_{k+1}=(k+1)\beta^k, k=1,2,\ldots,$ is now reduced to

$$\frac{f^{(n)}(z)}{n!} = \frac{f'(z)}{(1-|z|^2)^{n-1}}(\bar{z}+\beta)^{n-2}(\bar{z}+n\beta). \tag{2.7}$$

Since

$$|\bar{z}+\beta| = |z|+1 \quad \text{and} \quad |\bar{z}+n\beta| = |z|+n,$$

if and only if $z \in \Lambda(\beta)$, we can conclude that $z \in \Lambda(\beta)$. Furthermore, for the present $z \in \Lambda(\beta)$, the equality holds in (2.2) for all $n \geq 2$.

Consequently, if the equality holds in (2.2) for an $n \geq 2$, then it holds for all $n \geq 2$, and, furthermore,

$$f(w) \equiv (1-|z|^2)f'(z)K_\beta\left(\frac{w-z}{1-\bar{z}w}\right) + f(z)$$

for a $\beta \in \partial D$ with $z \in \Lambda(\beta)$.

On the other hand, setting

$$A'(c) = \frac{(1+\beta c)^3}{(1-|c|^2)(1-\beta c)} \quad \text{and} \quad B'(c) = \frac{c(1+\beta c)}{(1-|c|^2)(1-\beta c)}$$

for c on the diameter

$$\Xi(\beta) = \{\bar{\beta}t; -1 < t < 1\}, \quad \beta \in \partial D, \tag{2.8}$$

one can prove that

$$K_\beta(w) \equiv A'(c)K_\beta\left(\frac{w-c}{1-\bar{c}w}\right) + B'(c). \tag{2.9}$$

Since $z \in \Lambda(\beta) \subset \Xi(\beta)$, we have (1.5) with

$$A = \frac{(1-|z|^2)f'(z)}{A'(z)} = \frac{(1-\beta z)(1-|z|^2)^2 f'(z)}{(1+\beta z)^3} = \frac{(1-|z|)^3 f'(z)}{1+|z|}$$

and

$$B = f(z) - \frac{B'(z)}{A'(z)}(1-|z|^2)f'(z) = f(z) - \frac{z(1-|z|^2)f'(z)}{(1+\beta z)^2} = f(z) - \frac{z(1-|z|)f'(z)}{1+|z|}.$$

Given f of (1.5) and $n \geq 2$ we have

$$\frac{n!|n+\beta z|}{|1-\beta z|^{n-1}|1+\beta z|} = \left|\frac{K_\beta^{(n)}(z)}{K_\beta'(z)}\right| = \left|\frac{f^{(n)}(z)}{f'(z)}\right| = \frac{K^{(n)}(|z|)}{K'(|z|)}$$

if and only if

$$1-|z| = |1-\beta z|, \quad 1+|z| = |1+\beta z|, \quad \text{and} \quad n+|z| = |n+\beta z|,$$

if and only if $\mathrm{Re}(\beta z) = |z|$, hence, if and only if $z \in \Lambda(\beta)$. The remaining part of the proof of Theorem 1 is now obvious.

3. Radius of convexity

Suppose that $f'(z) \neq 0$ at a point $z \in D$ for f holomorphic in D. Then there exists $\rho_c(z, f) > 0$, the greatest r such that $0 < r \leq 1$ and f is univalent in the disk of (2.1) the image of which by f is convex. We call $\rho_c(z, f)$ the radius of convexity of f at z. With the aid of the known theorem [G, p. 119] one can prove that

$$(2-\sqrt{3})\rho(z, f) \leq \rho_c(z, f) \leq \rho(z, f).$$

As a generalized form of Theorem B we shall prove

THEOREM 2. *Let f be holomorphic in D and suppose that $f'(z) \neq 0$ at a point $z \in D$, so that $\rho_c = \rho_c(z, f) > 0$. Then*

$$\left|\frac{f^{(n)}(z)}{f'(z)}\right| \leq \mathscr{R}(z, \rho_c)^{1-n}\frac{L^{(n)}(\rho_c|z|)}{L'(\rho_c|z|)} = \frac{n!(\rho_c|z|+1)^{n-1}}{\rho_c^{n-1}(1-|z|^2)^{n-1}} \tag{3.1}$$

