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ON THE EXISTENCE OF CERTAIN QUADRATIC DIFFERENTIALS ON FOUR TIMES PUNCTURED SPHERES AND ONCE PUNCTURED TORI

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Let R be a Riemann surface. Let $\{\gamma_j\}_{j=1}^{p}$ be a set of homotopically nontrivial Jordan curves on R which are mutually disjoint and belong to different free homotopy classes. We consider the following problem.

PROBLEM. Find conditions of non-negative numbers $l_1, \dots, l_p$ such that there exists the holomorphic quadratic differential φ with closed trajectories on R which has following properties:

(a) Each of closed trajectories of φ is homotopic to one of the curves $\{\gamma_j\}_{j=1}^{p}$.

(b) For any $j = 1, \dots, p$, φ has closed trajectories homotopic to γ_j.

(c) For $j = 1, \dots, p$, the φ-length of closed trajectories homotopic to γ_j is equal to l_j.

In this paper, we shall give answers for this problem in the case where R is either a four times punctured sphere or a once punctured torus (see Sections 3 and 5). An essential tool in obtaining our results is the deformation space of Riemann surfaces with nodes due to Bers.

This problem is related to the following Strebel's result (cf. Theorem 23.5 in [18, p. 150]).

THEOREM. *Given a Riemann surface R with marked points $P_j, j = 1, \dots, p$ $p \geq 2$ and $\dot{R} = R \backslash \{P_j\}$ not the twice punctured sphere. We consider the quadratic differentials φ on $\dot{R}$ with closed trajectories the characteristic ring domains of which are punctured discs R_j, with punctures P_j. Then, the lengths $a_j > 0$ of the closed trajectories α_j around the P_j can be prescribed arbitrarily. The solution φ is uniquely determined.*

Strebel proved this result by using the convexity of the surface of reduced moduli (cf. [18, p. 148]). This theorem implies that our problem is solved in the case where every γ_j is homotopic to a small loop around a puncture. In this case, constants $l_1, \dots, l_p$ are prescribed arbitrarily. Therefore we only consider

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our problem in the case where for some $j = 1, \dots, p$, γ_j is essential (cf. Section 1.2).

This paper is organized as follows. In Section 1, we recall the definition of the deformation space of Riemann surfaces with nodes and give some notations used in this paper. Section 2 contains a detailed discussion of the deformation space of Riemann surfaces of type $(0,4)$ with a node. We shall treat this space by using a plumbing procedure. In Section 3, our problem is solved in case where a given Riemann surface R is a four times punctured sphere. As in Sections 2 and 3, Section 4 deals with the deformation space of Riemann surfaces of type $(1,1)$ with a node, and Section 5 contains an answer for our problem in case where R is a once punctured torus. In Section 6, we give results related to our problem.

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1. Preliminaries

1.1. We recall the deformation space of Riemann surfaces with nodes (cf. [2], [3], and [4]). A *Riemann surface with nodes*, S, is a connected complex space such that every $P \in S$ has either a fundamental system of neighborhoods isomorphic to the unit disc $|z| < 1$, or a fundamental system of neighborhoods isomorphic to the set $z_1 z_2 = 0$ in the unit bicylinder $|z_1| < 1$, $|z_2| < 1$. In the latter case P is called a *node*. Every component Σ of the complement of the set of nodes is called a *part* of S, and S is called *stable* if every part has the upper half-plane as its universal covering surface, and therefore carries a canonical Poincaré metric.

By a Riemann surface S of *finite type* we mean a stable Riemann surface with or without nodes, such that either $n = 0$ and S is compact, or $n > 0$ and S is compact expect for n punctures. (A puncture can never be at a node.) Such an S has finitely many parts $\Sigma^1, \dots, \Sigma^r$, each part Σ^j is compact of some genus p_j, except for n_j punctures, $3p_j - 3 - n_j \geq 0$ (this is the stability condition), and

$$\sum_{j=1}^{r} n_j = 2k + n$$

where k is the number of nodes. Also, the total Poincaré area of S equals

$$A = 2\pi \sum_{j=1}^{r} (2p_j - 2 + n_j).$$

The genus p of S is defined by the relation

$$A = 2\pi(2p - 2 + 2).$$

If one "thickens" each node so as to obtain a smooth surface $\check{S}$, $\check{S}$ is homeomorphic to a compact surface of genus p with n punctures. The pair

$$(p, n)$$

is called the type of S.

