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SOME REMARKS ON ALMOST KÄHLER 4-MANIFOLDS OF POINTWISE CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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1. Introduction

An almost Kähler manifold is an almost Hermitian manifold $M = (M, J, g)$ whose Kähler form Ω is closed, or equivalently,

$$g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y) = 0,$$

for all smooth vector fields X, Y, Z on M.

Concerning the integrability of the almost complex structure of an almost Kähler manifold, S. I. Goldberg [3] conjectured that the almost complex structure of a compact Einstein almost Kähler manifold is integrable (and the manifold is necessarily Kähler). In connection with this conjecture, some results have been proved under a variety of curvature conditions (e.g. [14, 15, 7, 8, 9, 1]).

In particular, the author [12] proved that a 4-dimensional compact almost Kähler manifold of constant holomorphic sectional curvature which satisfies the *RK*-condition is a Kähler manifold. More strongly, J. T. Cho and K. Sekigawa [2] asserted that a 4-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature is a Kähler manifold of constant holomorphic sectional curvature. But its proof have an error in computation of the square norm of the curvature tensor $\|R\|^2$. In fact, P. Nurowski and M. Przanowski [5] recently constructed a non-compact example of strictly almost Kähler, Ricci-flat manifold which is of pointwise constant holomorphic sectional curvature. However, the idea of [2] is still interesting.

The purpose of the present paper is to correct the calculations in [2] and to get some related results. In §2, after preparing some definitions and notations, we shall give a curvature expression in a 4-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature by using A_{ij} introduced by [2]. In §3, making use of the above curvature expression, we shall calculate $\|R\|^2$ and some relational quantities. We shall show some results on an almost Kähler 4-manifold of pointwise constant holomorphic sectional curvature under additional conditions. In §4, we introduce the example constructed by Nurowski and Przanowski and discuss it.

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Throughout this paper, we assume that the manifold under consideration to be connected and of class C^∞.

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2. Preliminaries

Let $M = (M, J, g)$ be an $m(= 2n)$-dimensional almost Hermitian manifold with an almost Hermitian structure (J, g). We denote by Ω and N the Kähler form and the Nijenhuis tensor of M defined respectively by $\Omega(X, Y) = g(X, JY)$ and $N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$ for $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of all smooth vector fields on M. The Nijenhuis tensor N satisfies

$$N(JX, Y) = N(X, JY) = -JN(X, Y), \quad X, Y \in \mathfrak{X}(M).$$

We assume that M is oriented by the volume form $dM = ((-1)^n/n!)\Omega^n$. Further we denote by ∇, $R = (R_{ijk}{}^l)$, $\rho = (\rho_{ij})$, τ, $\rho^* = (\rho^*_{ij})$ and τ^* the Riemannian connection, the Riemannian curvature tensor, the Ricci tensor, the scalar curvature, the Ricci $*$-tensor and the $*$-scalar curvature of M, respectively:

$$\begin{aligned} R(X, Y)Z &= [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z, \\ R(X, Y, Z, W) &= g(R(X, Y)Z, W), \\ \rho(x, y) &= \text{trace of } [z \mapsto R(z, x)y], \\ \tau &= \text{trace of } \rho, \\ \rho^*(x, y) &= \text{trace of } [z \to R(x, Jz)Jy], \\ \tau^* &= \text{trace of } \rho^*, \end{aligned}$$

where $X, Y, Z, W \in \mathfrak{X}(M)$, $x, y, z \in T_pM$, $p \in M$. The Ricci $*$-tensor ρ^* satisfies

$$\rho^*(JX, JY) = \rho^*(Y, X), \quad X, Y \in \mathfrak{X}(M).$$

An almost Hermitian manifold M is called a *weakly $*$-Einstein manifold* if it satisfies $\rho^* = \lambda^* g$ for some function λ^* on M. In particular, if λ^* is constant on M, then M is called a *$*$-Einstein manifold.*

We define a tensor field $G = (G_{ijkl})$ by

$$G(X, Y, Z, W) = R(X, Y, Z, W) - R(X, Y, JZ, JW).$$

Then we have

$$\sum_{i=1}^{m} G(E_i, X, Y, E_i) = \rho(X, Y) - \rho^*(X, Y), \tag{2.1}$$

where $\{E_i\}_{i=1,\ldots,m}$ is a local orthonormal frame of M.

An almost Hermitian manifold M is called an *RK-manifold* if it satisfies

$$R(JX, JY, JZ, JW) = R(X, Y, Z, W) \tag{2.2}$$

for $X, Y, Z, W \in \mathfrak{X}(M)$ ([16]).