for each $n \geq 2$. If the equality holds in (3.1) *for an $n \geq 2$, then $\rho_c(z, f) = 1$, so that $f \in \mathscr{U}$ and $f(D)$ is convex. Furthermore, f is of the form* (1.9). *Conversely for f of* (1.9) *the equality holds* (*in* (3.1), *i.e.,*) *in* (1.8) *for all $n \geq 2$ and at all points of $\Lambda(\beta)$, whereas the inequality* (1.8) *is strict for all $n \geq 2$ and at all points of $D \backslash \Lambda(\beta)$.*

Proof. We have, this time,

$$\frac{f^{(n)}(z)}{n!} = \frac{f'(z)}{\rho_c^{n-1}(1-|z|^2)^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}(\bar{z}\rho_c)^{n-1-k}b_{k+1}, \tag{3.2}$$

where

$$g(w) = \frac{f\left(\frac{\rho_c w + z}{1 + \bar{z}\rho_c w}\right) - f(z)}{\rho_c(1 - |z|^2)f'(z)} = \sum_{k=1}^{\infty} b_k w^k \tag{3.3}$$

is in $\mathscr{S}$ with convex $g(D)$. The well known coefficient theorem for g then reads that $|b_k| \leq 1$ for all $k \geq 2$; furthermore, if $|b_k| = 1$ for a $k \geq 2$, then

$$g(w) \equiv L_\beta(w) = \sum_{k=1}^{\infty} \beta^{k-1} w^k \tag{3.4}$$

for a $\beta \in \partial D$, so that $|b_k| = 1$ for all $k \geq 2$. Hence, (3.2) shows that

$$\frac{|f^{(n)}(z)|}{n!} \leq \frac{|f'(z)|}{\rho_c^{n-1}(1 - |z|^2)^{n-1}} (\rho_c|z| + 1)^{n-1},$$

from which follows (3.1).

If the equality holds in (3.1) for an $n \geq 2$, then g is of the form (3.4). Hence $\rho_c = 1$; otherwise, f has $(\rho_c\bar{\beta} + z)/(1 + \bar{z}\rho_c\bar{\beta}) \in D$ as a pole. We thus obtain

$$\frac{f^{(n)}(z)}{n!} = \frac{f'(z)}{(1 - |z|^2)^{n-1}} (\bar{z} + \beta)^{n-1},$$

because $b_{k+1} = \beta^k$. Note that $|\bar{z} + \beta| = 1 + |z|$ if and only if $z \in \Lambda(\beta)$.

Consequently, if the equality holds in (3.1) for an $n \geq 2$, then it holds for all $n \geq 2$, and furthermore

$$f(w) \equiv (1 - |z|^2)f'(z)L_\beta\left(\frac{w - z}{1 - \bar{z}w}\right) + f(z)$$

for a $\beta \in \partial D$ with $z \in \Lambda(\beta)$. By the similar reasoning as in the proof of Theorem 2 we have

$$A = \frac{(1 - |z|^2)^2 f'(z)}{(1 + \beta z)^2} = (1 - |z|)^2 f'(z)$$

and

$$B = f(z) - \frac{z(1 - |z|^2)f'(z)}{1 + \beta z} = f(z) - z(1 - |z|)f'(z)$$

for $z \in \Lambda(\beta) \subset \Xi(\beta)$ in (1.9) because

$$\frac{(1 + \beta c)^2}{1 - |c|^2} L_\beta\left(\frac{w - c}{1 - \bar{c}w}\right) + \frac{c(1 + \beta c)}{1 - |c|^2} \equiv L_\beta(w)$$

for $c \in \Xi(\beta)$. The rest of the proof is the same as that of Theorem 1 with K replaced by L.

4. Estimates containing $f', f'', \ldots, f^{(n)}, n \geq 2$

Two sharp inequalities containing $f', f'', \ldots, f^{(n)}$, at the same time will be proved.