Let S and S' be two Riemann surfaces of the same type. A continuous surjection $f : S' \to S$ is called a *deformation* if the inverse image of every node of S is either a node of S' or a Jordan curve on a part of S', if, for every part Σ of S, the restriction $f^{-1}|_{\Sigma}$ is an orientation preserving homeomorphism onto $f^{-1}(\Sigma)$, and if every puncture of S' corresponds, under f, to a puncture of S. A holomorphic deformation is called an *isomorphism.*

The equivalence class $[f]$ of a deformation $f : S' \to S$ consists of all deformations $S'' \to S$ of the form $\psi \circ f \circ \phi^{-1}$ where $\phi : S' \to S''$ is a deformation isotopic to an isomorphism and $\psi : S \to S$ is a deformation isotopic to the identity. The *deformation space* $D(S)$ consists of all equivalence classes $[f]$ of deformations onto S. It is known that $D(S)$ carries the natural complex structure compatible with that of the Teichmüller spaces (see [4, Section 16]).

1.2. Let R be a Riemann surface and γ a simple closed curve on R. In this paper, γ is called *essential* or an *essential curve* if γ does not bound either a disc or a punctured disc on R. For an essential curve γ on R, we denote by $\operatorname{mod}_R(\gamma)$ the modulus of the family of the curves homotopic to γ on R, that is, the reciprocal of the extremal length of the family of curves homotopic to γ on R (cf. [10, p. 13] and [1, p. 220]). In this paper, $\operatorname{mod}_R(\gamma)$ is said to be *the modulus of* γ for short. For a doubly connected domain D on a Riemann surface, $\operatorname{mod}(D)$ denotes the modulus of a simple closed curve in D which separates the boundary components of D. We know that if $D = \{z \in \boldsymbol{C} \mid r < |z| < 1\}$, it holds that $\operatorname{mod}(D) = -(1/2\pi)\log r$ (cf. [10, p. 17, Theorem 2.4]).

We know that for an essential curve γ on R, there exists a quadratic differential ψ on R with the following properties: (1) The non-critical horizontal trajectories of ψ are closed and homotopic to γ: (2) The modulus of its characteristic ring domain is equal to the modulus of γ (cf. Theorem 21.1 in [18, p. 107]). In this paper, such a quadratic differential is said to be the *J-S* (*Jenkins-Strebel*) *differential* on R with respect to γ. *Throughout this paper, we shall take the notation of* [18] *for granted, and restrict our attention to quadratic differentials that are holomorphic.*

2. The deformation space of Riemann surfaces of type $(0, 4)$ with a node

2.1. We consider a global coordinate of the deformation space of Riemann surfaces of type $(0, 4)$ with a node as follows.

We first construct a Riemann surface S_0 of type $(0, 4)$ with a node. Let Σ^1 and Σ^2 be two copies of a three punctured sphere $\boldsymbol{C}\backslash\{-1, 1\}$. Then S_0 is obtained by identifying the puncture ∞ of Σ^1 and the puncture ∞ of Σ^2. We denote by N_0 the node of S_0.

Next, for $t \in \Delta^* := \{0 < |z| < 1\}$, we construct a Riemann surface S_t of type $(0,4)$ by a "plumbing procedure". Let U^1 and U^2 be two copies of a domain $\boldsymbol{C} \backslash \{t \in \boldsymbol{R} \mid -1 \leq t \leq 1\}$. For $k = 1, 2$, we regard U^k as a subset of Σ^k. Let $Z^k(z) = z - \sqrt{z^2 - 1}$, where the branch of the square root is taken as $0 < Z^k(t) < 1$ for $t > 1$. Z^k is defined on $U^k \cup \{\infty\}$ by $Z^k(\infty) = 0$ and maps conformally $U^k \cup \{\infty\}$ onto $\Delta := \{z \in \boldsymbol{C} \mid |z| < 1\}$. For $k = 1, 2$, we define $U_t^k = \{P \in U^k \mid |Z^k(P)| > |t|\}$ and $S_t^k = U_t^k \cup \{t \in \boldsymbol{R} \mid -1 < t < 1\}$. A Riemann surface S_t is obtained by introducing an equivalence relation on the disjoint union $S_t^1 \cup S_t^2$. A point $P \in U_t^1$ is identified with a point $Q \in U_t^2$ if and only if $Z^1(P)Z^2(Q) = t$. Let σ_t be a central curve $\{P \in U_t^1 \mid |Z^1(P)| = |t|^{1/2}\} = \{P \in U_t^2 \mid |Z^2(P)| = |t|^{1/2}\}$ in S_t. For $|t| < 1$ and $k = 1, 2$, we denote by P_t^{2k-1} and P_t^{2k} the punctures of S_t corresponding to 1 and -1 in S_t^k respectively. Then there exists the deformation f_t from S_t to S_0 with $f_t(P_t^m) = P_0^m$ for $m = 1, \ldots, 4$ and $f_t^{-1}(N_0) = \sigma_t$. We define a holomorphic mapping Ψ from Δ to $D(S_0)$ by