Now we assume that $M = (M, J, g)$ is an almost Kähler manifold. Then we have

$$2g((\nabla_X J)Y, Z) = g(JX, N(Y, Z)).$$

It is well-known that ([4])

$$\begin{aligned} G(X, Y, Z, W) &+ G(JX, JY, JZ, JW) \\ &+ G(JX, Y, JZ, W) + G(X, JY, Z, JW) \\ &= 2g((\nabla_X J)Y - (\nabla_Y J)X, (\nabla_Z J)W - (\nabla_W J)Z). \end{aligned} \tag{2.3}$$

By (2.1) and (2.3), we have

$$\begin{aligned} \rho(X, Y) &+ \rho(JX, JY) - \rho^*(X, Y) - \rho^*(JX, JY) \\ &= -\sum_{i=1}^{m} g((\nabla_{E_i} J)X, (\nabla_{E_i} J)Y), \end{aligned} \tag{2.4}$$

$$\tau - \tau^* = -\frac{1}{2}\|\nabla J\|^2 = -\frac{1}{8}\|N\|^2. \tag{2.5}$$

In the sequel, we shall adopt the following notational convention: for an orthonormal basis $\{e_i\}$ of a tangent space T_pM,

$$\begin{aligned} &J_{ij} = g(Je_i, e_j), && \nabla_i J_{jk} = g((\nabla_{e_i} J)e_j, e_k), \\ &N_{ijk} = g(e_i, N(e_j, e_k)), && R_{ijkl} = R(e_i, e_j, e_k, e_l), \\ &G_{ijkl} = G(e_i, e_j, e_k, e_l), && \nabla_{\bar{i}} J_{jk} = g((\nabla_{Je_i} J)e_j, e_k), \\ &N_{\bar{i}jk} = g(Je_i, N(e_j, e_k)), && N_{\bar{i}\bar{j}k} = g(Je_i, N(Je_j, e_k)), \\ &R_{ijk\bar{l}} = R(e_i, e_j, e_k, Je_l), && G_{ij\bar{k}\bar{l}} = G(e_i, e_j, Je_k, Je_l), \quad \text{etc.} \end{aligned}$$

Then it is easy to see that

$$\nabla_i J_{jk} + \nabla_{\bar{i}} J_{\bar{j}k} = 0,$$

$$\nabla_{\bar{i}} J_{jk} = \nabla_i J_{\bar{j}k} = \nabla_i J_{j\bar{k}},$$

$$N_{ijk} = -2\nabla_{\bar{i}} J_{jk}, \quad 2\nabla_i J_{jk} = N_{\bar{i}jk}.$$

By the Ricci identity, we have

$$(2.6)\qquad \begin{aligned} 2G_{ijkl} &= \frac{1}{2}N^t{}_{ij}N_{tkl} - \nabla_i N_{\bar{j}\bar{k}l} + \nabla_j N_{\bar{i}\bar{k}l} \\ &= \nabla_i N_{jkl} - \nabla_j N_{ikl} + \frac{1}{2}(N_{ila}N_{jk}{}^a - N_{ika}N_{jl}{}^a). \end{aligned}$$

Now, let $M = (M, J, g)$ be an almost Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)$ ($p \in M$). Then, taking account of [11] and (2.6), we have

$$(2.7)\qquad \begin{aligned} R_{ijkl} &= \frac{c}{4}\{g_{il}g_{jk} - g_{ik}g_{jl} + J_{il}J_{jk} - J_{ik}J_{jl} - 2J_{ij}J_{kl}\} \\ &\quad + \frac{1}{16}\{N^t{}_{ik}N_{tjl} - N^t{}_{il}N_{tjk} + 2N^t{}_{ij}N_{tkl}\} \\ &\quad + \frac{1}{96}Q_{ijkl}, \end{aligned}$$

where

$$\begin{aligned} Q_{ijkl} = &-13\{\nabla_i N_{\bar{j}\bar{k}l} - \nabla_j N_{\bar{i}\bar{k}l} + \nabla_k N_{\bar{l}\bar{i}j} - \nabla_l N_{\bar{k}\bar{i}j}\} \\ &+ 3\{\nabla_{\bar{i}} N_{jk\bar{l}} - \nabla_{\bar{j}} N_{ik\bar{l}} + \nabla_{\bar{k}} N_{li\bar{j}} - \nabla_{\bar{l}} N_{ki\bar{j}}\} \\ &- \frac{13}{2}\{\nabla_i N_{\bar{k}\bar{j}l} - \nabla_k N_{\bar{i}\bar{j}l} + \nabla_j N_{\bar{l}\bar{i}k} - \nabla_l N_{\bar{j}\bar{i}k} \\ &\qquad - \nabla_i N_{\bar{l}\bar{j}k} + \nabla_l N_{\bar{i}\bar{j}k} - \nabla_j N_{\bar{k}\bar{i}l} + \nabla_k N_{\bar{j}\bar{i}l}\} \\ &+ \frac{3}{2}\{\nabla_{\bar{i}} N_{kj\bar{l}} - \nabla_{\bar{k}} N_{ij\bar{l}} + \nabla_{\bar{j}} N_{li\bar{k}} - \nabla_{\bar{l}} N_{ji\bar{k}} \\ &\qquad - \nabla_{\bar{i}} N_{lj\bar{k}} + \nabla_{\bar{l}} N_{ij\bar{k}} - \nabla_{\bar{j}} N_{ki\bar{l}} + \nabla_{\bar{k}} N_{ji\bar{l}}\} \\ &+ 2\{\nabla_i N_{j\bar{k}\bar{l}} + \nabla_{\bar{i}} N_{\bar{j}kl} + \nabla_j N_{ikl} + \nabla_{\bar{j}} N_{\bar{i}\bar{k}\bar{l}}\} \\ &+ \nabla_i N_{k\bar{j}\bar{l}} + \nabla_{\bar{i}} N_{\bar{k}jl} + \nabla_k N_{ijl} + \nabla_{\bar{k}} N_{\bar{i}\bar{j}\bar{l}} \\ &- \nabla_i N_{l\bar{j}\bar{k}} - \nabla_{\bar{i}} N_{\bar{l}jk} - \nabla_l N_{ijk} - \nabla_{\bar{l}} N_{\bar{i}\bar{j}\bar{k}}. \end{aligned}$$