For $z \in D$ and for $\beta \in \partial D$ we set

$$\Xi(z, \beta) = \left\{ \frac{\bar{\beta}t + z}{1 + \bar{z}\bar{\beta}t}; -1 < t < 1 \right\}.$$

The set $\Xi(z, \beta)$ is the non-Euclidean (geodesic) line in D ending at points $(z - \bar{\beta})/(1 - \bar{z}\bar{\beta})$ and $(z + \bar{\beta})/(1 + \bar{z}\bar{\beta})$ of ∂D, or, a circular arc in (possibly, a diameter of) D orthogonal to ∂D at the two points. Note that $\Xi(z, \beta) = \Xi(\beta)$ if and only if $z \in \Xi(\beta)$. In particular, $\Xi(\beta) = \Xi(0, \beta)$.

THEOREM 3. *Let f be holomorphic in D and suppose that $f'(z) \neq 0$ at a point $z \in D$. Then*

$$\rho(z, f)^{n-1} \left| \sum_{k=1}^{n} \frac{1}{k!} \binom{n-1}{k-1} (-\bar{z})^{n-k} (1 - |z|^2)^{k-1} \frac{f^{(k)}(z)}{f'(z)} \right| \leq n \tag{4.1}$$

for each $n \geq 2$. If the equality holds in (4.1) *for an $n \geq 2$, then f is of the form*

$$f(w) \equiv AK_{\beta}\left(\frac{w - z}{1 - \bar{z}w}\right) + B, \tag{4.2}$$

where $A \neq 0, B$ and $\beta \in \partial D$ are constants. Conversely for f of (4.2) *the equality holds in* (4.1) (*with $\rho(z, f) = 1$*) *for all $n \geq 2$ and at all points of $\Xi(z, \beta)$. The inequality* (4.1) *is, furthermore, strict for all $n \geq 2$ and at all points of $D \backslash \Xi(z, \beta)$.*

THEOREM 4. *Let f be holomorphic in D and suppose that $f'(z) \neq 0$ at a point $z \in D$. Then*

$$\rho_c(z, f)^{n-1} \left| \sum_{k=1}^{n} \frac{1}{k!} \binom{n-1}{k-1} (-\bar{z})^{n-k} (1 - |z|^2)^{k-1} \frac{f^{(k)}(z)}{f'(z)} \right| \leq 1 \tag{4.3}$$

for each $n \geq 2$. If the equality holds in (4.3) *for an $n \geq 2$, then f is of the form*

$$f(w) \equiv AL_{\beta}\left(\frac{w - z}{1 - \bar{z}w}\right) + B, \tag{4.4}$$

where $A \neq 0, B$ and $\beta \in \partial D$ are constants. Conversely for f of (4.4) *the equality holds in* (4.3) (*with $\rho_c(z, f) = 1$*) *for all $n \geq 2$ and at all points of $\Xi(z, \beta)$. The inequality* (4.3) *is, furthermore, strict for all $n \geq 2$ and at all points of $D \backslash \Xi(z, \beta)$.*

The proof of Theorem 4 is similar to that of Theorem 3, and hence is omitted.

Proof of Theorem 3. First of all we claim that, for a complex λ and $1 \leq k \leq n$, the expansion

$$\left(\frac{w}{1+\lambda w}\right)^k = \sum_{n=k}^{\infty}(-\lambda)^{n-k}\binom{n-1}{k-1}w^n, \tag{4.5}$$

holds provided that $|\lambda w| < 1$. This identity follows immediately from

$$\frac{1}{(1-\zeta)^k} = \sum_{n=0}^{\infty}\binom{n+k-1}{k-1}\zeta^n$$

for $|\zeta| < 1$ and $1 \le k \le n$.