$$\Psi(t) = [f_t]. \tag{1}$$

We can observe that Ψ is a biholomorphic mapping from Δ to $D(S_0)$, and hence Ψ is a global coordinate of $D(S_0)$. Indeed, the injectivity of Ψ follows from Lemma 5 in Section 3.3, and the surjectivity of that is given by the "open up" process in [13, Section 5.1].

2.2. For $k = 1, 2$, let (V^k, W^k) be a local uniformizing parameter at ∞, the puncture of Σ^k, such that $W^k(V^k) = \Delta^*$ and that ∞ corresponds to the origin. For $t \in \Delta^*$, a Riemann surface M_t of type $(0,4)$ is obtained by a plumbing procedure using given coordinates $\{(V^k, W^k)\}_{k=1,2}$. We also obtain the canonical deformation g_t from M_t to S_0 as in Section 2.1. Thus, we define the holomorphic mapping Φ from Δ to $D(S_0)$ by

$$\Phi(t) = [g_t]. \tag{2}$$

In this paper, we call Φ the *representation of pluming constructions using* $\{(V^k, W^k)\}_{k=1,2}$.

2.3. To compute the derivative of $T(t) := \Psi^{-1} \circ \Phi(t)$ at the origin (see Proposition 2 of Section 2.4), we shall give some notation and prove a lemma.

Let i_t^k be the canonical inclusion from S_t^k to S_t. Since $(Z^k)^*(-dZ^2/Z^2) = \psi^k := -dz^2/(z^2 - 1)$ for $k = 1, 2$, Z^k maps a closed trajectory of ψ^k to a circle with center 0 in Δ. Thus the J-S differential ψ_t on S_t with respect to σ_t is obtained by setting $(i_t^k)^*(\psi_t) = \psi^k$ on the image under i_t^k. The characteristic ring domain U_t of ψ_t coincides with $i_t^k(U_t^k)$. We define a conformal mapping Z_t from U_t to an annulus $A_{|t|} := \{|t| < |z| < 1\}$ by $Z_t(i_t^1(P)) = Z^1(P)$ for $P \in U_t^1$. Then, for $t \in \Delta^*$,

$$\operatorname{mod}_{S_t}(\sigma_t) = -\frac{1}{2\pi} \log |t|. \tag{3}$$

Let $V_t^k = \{P \in V^k \mid |W^k(P)| > |t|\}$ and $M_t^k = V_t^k \cup \overline{\Sigma^k \setminus V^k}$. We denote by j_t^k the canonical inclusion from M_t^k to M_t. Let V_t be the image of V_t^k under j_t^k. A conformal mapping W_t from V_t to an annulus $A_{|t|}$ is defined by $W_t(j_t^1(P)) = W^1(P)$ for $P \in V_t^1$. Let γ_t denote the central curve in V_t. For $|t| < 1$ and $k = 1, 2$, we denote by Q_t^{2k-1} and Q_t^{2k} punctures of M_t corresponding to 1 and -1 in M_t^k respectively. Then, by definition, the deformation g_t satisfies $g_t(Q_t^m) = P_0^m$ for $m = 1, \ldots, 4$ and $g_t^{-1}(N_0) = \gamma_t$.

Let η_t be the J-S differential on M_t with respect to γ_t such that the η_t-length of closed trajectories homotopic to γ_t is equal to 2π. Suppose that $\mathscr{A}_t$ is the characteristic ring domain of η_t. Then, (1), (2), and (3) imply

$$\text{(4)} \qquad \operatorname{mod}(\mathscr{A}_t) = \operatorname{mod}_{M_t}(\gamma_t) = \operatorname{mod}_{S_{T(t)}}(\sigma_{T(t)}) = -\frac{1}{2\pi}\log|T(t)|,$$

for $t \in \Delta^*$. We denote by ζ_t a conformal mapping from $\mathscr{A}_t$ to $A_{|T(t)|}$.