Then, from (2.7), we have

$$(2.8)\qquad \tau = n(n+1)c - \frac{3}{32}\|N\|^2,$$

$$(2.9)\qquad \tau^* = n(n+1)c + \frac{1}{32}\|N\|^2,$$

$$(2.10)\qquad \tau + 3\tau^* = m(m+2)c.$$

In the sequel, we shall consider the case $\dim M = 4$. We set

$$A_{ij} = g(e_i, (\nabla_{e_j} N)(e_1, e_3)) = \nabla_j N_{i13},$$

for a unitary basis $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ of $T_pM, p \in M$. We note that

$$A_{ij} - A_{ji} = -2G_{ij13}.$$

Then, from (2.7), J. T. Cho and K. Sekigawa obtained the following

PROPOSITION 2.1 ([2]). *Let M be a 4-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)$ $(p \in M)$. Then*

$$R_{1212} = R_{3434} = -c(p),$$

$$R_{1234} = -\frac{c(p)}{2} - \frac{1}{16}(\tau^* - \tau),$$

$$R_{1324} = -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) + \frac{1}{8}(A_{13} - A_{31} - A_{24} + A_{42}),$$

$$R_{1423} = \frac{c(p)}{4} + \frac{3}{32}(\tau^* - \tau) + \frac{1}{8}(A_{13} - A_{31} - A_{24} + A_{42}),$$

$$R_{1313} = -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) - \frac{3}{8}(A_{13} - A_{31}) - \frac{1}{8}(A_{24} - A_{42}),$$

$$R_{1414} = -\frac{c(p)}{4} + \frac{5}{32}(\tau^* - \tau) - \frac{1}{8}(A_{13} + A_{42}) - \frac{3}{8}(A_{31} + A_{24}),$$

$$R_{2323} = -\frac{c(p)}{4} + \frac{5}{32}(\tau^* - \tau) + \frac{1}{8}(A_{31} + A_{24}) + \frac{3}{8}(A_{13} + A_{42}),$$

$$R_{2424} = -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) + \frac{3}{8}(A_{24} - A_{42}) + \frac{1}{8}(A_{13} - A_{31}),$$

$$R_{1334} = -R_{2434} = -\frac{1}{4}(A_{34} - A_{43}),$$

$$R_{1213} = -R_{1224} = -\frac{1}{4}(A_{12} - A_{21}),$$

$$R_{1434} = R_{2334} = -\frac{1}{4}(A_{33} + A_{44}),$$

$$R_{1214} = R_{1223} = -\frac{1}{4}(A_{11} + A_{22}),$$

$$R_{1323} = \frac{1}{8}(A_{14} + A_{41} + A_{32} - 3A_{23}),$$

$$R_{2324} = \frac{1}{8}(A_{14} + A_{41} + A_{23} - 3A_{32}),$$

$$R_{1314} = -\frac{1}{8}(A_{23} + A_{32} + A_{14} - 3A_{41}),$$

$$R_{1424} = -\frac{1}{8}(A_{23} + A_{32} + A_{41} - 3A_{14}),$$

for any unitary basis $\{e_i\}$ *of* T_pM *at each point* $p \in M$.

From Proposition 2.1, we have the following

LEMMA 2.2. *Let* M *be a* 4*-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature. Then* M *is an RK-manifold if and only if*

$$(A_{13} - A_{31}) + (A_{24} - A_{42}) = 0,$$

$$(A_{13} + A_{24}) + (A_{31} + A_{42}) = 0,$$

$$A_{12} = A_{21},$$

$$A_{34} = A_{43},$$

$$A_{14} = A_{23},$$

$$A_{41} = A_{32},$$

$$A_{11} + A_{22} = 0,$$

$$A_{33} + A_{44} = 0,$$

for a unitary basis $\{e_i\}$ *of* T_pM *at each point* $p \in M$.

Further, by Proposition 2.1, we have easily

$$\rho_{11} = \frac{\tau}{4} + \frac{1}{2}(A_{13} + A_{24}),$$

$$\rho_{22} = \frac{\tau}{4} - \frac{1}{2}(A_{13} + A_{24}),$$

$$\rho_{33} = \frac{\tau}{4} - \frac{1}{2}(A_{31} + A_{42}),$$

$$\rho_{44} = \frac{\tau}{4} + \frac{1}{2}(A_{31} + A_{42}), \tag{2.11}$$

$$\rho_{12} = \frac{1}{2}(A_{23} - A_{14}),$$

$$\rho_{13} = -\rho_{24} = -\frac{1}{4}\{(A_{11} + A_{22}) - (A_{33} + A_{44})\},$$

$$\rho_{14} = \rho_{23} = \frac{1}{4}\{(A_{12} - A_{21}) - (A_{34} - A_{43})\},$$

$$\rho_{34} = \frac{1}{2}(A_{32} - A_{41}),$$

(2.12)
$$\|\rho\|^2 = \frac{\tau^2}{4} + \frac{1}{2}\{(A_{13}+A_{24})^2+(A_{31}+A_{42})^2+(A_{23}-A_{14})^2+(A_{32}-A_{41})^2\}$$
$$+\frac{1}{4}\{(A_{11}+A_{22})-(A_{33}+A_{44})\}^2+\frac{1}{4}\{(A_{12}-A_{21})-(A_{34}-A_{43})\}^2,$$