Set $\rho = \rho(z, f)$ and consider g of (2.4). Set

$$\phi(w) = \frac{w}{1+\rho\bar{z}w}$$

for $w \in D$ and

$$\begin{aligned} F(\zeta) &= \frac{f(\rho(1-|z|^2)\zeta + z) - f(z)}{\rho(1-|z|^2)f'(z)} \\ &= \sum_{k=1}^{\infty}\frac{f^{(k)}(z)}{k!}\frac{[\rho(1-|z|^2)\zeta]^k}{\rho(1-|z|^2)f'(z)}. \end{aligned}$$

Then

$$g(w) = F\circ\phi(w) = \frac{1}{f'(z)}\sum_{k=1}^{\infty}\frac{f^{(k)}(z)}{k!}[\rho(1-|z|^2)]^{k-1}\phi(w)^k,$$

so that, with the aid of (4.5) for $\lambda = \rho\bar{z}$, we have

$$g(w) = \sum_{n=1}^{\infty} b_n w^n$$

with

$$b_n = \rho^{n-1}\sum_{k=1}^{n}\frac{1}{k!}\binom{n-1}{k-1}(-\bar{z})^{n-k}(1-|z|^2)^{k-1}\frac{f^{(k)}(z)}{f'(z)}.$$

Applying the coefficient theorem $|b_n| \le n, n \ge 2$, to $g \in \mathscr{S}$ we immediately have (4.1).

If the equality holds in (4.1) for an $n \ge 2$, then it holds for all $n \ge 2$, $\rho(z, f) = 1$, and f is of the form (4.2) with $A = (1-|z|^2)f'(z)$ and $B = f(z)$.

Conversely, given f of (4.2) we suppose that the equality holds in (4.1) at $c \in D$ and for an (hence, all) $n \ge 2$. In particular, for $n = 2$ we have $|Q(c)| = 2$ for

$$Q(c) = -\bar{c} + \frac{1}{2}(1-|c|^2)\frac{f''(c)}{f'(c)}.$$

Setting $\psi(w) = \beta(w-z)/(1-\bar{z}w), w \in D$, and recalling

$$1 - |\psi(c)|^2 = \frac{(1-|z|^2)(1-|c|^2)}{(1-\bar{z}c)(1-z\bar{c})},$$

we have that

$$Q(c) = \frac{\beta(1-z\bar{c})}{1-\bar{z}c}\left(-\overline{\psi(c)} + \frac{1}{2}(1-|\psi(c)|^2)\frac{K''(\psi(c))}{K'(\psi(c))}\right).$$

Hence

$$\left|-\overline{\psi(c)} + \frac{1}{2}(1-|\psi(c)|^2)\frac{K''(\psi(c))}{K'(\psi(c))}\right| = 2. \tag{4.6}$$

On the other hand,

$$\left|-\bar{\zeta} + \frac{1}{2}(1-|\zeta|^2)\frac{K''(\zeta)}{K'(\zeta)}\right| = 2$$

for $\zeta \in D$ if and only if $1-|\zeta|^2 = |1-\zeta^2|$ or if and only if $\zeta \in \Xi(1) = (-1,1)$. It then follows from (4.6) that $\psi(c) \in \Xi(1)$, so that $c \in \Xi(z,\beta)$. Given $c' \in \Xi(z,\beta)$ for f of (4.2) we may trace back the above argument on replacing c with c' to observing that the equality holds in (4.1) at c' for all $n \geq 2$. The remaining part of the proof is now obvious.

For $f \in \mathscr{U}$ at $z = 0$, the inequality (4.1) is just (1.1). One can call Theorem 3, therefore, the second localization of the coefficient theorem; similarly for Theorem 4.

The case $n = 2$ in (4.1) reads

$$\rho(z,f)\left|-\bar{z} + \frac{1}{2}(1-|z|^2)\frac{f''(z)}{f'(z)}\right| \leq 2,$$

which is familiar in case $\rho(z,f) = 1$ or $f \in \mathscr{U}$; see [G, (5), p. 63].

5. Hyperbolic domain

A domain Ω in the plane $\boldsymbol{C} = \{|z| < +\infty\}$ is called hyperbolic if $\boldsymbol{C}\backslash\Omega$ contains at least two points. Let ϕ be a universal covering projection from D onto a hyperbolic domain Ω in $\boldsymbol{C}$; ϕ is holomorphic and ϕ' is zero-free in D. The Poincaré density P_Ω is then the function in Ω defined by

$$P_\Omega(z) = \frac{1}{(1-|w|^2)|\phi'(w)|}, \quad z \in \Omega,$$

where $z = \phi(w)$; the choice of ϕ and w is immaterial as far as $z = \phi(w)$ is satisfied.