LEMMA 1. (i) *Let $l(t)$ be the η_t-length of γ_t. Then $l(t) = O(1)$ as $|t| \to 0$.* (ii) *For any $0 < r < 1$, there exists $\delta_1 > 0$ with $\delta_1 < r^2$ such that*

$$\gamma_t \cap \{P \in \mathscr{A}_t \mid |T(t)|/r < |\zeta_t(P)| < r\} \neq \emptyset \quad \textit{whenever } 0 < |t| < \delta_1.$$

(iii) *There exists $\delta_0 > 0$ such that $\gamma_t \subset \mathscr{A}_t$ whenever $0 < |t| < \delta_0$.*

Proof. (i) Fix $t \in \Delta^*$. Suppose that η_t has the representation $\eta_t(w)\,dw^2$ in terms of the local uniformizing parameter (V_t, W_t). We set

$$L(x) = \int_{|w|=x} |\eta_t(w)|^{1/2}|dw|$$

for $|t| < x < 1$. Since $\eta_t(w)$ is a holomorphic function on $A_{|t|}$, $L(x)$ is a convex function of $\log x$. Therefore, we obtain

$$\frac{1}{x}l(t) = \frac{1}{x}L(|t|^{1/2}) \le \frac{1}{2x}\left(L\left(\frac{|t|}{x}\right) + L(x)\right)$$

for $|t|^{1/2} \le x < 1$. Integration over $|t|^{1/2} \le x \le 1$ yields

$$\begin{aligned}\frac{\log(1/|t|)}{2}l(t) &\le \int_{|t|^{1/2}}^{1} \frac{1}{2}\left(L\left(\frac{|t|}{x}\right) + L(x)\right)\frac{dx}{x} \\ &\le \frac{1}{2}\left\{\int_{V_t}|\eta_t|\right\}^{1/2} \cdot \{2\pi\log(1/|t|)\}^{1/2} \\ &\le \frac{1}{2}\{2\pi\log(1/|T(t)|)\}^{1/2} \cdot \{2\pi\log(1/|t|)\}^{1/2}.\end{aligned}$$

Hence we have

$$l(t) \le 2\pi\{\log(1/|T(t)|)\}^{1/2}/\{\log(1/|t|)\}^{1/2} = O(1), \quad \text{as } |t| \to 0.$$

(ii) Assume that there exist $0 < r_0 < 1$ and a sequence $\{t_n\}_{n=1}^{\infty}$ such that

$$|t_n| \to 0 \quad \text{as } n \to \infty, \tag{5}$$

and that $\gamma_{t_n} \cap \{P \in \mathscr{A}_{t_n} \mid |T(t_n)|/r_0 < |\zeta_{t_n}(P)| < r_0\} = \emptyset$. Then, some component C_n of $M_{t_n} \setminus \{P \in \mathscr{A}_{t_n} \mid |T(t_n)|/r_0 < |\zeta_{t_n}(P)| < r_0\}$ contains one of the conponents of $M_{t_n} \setminus \gamma_{t_n}$. By definition, $\eta_{t_n}|_{C_n}$ in the J-S differential on C_n with respect to a simple closed curve τ_n homotopic to the boundary contour $\overline{C_n} \cap \{P \in \mathscr{A}_{t_n} \mid |T(t_n)|/r_0 \leq |\zeta_{t_n}(P)| \leq r_0\}$ of C_n. Hence

$$\mathrm{mod}_{C_n}(\tau_n) = -\frac{1}{2\pi} \log|r_0|. \tag{6}$$

On the other hand, since C_n contains some component of $M_{t_n} \setminus \gamma_{t_n}$, there exists an injective holomorphic mapping h_n from $A_{|t_n|^{1/2}}$ to C_n such that $h_n(\{w \in A_{|t_n|^{1/2}} \mid |w| = |t_n|^{1/4}\})$ is homotopic to τ_n. Therefore

$$\mathrm{mod}_{C_n}(\tau_n) \geq -\frac{1}{2\pi} \log |t_n|^{1/2}.$$

By (5), this contradicts (6).