(2.13)
$$\rho^*_{11} = \rho^*_{22} = \rho^*_{33} = \rho^*_{44} = \frac{\tau^*}{4},$$

$$\rho^*_{12} = \rho^*_{21} = \rho^*_{34} = \rho^*_{43} = 0,$$

$$\rho^*_{13} = \rho^*_{42} = -\rho^*_{24} = -\rho^*_{31} = \frac{1}{4}(A_{11} + A_{22} + A_{33} + A_{44}),$$

$$\rho^*_{14} = -\rho^*_{32} = \rho^*_{23} = -\rho^*_{41} = -\frac{1}{4}(A_{12} - A_{21} + A_{34} - A_{43})$$

and

(2.14)
$$\|\rho^*\|^2 = \frac{(\tau^*)^2}{4} + \frac{1}{4}(A_{11} + A_{22} + A_{33} + A_{44})^2$$
$$+\frac{1}{4}(A_{12} - A_{21} + A_{34} - A_{43})^2.$$

3. Some results

In this section, we show some results on a 4-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature. By (2.11) and (2.13), we have the following lemmas.

Lemma 3.1. *Let M be a 4-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature. Then M is Einstein if and only if*

$$A_{13} + A_{24} = 0,$$
$$A_{31} + A_{42} = 0,$$
$$A_{14} = A_{23},$$
$$A_{41} = A_{32},$$
$$A_{11} + A_{22} = A_{33} + A_{44},$$
$$A_{12} - A_{21} = A_{34} - A_{43},$$

for a unitary basis $\{e_i\}$ *of* T_pM *at each point* $p \in M$.

LEMMA 3.2. *Let M be a* 4*-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature. Then M is weakly* $*$*-Einstein if and only if*

$$A_{11} + A_{22} + A_{33} + A_{44} = 0,$$
$$A_{12} - A_{21} + A_{34} - A_{43} = 0,$$

for a unitary basis $\{e_i\}$ *of* T_pM *at each point* $p \in M$.

If ρ (resp. ρ^*) is J-invariant, then it is obvious that the conditions in Lemma 3.1 (resp. Lemma 3.2) are satisfied. So we have the following

PROPOSITION 3.3. *Let M be a* 4*-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature. If the Ricci tensor* ρ (*resp. the Ricci* $*$*-tensor* ρ^*) *is J-invariant, then M is an Einstein* (*resp. a weakly* $*$*-Einstein*) *manifold.*

By comparing with above Lemmas 2.2, 3.1 and 3.2, we have the following

PROPOSITION 3.4. *Let M be a* 4*-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature. Then M is an RK-manifold if and only if it is an Einstein and weakly* $*$*-Einstein manifold.*

From these results, we get again the following

THEOREM 3.5 ([12]). *Let M be a* 4*-dimensional almost Kähler manifold of constant holomorphic sectional curvature. If M satisfies the RK-condition, then M is a Kähler manifold.*

Proof. By Proposition 3.4, we see that M is an Einstein and weakly $*$-Einstein manifold. Since c and τ are constant on M, τ^* is also constant by (2.10), that is, M is $*$-Einstein. Then, taking account of the theorem of Oguro and Sekigawa [9], we can conclude that M is Kählerian. □

Note that the compactness assumption in the above theorem is not necessary due to [9]. The example in §4 shows that the condition of constant holomorphic sectional curvature can't be replaced by that of pointwise constant holomorphic sectional curvature.

Next, we shall calculate the square norm $\|R\|^2$ of the curvature tensor at an arbitrary point p of M. By the same way as J. T. Cho and K. Sekigawa [2], we have

$$\begin{aligned} \|R\|^2 = {} & 4\sum_{a,b}(R_{1a1b}^2 + R_{2a2b}^2 + R_{3a3b}^2 + R_{4a4b}^2) \\ & - 4(R_{1212}^2 + R_{1313}^2 + R_{1414}^2 + R_{2323}^2 + R_{2424}^2 + R_{3434}^2) \\ & + 8(R_{1234}^2 + R_{1324}^2 + R_{1423}^2), \end{aligned} \tag{3.1}$$

$$\begin{aligned} & 4\sum_{a,b}(R_{1a1b}^2 + R_{2a2b}^2 + R_{3a3b}^2 + R_{4a4b}^2) \\ & = 18c^2 + \frac{13}{32}(\tau^* - \tau)^2 - \frac{3c}{2}(\tau^* - \tau) + \frac{\tau^* - \tau}{2}(A_{13} - A_{31} - A_{24} + A_{42}) \\ & + (A_{12} - A_{21})^2 + (A_{11} + A_{22})^2 + (A_{34} - A_{43})^2 + (A_{33} + A_{44})^2 \\ & + \frac{5}{4}\{(A_{13} - A_{31})^2 + (A_{24} - A_{42})^2 + (A_{13} + A_{42})^2 + (A_{31} + A_{24})^2\} \\ & + \frac{3}{2}\{(A_{13} - A_{31})(A_{24} - A_{42}) + (A_{13} + A_{42})(A_{31} + A_{24})\} \\ & + \frac{1}{8}\{(A_{23} + A_{32} + A_{14} - 3A_{41})^2 + (A_{14} + A_{41} + A_{23} - 3A_{32})^2 \\ & \quad + (A_{14} + A_{41} + A_{32} - 3A_{23})^2 + (A_{23} + A_{32} + A_{41} - 3A_{14})^2\}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} & - 4(R_{1212}^2 + R_{1313}^2 + R_{1414}^2 + R_{2323}^2 + R_{2424}^2 + R_{3434}^2) \\ & = -9c^2 - \frac{13}{64}(\tau^* - \tau)^2 + \frac{3c}{4}(\tau^* - \tau) - \frac{\tau^* - \tau}{4}(A_{13} - A_{31} - A_{24} + A_{42}) \\ & - \frac{5}{8}\{(A_{13} - A_{31})^2 + (A_{24} - A_{42})^2 + (A_{13} + A_{42})^2 + (A_{31} + A_{24})^2\} \\ & - \frac{3}{4}\{(A_{13} - A_{31})(A_{24} - A_{42}) + (A_{13} + A_{42})(A_{31} + A_{24})\} \end{aligned} \tag{3.3}$$