We next set $\rho_\Omega(z) = \rho(w,\phi)$ for $z = \phi(w) \in \Omega$. Again, $\rho_\Omega(z)$ is independent of the particular choice of ϕ and w as far as $z = \phi(w)$ is satisfied. We call $\rho_\Omega(z)$ the radius of univalency of Ω at z.

Let $\mathscr{U}(\Omega)$ be the family of all the functions holomorphic and univalent in Ω; in particular, $\mathscr{U} = \mathscr{U}(D)$.

As another application of the coefficient theorem we propose

THEOREM 5. *For $f \in \mathscr{U}(\Omega)$ of a hyperbolic domain $\Omega \subset \boldsymbol{C}$ the inequality*

$$\left(\frac{\rho_\Omega(z)}{P_\Omega(z)}\right)^{n-1}\left|\frac{f^{(n)}(z)}{f'(z)}\right| \le n!4^{n-1} \tag{5.1}$$

holds for each $n \ge 2$ and at each $z \in \Omega$. If the equality holds in (5.1) *at a point $z \in \Omega$ and for an $n \ge 2$, then the following items* (I) *and* (II) *hold.*

(I) *There exist complex constants $Q \ne 0$ and R such that Ω is the slit domain*

$$\Omega = \boldsymbol{C}\backslash\left\{Qt + R; t \le -\frac{1}{4}\right\}; \tag{5.2}$$

in particular, $\rho_\Omega(z) \equiv 1$.

(II) *The function f is of the form*

$$f(w) = \frac{S(R-w)}{4w+Q-4R} + T, \tag{5.3}$$

where $S \ne 0$ and T are complex constants.

Conversely, suppose that f of (5.3) *is given in Ω of* (5.2). *Then the equality holds in* (5.1) *at each point of the half-line*

$$\mathscr{L} = \left\{Qt + R; t > -\frac{1}{4}\right\}$$

and for each $n \ge 2$, whereas the inequality (5.1) *is strict at each point of $\Omega\backslash\mathscr{L}$ and for each $n \ge 2$.*

The extremal function f of (5.3) maps Ω of (5.2) univalently onto the slit domain

$$\boldsymbol{C}\bigg\backslash\left\{St + T; t \le -\frac{1}{4}\right\}.$$

K. S. Chua [C, Theorem 1] proved (5.1) in case $\rho_\Omega(z) \equiv 1$, namely, in case Ω is a simply connected, proper subdomain of $\boldsymbol{C}$; his equality condition is not complete enough. Chua actually proved that the equality holds in (5.1) at 0 for f of (5.3) with $Q = 1, R = 0, S = (-1)^n$, and $T = 0$ in Ω of (5.2) [C, p. 69]. In case $\Omega = D$ and $f \in \mathscr{U}$, the inequality (5.1) at $z = 0$ reads

$$\left|\frac{f^{(n)}(0)}{f'(0)}\right| \le n!4^{n-1}, \tag{5.4}$$

a worse result than (1.1) for $n \ge 2$. Theorem 5 is, in this sense, never an extension of Theorem A.

Theorem 5 for the fixed $n = 2$ is known; see [Y2, Théorème *et seq.*].

The inverse function of $h \in \mathscr{S}$ in $h(D)$ is always denoted by h^*. The function $h^{*k} \equiv (h^*)^k$, the k-th power of $h^*, k = 1, 2, \ldots$, in $h(D)$, then has the expansion

$$h^{*k}(\zeta) = \sum_{n=k}^{\infty} B_{nk}(h)\zeta^n$$

in a neighborhood of $0 \in h(D)$ and $B_{kk}(h) = 1$. An important case is that $h = K$,

$$B_{nk}(K) = (-1)^{n-k}\frac{k}{n}\binom{2n}{n-k}, \quad 1 \le k \le n,$$

for which

$$\sum_{k=1}^{n} k|B_{nk}(K)| = \sum_{k=1}^{n} \frac{k^2}{n}\binom{2n}{n-k} = 4^{n-1};$$

see [C, (8) and (14)]. Moreover, for $\gamma \in \partial D$ one has

$$B_{nk}(K_\gamma) = B_{nk}(K)\gamma^{n-k}, \quad 1 \le k \le n.$$

Notice that

$$(K_\gamma)^*(\zeta) = \bar{\gamma}K^*(\gamma\zeta), \quad \zeta \in K_\gamma(D).$$