(iii) By (i), there exists δ_1 and $L > 0$ such that $l(t) < L$ for $|t| < \delta_1$. Let $r = e^{-2L}$. In view of (ii), we obtain a number δ_0 with $0 < \delta_0 < \min\{\delta_1, r^2\}$ such that

$$\gamma_t \cap \{P \in \mathscr{A}_t \mid |T(t)|/r < |\zeta_t(P)| < r\} \neq \emptyset, \quad \text{whenever } 0 < |t| < \delta_0. \tag{7}$$

Let d_{η_t} be the η_t-distance on M_t. We denote by $\mathscr{C}_t$ the union of critical trajectories of η_t. Then for $x \in \{P \in \mathscr{A}_t \mid |T(t)|/r < |\zeta_t(P)| < r\} \cap \gamma_t$ and $y \in \mathscr{C}_t$, we have $d_{\eta_t}(x, y) \geq 2L$. By the definition of the constant L, (7) implies that points $x \in \{P \in \mathscr{A}_t \mid |T(t)|/r < |\zeta_t(P)| < r\} \cap \gamma_t$ and $z \in \gamma_t$ satisfy $d_{\eta_t}(x, z) \leq L$. Therefore we have $d_{\eta_t}(y, z) \geq L > 0$, for $y \in \mathscr{C}_t$ and $z \in \gamma_t$. Thus $\gamma_t \cap \mathscr{C}_t = \emptyset$ whenever $0 < |t| < \delta_0$. Since $\mathscr{A}_t = M_t \setminus \mathscr{C}_t$, γ_t is contained in $\mathscr{A}_t$ whenever $0 < |t| < \delta_0$. □

2.4.

PROPOSITION 2. *For $k = 1, 2$, let (V^k, W^k) be a local parameter around ∞ in Σ^k such that $W^k(V^k) = \Delta^*$ and that ∞ corresponds to the origin. Let Φ be the representation of plumbing constructions using $\{(V^k, W^k)\}_{k=1,2}$. Suppose that W^k has the expansion*

$$W^k(z) = \frac{A^k}{z} + \cdots$$

in terms of the global coordinate z of Σ^k near ∞. Then the derivative of $T = \Psi^{-1} \circ \Phi$ at the origin is given by

$$\frac{dT}{dt}(0) = \frac{1}{4A^1A^2}.$$

Proof. By definition, for $t \in \Delta^*$, there exists a biholomorphic mapping h_t from M_t to $S_{T(t)}$ such that $h_t(Q_t^m) = P_{T(t)}^m$ for $m = 1, \ldots, 4$ and that $h_t(\gamma_t)$ is homotopic to $\sigma_{T(t)}$.

We first show that there exists $\delta, r_0 > 0$ such that

$$h_t(\{P \in V_t \mid |t|/r_0 < |W_t(P)| < r_0\}) \subset U_{T(t)}$$

whenever $0 < |t| < \delta$. Since $U_{T(t)}$ is the characteristic ring domain of $\psi_{T(t)}$, Lemma 1 guarantees that there exists $\delta_0 > 0$ such that $h_t(\gamma_t) \subset U_{T(t)}$ whenever $0 < |t| < \delta_0$. This implies $h_t(j_t^k(M_{|t|^{1/2}}^k)) \subset i_{T(t)}^k(S_{T(t)}^k)$. Here, we regard $M_{|t|^{1/2}}^k$ as a subset of M_t^k. We remark that $j_{|t|^{1/2}}^k = j_t^k|_{M_{|t|^{1/2}}^k}$. Hence, for $k = 1, 2$ and $0 < |t| < \delta_0$, an injective holomorphic mapping h_t^k from $M_{|t|^{1/2}}^k$ to $S_{T(t)}^k$ is defined by $h_t^k = (i_{T(t)}^k)^{-1} \circ h_t \circ j_{|t|^{1/2}}^k$.

Since $\{S_{T(t)}^k\}_{0<|t|<\delta_0}$ and $\{M_{|t|^{1/2}}^k\}_{0<|t|<\delta_0}$ are exhaustions of Σ^k and $h_t^k(1) = 1$ and $h_t^k(-1) = -1$, h_t^k converges uniformly to the identity mapping of Σ^k on every compact set of Σ^k as $t \to 0$. Therefore there exist δ and $r_0 > 0$ such that $h_t^k(V_t^k(r_0)) \subset U_{T(t)}^k$ whenever $0 < |t| < \delta$, where $V_t^k(r_0) := \{P \in M_{|t|^{1/2}}^k \mid |t|^{1/2} \le |W^k(P)| < r_0\}$. Since $j_{|t|^{1/2}}^1(V_t^1(r_0)) \cup j_{|t|^{1/2}}^2(V_t^2(r_0)) = \{P \in V_t \mid |t|/r_0 < |W_t(P)| < r_0\}$ and $i_{T(t)}^k(U_{T(t)}^k) = U_{T(t)}$, we have the first assertion.