and

$$(3.4)\quad 8(R_{1234}^2 + R_{1324}^2 + R_{1423}^2)$$
$$= 3c^2 + \frac{7}{64}(\tau^* - \tau)^2 + \frac{3c}{4}(\tau^* - \tau) + \frac{1}{4}(A_{13} - A_{31} - A_{24} + A_{42})^2$$
$$+ \frac{\tau^* - \tau}{4}(A_{13} - A_{31} - A_{24} + A_{42}).$$

By (3.1), (3.2), (3.3) and (3.4), as a correction of [2], we have the following

LEMMA 3.6. *Let M be a 4-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)$ $(p \in M)$. Then we have*

$$(3.5)\quad \|R\|^2 = 12c^2 + \frac{5}{16}(\tau^* - \tau)^2 + \frac{\tau^* - \tau}{2}(A_{13} - A_{31} - A_{24} + A_{42})$$
$$+ (A_{12} - A_{21})^2 + (A_{11} + A_{22})^2 + (A_{34} - A_{43})^2 + (A_{33} + A_{44})^2$$
$$+ \frac{3}{4}\{(A_{13} - A_{31})^2 + (A_{24} - A_{42})^2 + (A_{13} + A_{42})^2 + (A_{31} + A_{24})^2\}$$
$$+ \frac{1}{2}\{(A_{13} - A_{31})(A_{24} - A_{42}) + (A_{13} + A_{42})(A_{31} + A_{24})\}$$
$$+ \frac{1}{2}\{(A_{23} - A_{32})^2 + (A_{23} - A_{14})^2 + (A_{23} - A_{41})^2$$
$$+ (A_{32} - A_{14})^2 + (A_{32} - A_{41})^2 + (A_{14} - A_{41})^2\}.$$

It is assured, after long and tedious calculations, that the right hand side of (3.5) is invariant under the change of unitary bases.

In particular, if M is Einstein, then by Lemma 3.1, (3.5) reduces to

$$(3.6)\quad \|R\|^2 = 12c^2 + \frac{5}{16}(\tau^* - \tau)^2 + (\tau^* - \tau)(A_{13} - A_{31})$$
$$+ 2\{(A_{12} - A_{21})^2 + (A_{11} + A_{22})^2 + (A_{13} - A_{31})^2 + (A_{14} - A_{41})^2\}.$$

From now on we assume that the manifold M is compact. We shall calculate the characteristic classes of M by using A_{ij}. By (2.12), (3.5) and the Gauss-Bonnet formula, we have the following

PROPOSITION 3.7. *Let M be a 4-dimensional compact almost Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)$ $(p \in M)$. Then the*

Euler class of M is given by

$$(3.7)\ \chi(M) = \frac{1}{32\pi^2}\int_M \Bigg[12c^2 + \frac{5}{16}(\tau^* - \tau)^2 + \frac{\tau^* - \tau}{2}(A_{13} - A_{31} - A_{24} + A_{42})$$

$$+ \frac{3}{4}\{(A_{13} - A_{31})^2 + (A_{24} - A_{42})^2 + (A_{13} + A_{42})^2 + (A_{31} + A_{24})^2\}$$

$$+ \frac{1}{2}\{(A_{23} - A_{32})^2 + (A_{23} - A_{41})^2 + (A_{32} - A_{14})^2 + (A_{14} - A_{41})^2\}$$

$$+ 2\{(A_{11} + A_{22})(A_{33} + A_{44}) + (A_{12} - A_{21})(A_{34} - A_{43})\}$$

$$+ \frac{1}{2}\{(A_{13} - A_{31})(A_{24} - A_{42}) + (A_{13} + A_{42})(A_{31} + A_{24})\}$$

$$-2\{(A_{13}+A_{24})^2+(A_{31}+A_{42})^2\}-\frac{3}{2}\{(A_{23}-A_{14})^2+(A_{32}-A_{41})^2\}\Bigg]\, dM.$$

PROPOSITION 3.8. *Let M be a* 4*-dimensional compact almost Kähler manifold of pointwise constant holomorphic sectional curvature with J-invariant Ricci tensor. Then the square of the first Chern class of M is given by*

$$(3.8)\qquad c_1^2(M) = \frac{1}{32\pi^2}\int_M [\tau\tau^* + 4(A_{11} + A_{22})^2 + 4(A_{12} - A_{21})^2]\, dM.$$

If τ and τ^* have same sign, then $c_1^2(M)$ is non-negative.