Proof of Theorem 5. We first suppose that $0 \in \Omega$ and $\phi(0) = \phi'(0) - 1 = 0$ for a projection $\phi : D \to \Omega$. Then $P_\Omega(0) = 1$. Supposing further that $f(0) = f'(0) - 1 = 0$ we shall prove that

$$\rho^{n-1}|f^{(n)}(0)| \le n!4^{n-1}, \tag{5.5}$$

where $\rho = \rho_\Omega(0)$. The functions

$$\Phi(z) = \rho^{-1}\phi(\rho z) \quad \text{and} \quad F(z) = \rho^{-1}f(\phi(\rho z)) = \rho^{-1}f(\rho\Phi(z)) \quad \text{for } z \in D$$

both are in $\mathscr{S}$. Since

$$\rho^{-1}f(\rho\zeta) = F \circ \Phi^*(\zeta), \quad \zeta = \Phi(z) \in \Phi(D),$$

it follows from [T, Theorem 1, p. 220] for $F \circ \Phi^*$ defined in $\Phi(D)$ that

$$\rho^{-1}\frac{d^n}{d\zeta^n}f(\rho\zeta) = \sum_{k=1}^{n} A_{nk}(\zeta)F^{(k)}(\Phi^*(\zeta)),$$

where

$$A_{nk}(\zeta) = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}(\Phi^*)^{k-j}(\zeta)(\Phi^{*j})^{(n)}(\zeta), \quad n = 1, 2, \ldots.$$

Since $\Phi^*(0) = 0$ it then follows that

(5.6)
$$\rho^{n-1}f^{(n)}(0) = \sum_{k=1}^{n} n!B_{nk}(\Phi)\frac{F^{(k)}(0)}{k!}.$$

On the other hand, it follows from Chua's theorem [C, Theorem 2], applied to $\Phi \in \mathscr{S}$, that

(5.7)
$$|B_{nk}(\Phi)| \le |B_{nk}(K)|, \quad 1 \le k \le n.$$

Recalling the coefficient theorem for $F \in \mathscr{S}$, one finally has (5.5) from (5.6). Observe that if $n \ge 2$ and if the equality holds in (5.7) for a pair, n, k with $k < n$, then $\Phi = K_\beta$ for a $\beta \in \partial D$, so that the equality holds in (5.7) for all pairs of n, k with $1 \le k \le n$.

Suppose that the equality holds in (5.5) for an $n \ge 2$. Then

$$F = K_\alpha \quad \text{and} \quad \Phi = K_\beta$$

for $\alpha, \beta \in \partial D$. If $\rho < 1$, then f has a pole $\phi(\rho\bar{\alpha}) \in \Omega$. This contradiction shows that $\rho = 1$, so that $\phi = \Phi = K_\beta$. Hence

$$\Omega = \boldsymbol{C}\Big\backslash\left\{\bar{\beta}t; t \le -\frac{1}{4}\right\},$$

so that $Q = \bar{\beta}$ and $R = 0$ in (5.2). On the other hand, it follows from (5.6) that

$$f^{(n)}(0) = n! \sum_{k=1}^{n} B_{nk}(K)\beta^{n-k}k\alpha^{k-1}$$

with $|f^{(n)}(0)| = n!4^{n-1}$. Setting $\gamma = -\alpha\bar{\beta}$ and $C_{nk} = k|B_{nk}(K)|, 1 \le k \le n$, we now have

$$\left|\sum_{k=1}^{n} C_{nk}\gamma^k\right| = \frac{|f^{(n)}(0)|}{n!} = 4^{n-1} = \sum_{k=1}^{n} C_{nk},$$

so that, on squaring the left- and the right-most sides, we have

$$\sum C_{nk}C_{nl}(1-\gamma^{k-l}) = 0 \quad \Big(\sum \quad \text{for } k \ne l, 1 \le k \le n, 1 \le l \le n\Big).$$