For $k = 1, 2$, a holomorphic mapping H^k from $D' := \{(t, w) \mid 0 < |t| < \delta$ and $|t|/r_0 < |w| < r_0\}$ to Δ^* is defined by

$$H^k(t, w) = Z^k \circ (i_{T(t)}^k)^{-1} \circ h_t \circ j_t^k \circ (W^k)^{-1}(w).$$

Since $\{(i_{T(t)}^k)^{-1} \circ h_t \circ j_t^k\}_{0<|t|<\delta}$ tends to the identity mapping on Σ^k as $t \to 0$, $\{P \in V^k \mid |W^k(P)| < r_0\}$ is contained in U^k and H^k is extended holomorphically on $D := \{(t, w) \mid |t| < \delta, |t|/r_0 < |w| < r_0\}$ by setting $H^k(0, w) = Z^k \circ (W^k)^{-1}(w)$.

Suppose that the Laurent development of H^k forms

$$H^k(t, w) = \sum_{m=-\infty}^{\infty} a_m^k(t) w^m. \tag{8}$$

We remark that for every integer m and $k = 1, 2$, $a_m^k(t)$ is holomorphic on $\{t \in \Delta \mid |t| < \delta\}$.

Here, we assume the following equation which will be proved later.

$$a_m^k(t) t^m = o(1), \quad \text{as } t \to 0 \text{ for } k = 1, 2 \text{ and } m \in \boldsymbol{Z}. \tag{9}$$

The definition of $S_{T(t)}$ implies

$$T(t) = H^1(t, w) H^2(t, t/w) = \sum_{m, l=-\infty}^{\infty} a_m^1(t) a_l^2(t) t^l w^{m-l}$$

on D. Since the left-hand side of this equality is independent on the parameter w, we obtain

$$T(t) = \sum_{l=-\infty}^{\infty} a_l^1(t) a_l^2(t) t^l$$

on $\{|t| < \delta\}$. Therefore the assumption (9) shows

$$\frac{T(t)}{t} = a_1^1(0)a_1^2(0) + o(1) = \frac{1}{4A^1A^2} + o(1). \qquad \square$$

Now, to prove Proposition 2, it remains to show the following lemma.

LEMMA 3. *The equation* (9) *holds.*

Proof. We may assume that $k = 1$. Fix $0 < |w| < \min\{\delta, \delta/r_0\}$. By the definition of S_t, if $0 < |t| < r_0|w|$, then $(t, w), (t, t/w) \in D$ and

$$H^1(t, t/w) = T(t)/H^2(t, w).$$

Since $H^2(0, w) \neq 0$, (8) implies

$$\sum_{m=-\infty}^{\infty} a_m^1(t)t^m/w^m \to 0$$

as $t \to 0$. Since w is arbitrary, we conclude (9). $\square$

2.5.

COROLLARY 1. *Let* $\{(V^k, W^k)\}_{k=1,2}$ *be as in Proposition* 2. *Let* R *be a Riemann surface of type* $(0,4)$. *Suppose that there exists an essential curve* γ *on* R *such that*

$$\sinh(2\pi \operatorname{mod}_R(\gamma))/2\pi \operatorname{mod}_R(\gamma) > 4|A^1A^2|.$$

Then R *is obtained by the plumbing construction using* $\{(V^k, W^k)\}_{k=1}^2$. *Namely,* R *is biholomorphic to* M_t *for some* $t \in \Delta^*$.

From (4) and the following lemma, we obtain this corollary by an argument similar to that of the proof of Theorem 1 (see Section 3.4). Hence we omit the proof.

LEMMA (cf. [15, p. 233]). *Let* f *be a holomorphic mapping on* $|z| < 1$ *such that* $f(0) = 0$ *and* $|f(z)| < 1$ *on* $|z| < 1$. *If* f *leaves out a value* α *such that* $|\alpha| <$ 1, *then*

$$|f'(0)| \leq 2|\alpha| \log(1/|\alpha|)/(1 - |\alpha|^2).$$

We know that every Riemann surface of type $(0,4)$ has an essential curve whose modulus is more than or equal to $\sqrt{3}/4$ (see (i) of Lemma 12). Therefore we have the following.