Proof. First of all, we notice that M is Einstein by virtue of Proposition 3.3. It is easy to see that

$$(3.9)\qquad \gamma \wedge \gamma = \frac{1}{4}\sum_{i,j,k,l}\{\gamma_{ij}\gamma_{kl}J_{ij}J_{kl} - 2\gamma_{ij}\gamma_{kl}J_{ik}J_{jl}\}\, dM,$$

and

$$(3.10)\qquad 8\pi\sum \gamma_{ij}J_{ij} = 2(\tau + \tau^*).$$

By the definition of the first Chern form γ (cf. [13]), we have

$$8\pi\gamma_{12} = -4\sum J_{2k}\rho^*_{1k} - \sum J_{kl}(\nabla_2 J_{kh})\nabla_1 J_{lh}$$

$$= 4\rho^*_{11} - \frac{1}{4}\sum J_{kl}N_{\bar{2}kh}N_{\bar{1}lh}$$

$$= 4\rho^*_{11} + \frac{1}{4}\sum N_{1kh}N_{\bar{1}\bar{k}h}$$

$$= 4\rho^*_{11} - \frac{1}{4}\sum (N_{1kh})^2$$

$$= \tau^* - \{(N_{113})^2 + (N_{213})^2)\},$$

$$8\pi\gamma_{13} = -4\rho^*_{14} + \frac{1}{4}\sum N_{1kh}N_{4kh}$$

$$= 2(A_{12} - A_{21}) + (N_{113}N_{413} - N_{213}N_{313}),$$

$$8\pi\gamma_{14} = 4\rho^*_{13} - \frac{1}{4}\sum N_{1kh}N_{3kh}$$

$$= 2(A_{11} + A_{22}) - (N_{113}N_{313} + N_{213}N_{413}),$$

$$8\pi\gamma_{23} = -4\rho^*_{24} + \frac{1}{4}\sum N_{2kh}N_{4kh}$$

$$= 2(A_{11} + A_{22}) + (N_{113}N_{313} + N_{213}N_{413}),$$

$$8\pi\gamma_{24} = 4\rho^*_{23} - \frac{1}{4}\sum N_{2kh}N_{3kh}$$

$$= -2(A_{12} - A_{21}) + (N_{113}N_{413} - N_{213}N_{313}),$$

$$8\pi\gamma_{34} = 4\rho^*_{33} - \frac{1}{4}\sum (N_{3kh})^2$$

$$= \tau^* - \{(N_{313})^2 + (N_{413})^2\}.$$

From these identities, we have

$$(3.11) \qquad \sum_{i,j,k,l} \gamma_{ij}\gamma_{kl}J_{ik}J_{jl}$$

$$= \sum (\gamma_{1j}\gamma_{kl}J_{1k} + \gamma_{2j}\gamma_{kl}J_{2k} + \gamma_{3j}\gamma_{kl}J_{3k} + \gamma_{4j}\gamma_{kl}J_{4k})J_{jl}$$

$$= 2\sum (\gamma_{1j}\gamma_{2l} + \gamma_{3j}\gamma_{4l})J_{jl}$$

$$= 2\{(\gamma_{12})^2 + 2\gamma_{13}\gamma_{24} - 2\gamma_{14}\gamma_{23} + (\gamma_{34})^2\}$$

$$= \frac{1}{32\pi^2}[(\tau^*)^2 - 2\{(N_{113})^2 + (N_{213})^2\}\tau^* + \{(N_{113})^2 + (N_{213})^2\}^2$$

$$+ 2\{(N_{113}N_{413} - N_{213}N_{313})^2 - 4(A_{12} - A_{21})^2\}$$

$$+ 2\{(N_{113}N_{313} + N_{213}N_{413})^2 - 4(A_{11} + A_{22})^2\}$$

$$+ (\tau^*)^2 - 2\{(N_{313})^2 + (N_{413})^2\}\tau^* + \{(N_{313})^2 + (N_{413})^2\}^2]$$

$$= \frac{1}{32\pi^2}[2(\tau^*)^2 - 2\|N(e_1,e_3)\|^2\tau^* + \|N(e_1,e_3)\|^4 - 8\{(A_{11}+A_{22})^2 + (A_{12}-A_{21})^2\}]$$

$$= \frac{1}{32\pi^2}[(\tau^*)^2 + \tau^2 - 8\{(A_{11}+A_{22})^2 + (A_{12}-A_{21})^2\}].$$

By (3.9), (3.10) and (3.11),

$$\gamma \wedge \gamma = \frac{1}{32\pi^2}\{\tau\tau^* + 4(A_{11}+A_{22})^2 + 4(A_{12}-A_{21})^2\}\, dM.$$

Thus, we obtain (3.8). □

PROPOSITION 3.9. *Let M be a 4-dimensional compact almost Kähler manifold of pointwise constant holomorphic sectional curvature with J-invariant Ricci tensor. Then the first Pontrjagin class of M is given by*

$$p_1(M) = \frac{1}{32\pi^2}\int_M \left[2\|R\|^2 - \frac{1}{3}\tau^2\right] dM. \tag{3.12}$$