Since $\operatorname{Re}(1-\gamma^{k-l}) \ge 0$ and $C_{nk}C_{nl} > 0$, it follows that $\operatorname{Re}\gamma^{k-l} = 1$ for $k \ne l$, $1 \le k \le n, 1 \le l \le n$. We may choose $k = 2$, and $l = 1$, so that

(5.8)
$$1 = \gamma = -\alpha\bar{\beta}.$$

Since

$$K^*(\zeta) = \frac{2\zeta + 1 - \sqrt{4\zeta + 1}}{2\zeta},$$

it follows that

$$-K(-K^*(\zeta)) = \frac{\zeta}{4\zeta + 1}, \quad \zeta \in K(D).$$

Consequently, for $w \in \Omega$, we have

$$f(w) = K_\alpha \circ (K_\beta)^*(w) = K_\alpha(\bar{\beta}K^*(\beta w)) = \bar{\alpha}K(-K^*(\beta w)) = \frac{\bar{\beta}w}{4w+\bar{\beta}}$$

by (5.8). Hence we have $S = -\bar{\beta}$ and $T = 0$ with $R = 0$ in (5.3).

To complete the proof of (5.1) at $z = a \in \Omega$ in the general case, we choose a projection ϕ with $\phi(0) = a$, and set

$$g(w) = \frac{f(a + \phi'(0)w) - f(a)}{\phi'(0)f'(a)} \tag{5.9}$$

for the variable w in the domain

$$\Sigma = \left\{\frac{z-a}{\phi'(0)}; z \in \Omega\right\}$$

onto which $\psi = (\phi - a)/\phi'(0)$ is a projection with $\psi(0) = \psi'(0) - 1 = 0$. Since

$$g^{(n)}(0) = \frac{f^{(n)}(a)\phi'(0)^{n-1}}{f'(a)}, \quad \rho_\Sigma(0) = \rho_\Omega(a) \quad \text{and} \quad |\phi'(0)| = 1/P_\Omega(a),$$

it follows from (5.5) applied to g at 0 with $\rho = \rho_\Sigma(0)$ that

$$\left(\frac{\rho_\Omega(a)}{P_\Omega(a)}\right)^{n-1}\left|\frac{f^{(n)}(a)}{f'(a)}\right| = \rho_\Sigma(0)^{n-1}|g^{(n)}(0)| \le n!4^{n-1}.$$

This is (5.1) for $z = a$.

Suppose that the equality holds at $z = a$ in (5.1). Then, in (I) and (II), we can set, with the aid of g of (5.9),

$$Q = \bar{\beta}\phi'(0), \quad R = a, \quad S = -\bar{\beta}\phi'(0)f'(a), \quad \text{and} \quad T = f(a)$$

for a $\beta \in \partial D$.

Conversely, given f of (5.3) in Ω of (5.2) and $n \ge 2$ we have

$$f^{(n)}(z) = \frac{n!(-4)^{n-1}(-SQ)}{(4z + Q - 4R)^{n+1}},$$

so that

$$\frac{f^{(n)}(z)}{f'(z)} = \frac{n!(-4)^{n-1}}{(4z + Q - 4R)^{n-1}}, \quad z \in \Omega.$$

Since $z = QK(\zeta) + R$ maps D univalently onto Ω, it follows that

$$\frac{1}{P_\Omega(z)} = \frac{|Q|(1 - |\zeta|^2)|1 + \zeta|}{|1 - \zeta|^3}$$

and

$$\left|\frac{f^{(n)}(z)}{f'(z)}\right| = \frac{n!4^{n-1}}{|Q|^{n-1}}\left|\frac{1-\zeta}{1+\zeta}\right|^{2n-2},$$

so that

$$P_\Omega(z)^{1-n}\left|\frac{f^{(n)}(z)}{f'(z)}\right| = n!4^{n-1}\left(\frac{1-|\zeta|^2}{|1-\zeta^2|}\right)^{n-1}.$$

Hence, for $n \geq 2$,

$$P_\Omega(z)^{1-n}\left|\frac{f^{(n)}(z)}{f'(z)}\right| = n!4^{n-1}$$

if and only if $1-|\zeta|^2 = |1-\zeta^2|$ or if and only if $\zeta \in \Xi(1)$. In conclusion, the equality holds in (5.1) at $z \in \Omega$ if and only if z is on $\mathscr{L}$, the image of $\Xi(1)$ by $z = QK(\zeta) + R$.