COROLLARY 2. *Let* $\{(V^k, W^k)\}_{k=1,2}$ *be as in Proposition* 2. *Assume that*

$$|A^1A^2| < \sinh(\sqrt{3}\pi/2)/2\sqrt{3}\pi = 0.694\ldots.$$

Then every four times punctured sphere is obtained by the plumbing construction using coordinates $\{(V^k, W^k)\}_{k=1}^2$.

3. A Solution to the problem in case of four times punctured spheres

In this section, we solve our problem in case of four times punctured spheres.

3.1.

THEOREM 1. *Let* $R = \hat{C} \setminus \{P_1, \dots, P_4\}$. *Let* γ *be a simple close curve on* R *which separates* $\{P_1, P_2\}$ *and* $\{P_3, P_4\}$. *Assume that non-negative numbers* $l_1, \dots, l_4$ *and a positive number* L *satisfy by condition*

$$\mathscr{M}(l_1/L, l_2/L, l_3/L, l_4/L) < 2\pi \operatorname{mod}_R(\gamma).$$

Then there exists a unique quadratic differential φ *with closed trajectories on* R *with following conditions*:

(a) φ *has a closed trajectory homotopic to* γ *and the* φ*-length of this trajectory is equal to* L.

(b) *For the local uniformizing parameter* w *at* P_j *such that* $w(P_j) = 0$, *the respective development of* $\varphi = \varphi(w)\,dw^2$ *is*

$$\varphi(w)\,dw^2 = \left(-\frac{l_j^2}{4\pi^2} \frac{1}{w^2} + \cdots \right) dw^2.$$

Here, the function $\mathscr{M}$ *is defined by* $\mathscr{M}(x, y, z, w) = m(D(x, y) \cdot D(z, w))$, *where*

$$m(D) = \log(2 - D + 2\sqrt{1 - D}) - \log D,$$

and

$$D(x, y) = \frac{|x - y - 1|^{(x-y-1)/2} |x + y - 1|^{(x+y-1)/2}}{|x - y + 1|^{(x-y+1)/2} |x + y + 1|^{(1+x+y)/2}}$$

with $0^0 = 1$.

This theorem will be proved in Section 3.4. The uniqueness of such a differential follows from Theorem 23.1 in [18, p. 143]. Hence we prove only the existence of the differential. Further, if φ is the quadratic differential on R which satisfies the conditions in Theorem 1 for constants $l_1/L, \dots, l_4/L$, and 1, $L^2\varphi$ is the quadratic differential on R satisfying the conditions in Theorem 1 for constants $l_1, \dots, l_4$, and L. Therefore, we may assume that $L = 1$.

3.2. We define a quadratic differential φ^k on Σ^k by

$$\varphi^k = \varphi^k(z)\,dz^2 = -\frac{1}{4\pi^2} \frac{z^2 - 2(l_{2k}^2 - l_{2k-1}^2)z + (2l_{2k-1}^2 + 2l_{2k}^2 - 1)}{(z + 1)^2 (z - 1)^2} dz^2. \tag{10}$$

φ^k has closed trajectories homotopic to a loop around ∞ with φ^k-length one. Moreover, if l_m ($m = 2k-1$ or $2k$) is positive, then φ^k has closed trajectories homotopic to small loops around either 1 if m is odd, or -1 if m is even.

Let V^k be the characteristic ring domain of φ^k with respect to ∞ and W^k a conformal mapping from $V^k \cup \{\infty\}$ to Δ such that $W^k(\infty) = 0$. We take notations made in Sections 2.2 and 2.3 for granted. For $\ell = (l_1, l_2, l_3, l_4) \in \boldsymbol{R}^4_{\geq 0}$, we denote by Φ_ℓ the representation of plumbing constructions using $\{(V^k, W^k)\}_{k=1,2}$. By definition, we obtain the following.