Proof. By definition,

$$p_1(M) = \frac{1}{32\pi^2}\int_M \sum\{R_{a\bar{a}ij}R_{b\bar{b}ij} - 2R_{abij}R_{\bar{a}\bar{b}ij}\}\, dM \tag{3.13}$$

$$= \frac{1}{32\pi^2}\int_M \{4\|\rho^*\|^2 + \|G\|^2 - 2\|R\|^2\}\, dM.$$

According to Koda [6], an almost Hermitian 4-manifold of pointwise constant holomorphic sectional curvature is self-dual. By making use of self-duality, we have

$$\sum\{(R_{12ij}-R_{34ij})^2 + (R_{13ij}+R_{24ij})^2 + (R_{14ij}-R_{23ij})^2\} = \|\rho\|^2 - \frac{1}{6}\tau^2. \tag{3.14}$$

On the other hand, we have

$$\begin{aligned} &\sum(R_{12ij}+R_{34ij})^2 = \|\rho^*\|^2, \\ &\sum(R_{13ij}-R_{24ij})^2 = \sum(G_{ij13})^2 = \sum(G_{ij24})^2, \\ &\sum(R_{14ij}+R_{23ij})^2 = \sum(G_{ij14})^2 = \sum(G_{ij23})^2. \end{aligned} \tag{3.15}$$

By (3.14) and (3.15), we have

$$\|R\|^2 = \|\rho\|^2 - \frac{1}{6}\tau^2 + \|\rho^*\|^2 + \frac{1}{4}\|G\|^2. \tag{3.16}$$

Consequently, we obtain (3.12) from (3.13), (3.16). □

From Proposition 3.9, we can see again that $p_1(M) \geqq 0$ and $p_1(M) = 0$ implies that M is of constant curvature (and hence, M is Kähler).

By (3.6), (3.8), (3.12), and the Wu's theorem (cf. [13]), we get the following

PROPOSITION 3.10. *Let M be a 4-dimensional compact almost Kähler manifold of pointwise constant holomorphic sectional curvature with J-invariant Ricci tensor. Then we have*

$$\begin{aligned}\int_M \Big[&\frac{1}{4}(\tau^* - \tau)\{\tau + 3\tau^* + 16(A_{13} - A_{31})\} + \frac{5}{4}(\tau^* - \tau)^2 \\ &+ 4\{(A_{11} + A_{22})^2 + (A_{12} - A_{21})^2\} + 8\{(A_{13} - A_{31})^2 + (A_{14} - A_{41})^2\}\Big]\, dM \\ &= 0.\end{aligned} \tag{3.17}$$

Taking account of $A_{13} - A_{31} = -2G_{1313}$, $\tau + 3\tau^* = 24c$ and the above Proposition, we have the following

THEOREM 3.11. *Let M be a 4-dimensional compact almost Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)$ $(p \in M)$ with J-invariant Ricci tensor. If*

$$G(x, y, x, y) \leqq \frac{3}{4}c(p),$$

for any unit vectors $x, y \in T_pM$ such that $g(x, y) = g(x, Jy) = 0$, at every point p of M, then M is a Kähler manifold.

Finally, we shall note a fact related to the theorem of J. Armstrong [1]. He has proved.

THEOREM 3.12 ([1]). *If (M, J, g) is a 4-dimensional compact Einstein almost Kähler manifold, then we must have that $\tau^* = \tau$ somewhere on the manifold.*

If M is of constant holomorphic sectional curvature, then from $\tau + 3\tau^* = 24c$, it can be $\tau^* = \tau$ on the whole of M owing to the above theorem. Consequently, we have the following

THEOREM 3.13. *Let M be a 4-dimensional compact almost Kähler manifold of constant holomorphic sectional curvature with J-invariant Ricci tensor. Then M is a Kähler manifold.*

4. The example of Nurowski and Przanowski

In this section, we shall describe the example of 4-dimensional Ricci-flat strictly almost Kähler manifold of pointwise constant holomorphic sectional curvature constructed by P. Nurowski and M. Przanowski [5]. Following T. Oguro, K. Sekigawa and A. Yamada [10], we write down explicitly this example.

Let M be a 4-dimensional real half space given by

$$M = \{(x_1, x_2, x_3, x_4) \in \boldsymbol{R}^4 \mid x_1 > 0, (x_2, x_3, x_4) \in \boldsymbol{R}^3\}.$$

We define a Riemannian metric g and almost complex structure J on M respectively by

$$g = x_1(dx_1^2 + dx_2^2 + dx_3^2) + \frac{1}{x_1}\left(\frac{x_3}{2}\,dx_2 - \frac{x_2}{2}\,dx_3 + dx_4\right)^2 \tag{4.1}$$

and

$$\begin{aligned} J\left(\frac{\partial}{\partial x_1}\right) &= \frac{\partial}{\partial x_3} + \frac{x_2}{2}\frac{\partial}{\partial x_4}, \\ J\left(\frac{\partial}{\partial x_2}\right) &= \frac{x_3}{2x_1}\frac{\partial}{\partial x_2} - \left(x_1 + \frac{x_3^2}{4x_1}\right)\frac{\partial}{\partial x_4}, \\ J\left(\frac{\partial}{\partial x_3}\right) &= -\frac{\partial}{\partial x_1} - \frac{x_2}{2x_1}\frac{\partial}{\partial x_2} + \frac{x_2 x_3}{4x_1}\frac{\partial}{\partial x_4}, \\ J\left(\frac{\partial}{\partial x_4}\right) &= \frac{1}{x_1}\frac{\partial}{\partial x_2} - \frac{x_3}{2x_1}\frac{\partial}{\partial x_4}. \end{aligned} \tag{4.2}$$