Remark. Let ϕ be a universal covering projection from D onto Ω and let $z = \phi(w), w \in D$. Set

$$\Delta(z) = \phi\left(\left\{\zeta; \left|\frac{\zeta - w}{1-\bar{w}\zeta}\right| < \rho_\Omega(z)\right\}\right);$$

possibly, $\Delta(z) = \Omega$. This simply connected domain is independent of the particular choice of ϕ and w as far as $z = \phi(w)$ is satisfied. We can replace, in Theorem 5, the condition on f that $f \in \mathscr{U}(\Omega)$ with the following weaker one. Namely, f is holomorphic in Ω and univalent in each $\Delta(z), z \in \Omega$.

6. Concluding remarks

For z of a hyperbolic domain Ω we set $\rho_{\Omega c}(z) = \rho_c(w, \phi)$, where $z = \phi(w)$ is a universal covering projection. Then $\rho_{\Omega c}$ is a function well defined in Ω and $\rho_{\Omega c}(z)$ is called the radius of convexity of Ω at z.

Suppose that $\Phi \in \mathscr{S}$ and $\Phi(D)$ is convex. Then,

$$|B_{nk}(\Phi)| \leq \binom{n-1}{k-1}, \quad n-3 \leq k \leq n;$$

[C, Lemma 2]. Hence if $2 \leq n \leq 4, z \in \Omega$, and $f \in \mathscr{U}(\Omega)$ with Ω hyperbolic, then

$$\left(\frac{\rho_{\Omega c}(z)}{P_\Omega(z)}\right)^{n-1}\left|\frac{f^{(n)}(z)}{f'(z)}\right| \leq (n+1)!2^{n-2}. \tag{6.1}$$

Note that

$$\sum_{k=1}^{n} k\binom{n-1}{k-1} = \sum_{k=0}^{n-1}(k+1)\binom{n-1}{k} = (n+1)2^{n-2},$$

the case $m = n - 1$ and $P = Q = 1$ in (2.3). In view of the $\rho_{\Omega c}$ version of (5.6) the proof of (6.1) is now obvious. One can loosen the condition $f \in \mathscr{U}(\Omega)$ for (6.1) on only supposing that f is univalent in each domain

$$\Delta_c(z) \equiv \phi\left(\left\{\zeta; \left|\frac{\zeta - w}{1 - \overline{w}\zeta}\right| < \rho_{\Omega c}(z)\right\}\right), \quad z = \phi(w) \in \Omega.$$

Chua proved in [C, Theorem 3] that for $f \in \mathscr{U}(\Omega)$ with Ω convex,

$$\left|\frac{f^{(n)}(z)}{f'(z)}\right| \leq (n+1)!2^{n-2}P_\Omega(z)^{n-1}, \quad z \in \Omega; n = 2, 3, 4, \tag{6.2}$$

and if $f(\Omega)$ is convex further, then

$$\left|\frac{f^{(n)}(z)}{f'(z)}\right| \leq n!2^{n-1}P_\Omega(z)^{n-1}, \quad z \in \Omega; n = 2, 3, 4. \tag{6.3}$$

We note that some results of Chua in the specified case $n = 2$ are proved already in [Y1]. First, the estimate (4) for $n = 2$ in [C, Theorem 1] is exactly $\|A\|_{CS} \leq 6$ in [Y1, Théorème 1]. The case $n = 2$ in (6.2) and (6.3) are known; see $\|A\|_S \leq 6$ and $\|A\|_{CS} \leq 4$ in [Y1, Théorème 2]. If $\rho_{\Omega c}(z) = 1$ in (6.1), then we have (6.2).

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