Then, we can see easily that (J, g) is an almost Hermitian structure on M and the Kähler form Ω is given by

$$\Omega = -x_1 dx_1 \wedge dx_3 - \frac{x_2}{2}\,dx_2 \wedge dx_3 + dx_2 \wedge dx_4. \tag{4.3}$$

From (4.3), we see immediately that Ω is closed, and hence (M, J, g) is an almost Kähler manifold. Now, we define a unitary frame field $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ on M by

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{x_1}}\frac{\partial}{\partial x_1}, \quad e_2 = \frac{1}{\sqrt{x_1}}\frac{\partial}{\partial x_3} + \frac{x_2}{2\sqrt{x_1}}\frac{\partial}{\partial x_4}, \\ e_3 &= \sqrt{x_1}\,\frac{\partial}{\partial x_4}, \quad e_4 = \frac{1}{\sqrt{x_1}}\frac{\partial}{\partial x_2} - \frac{x_3}{2\sqrt{x_1}}\frac{\partial}{\partial x_4}. \end{aligned} \tag{4.4}$$

With respect to this unitary frame $\{e_i\}_{i=1,2,3,4}$, we have

$$
(4.5)\qquad
\begin{aligned}
&\nabla_{e_1}e_1 = 0, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = 0, \quad \nabla_{e_1}e_4 = 0,\\
&\nabla_{e_2}e_1 = \frac{1}{2x_1\sqrt{x_1}}e_2, \qquad \nabla_{e_2}e_2 = -\frac{1}{2x_1\sqrt{x_1}}e_1,\\
&\nabla_{e_2}e_3 = \frac{1}{2x_1\sqrt{x_1}}e_4, \qquad \nabla_{e_2}e_4 = -\frac{1}{2x_1\sqrt{x_1}}e_3,\\
&\nabla_{e_3}e_1 = -\frac{1}{2x_1\sqrt{x_1}}e_3, \quad \nabla_{e_3}e_2 = \frac{1}{2x_1\sqrt{x_1}}e_4,\\
&\nabla_{e_3}e_3 = \frac{1}{2x_1\sqrt{x_1}}e_1, \qquad \nabla_{e_3}e_4 = -\frac{1}{2x_1\sqrt{x_1}}e_2,\\
&\nabla_{e_4}e_1 = \frac{1}{2x_1\sqrt{x_1}}e_4, \qquad \nabla_{e_4}e_2 = \frac{1}{2x_1\sqrt{x_1}}e_3,\\
&\nabla_{e_4}e_3 = -\frac{1}{2x_1\sqrt{x_1}}e_2, \quad \nabla_{e_4}e_4 = -\frac{1}{2x_1\sqrt{x_1}}e_1.
\end{aligned}
$$

From (4.5), by straightforward calculation, we have

$$
(4.6)\qquad
\begin{aligned}
\nabla_3 J_{14} &= -\nabla_3 J_{41} = \nabla_3 J_{23} = -\nabla_3 J_{32}\\
&= \nabla_4 J_{13} = -\nabla_4 J_{31} = \nabla_4 J_{42} = -\nabla_4 J_{24}\\
&= \frac{1}{x_1\sqrt{x_1}},\\
\nabla_i J_{jk} &= 0 \quad \text{(otherwise)},
\end{aligned}
$$

$$
(4.7)\qquad
\begin{aligned}
&A_{11} = 0, \quad A_{12} = 0, \quad A_{13} = -\frac{1}{x_1^3}, \quad A_{14} = 0,\\
&A_{21} = 0, \quad A_{22} = 0, \quad A_{23} = 0, \quad A_{24} = \frac{1}{x_1^3},\\
&A_{31} = \frac{3}{x_1^3}, \quad A_{32} = 0, \quad A_{33} = 0, \quad A_{34} = 0,\\
&A_{41} = 0, \quad A_{42} = -\frac{3}{x_1^3}, \quad A_{43} = 0, \quad A_{44} = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&R_{1212} = R_{3434} = -\frac{1}{2x_1^3}, \\
&R_{1234} = -\frac{1}{2x_1^3}, \\
&R_{1324} = -\frac{1}{x_1^3}, \\
&R_{1423} = -\frac{1}{2x_1^3}, \\
&R_{1313} = \frac{1}{x_1^3}, \\
&R_{1414} = -\frac{1}{2x_1^3}, \\
&R_{2323} = -\frac{1}{2x_1^3}, \\
&R_{2424} = \frac{1}{x_1^3}, \\
&R_{1334} = -R_{2434} = R_{1213} = -R_{1224} = R_{1434} = R_{2334} \\
&\quad = R_{1214} = R_{1223} = R_{1323} = R_{2324} = R_{1314} = R_{1424} = 0.
\end{aligned}
\tag{4.8}
$$

Taking account of Proposition 2.1, (4.7) and (4.8) show that (M, J, g) is a strictly almost Kähler manifold of pointwise constant holomorphic sectional curvature $c(p) = 1/2x_1^3 (p = (x_1, x_2, x_3, x_4) \in M)$ and satisfies the *RK*-condition. From (4.8), it is easy to see that

$$\rho = 0, \quad \text{and} \quad \rho^* = \frac{1}{x_1^3} g,$$

and hence (M, J, g) is a Ricci-flat, weakly $*$-Einstein manifold with $*$-scalar curvature $\tau^* = 4/x_1^3$.

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