Lemma 4. *There exists a quadratic differential φ_t on M_t with conditions in Theorem* 1 *for a curve γ_t, punctures $\{P_t^m\}_{m=1,\dots,4}$, constants ℓ and $L = 1$.*

3.3. The following is a key step to proving Theorem 1.

Lemma 5. *Φ_ℓ is injective.*

Proof. For $t_1, t_2 \in \Delta$, we assume that $\Phi_\ell(t_1) = \Phi_\ell(t_2)$. We may also assume that $t_i \neq 0$ for $i = 1, 2$. Then, by definition, there exists the biholomorphic mapping h from M_{t_1} to M_{t_2} such that $h(P_{t_1}^m) = P_{t_2}^m$ for $m = 1, \dots, 4$ and that $h(\gamma_{t_1})$ is homotopic to γ_{t_2}. Let $\eta_2 = h^*(\varphi_{t_2})$. By Theorem 23.1 in [18, p. 143], η_2 coincides with φ_{t_1}. This implies that $h(V_{t_1}) = V_{t_2}$. In the sequel, h leads the biholomorphic mapping h^k from $M_{t_1}^k$ to $M_{t_2}^k$ such that $h^k(V_{t_1}^k) = V_{t_2}^k$ for $k = 1, 2$. Since $h^*(\varphi_{t_2}) = \varphi_{t_1}$, $(h^k)^*(\varphi^k) = \varphi^k$. Therefore, $W^k \circ h^k = e^{i\theta} W^k$ on $V_{t_1}^k$ for some $\theta \in \boldsymbol{R}$. Hence, h^k can be extended to an automorphism of Σ^k such that $h^k(1) = 1$, $h^k(-1) = -1$ and $h^k(\infty) = \infty$ by setting $(W^k)^{-1}(e^{i\theta} W^k(x))$ for $x \in V^k$. Thus h^k is the identity mapping of Σ^k. Therefore for $x^k \in V_{t_1}^k$ such that $W^1(x^1) W^2(x^2) = t_1$,

$$t_2 = W^1(h^1(x^1)) W^2(h^2(x^2)) = W^1(x^1) W^2(x^2) = t_1. \qquad \square$$

Lemma 6. *Suppose that W^k has the expansion as in Proposition* 2. *Then*

$$|A^k| = 1/2D(l_{2k-1}, l_{2k}),$$

for $k = 1, 2$, where $D(-,-)$ is defined as in Theorem 1.

Proof. Fix $k = 1, 2$. We put $a = -2(l_{2k}^2 - l_{2k-1}^2)$, $b = 2l_{2k-1}^2 + 2l_{2k}^2 - 1$, and $z_0 := (-a + \sqrt{a^2 - 4b})/2$ where the branch of the square root is taken as $\sqrt{1} = 1$. Since $W^k(\infty) = 0$, we may assume that

$$W^k(z) = \exp\left\{ -\int_{z_0}^{z} \frac{\sqrt{x^2 + ax + b}}{x^2 - 1}\, dx \right\}.$$

Here, we only prove the case where $l_{2k} + l_{2k-1} \neq 1$ and $|l_{2k} - l_{2k-1}| \neq 1$. Another cases are proved by the similar arguments.

A direct calculation gives that for $z \in V^k$,

$$
\begin{aligned}
|z|\,|W^k(z)| &= |a^2-4b|^{1/2}\left|\frac{z}{a+2z+2\sqrt{z^2+az+b}}\right| \\
&\quad\times\left|\frac{a+2b+(2+a)z+2\sqrt{1+a+b}\sqrt{z^2+az+b}}{\sqrt{a^2-4b}(z-1)}\right|^{\sqrt{1+a+b}/2} \\
&\quad\times\left|\frac{\sqrt{a^2-4b}(z+1)}{a-2b+(2-a)z-2\sqrt{1-a+b}\sqrt{z^2+az+b}}\right|^{\sqrt{1-a+b}/2} \\
&= |a^2-4b|^{1/2}I_1(z)\times I_2(z)\times I_3(z).
\end{aligned}
$$

Since $\lim_{|z|\to\infty} I_1(z) = 1/4$,

$$
|a^2-4b|^{1/2} = |1+l_{2k}^4+l_{2k-1}^4-2(l_{2k}^2+l_{2k-1}^2+l_{2k}^2l_{2k-1}^2)|^{1/2},
$$

$$
\begin{aligned}
\lim_{|z|\to\infty} I_2(z) &= |(2+a+2\sqrt{1+a+b})/(\sqrt{a^2-4b})|^{\sqrt{1+a+b}/2} \\
&= |(\sqrt{a^2-4b})/(2+a-2\sqrt{1+a+b})|^{\sqrt{1+a+b}/2} \\
&= \left|\frac{(1+l_{2k}^4+l_{2k-1}^4-2(l_{2k}^2+l_{2k-1}^2+l_{2k}^2l_{2k-1}^2))^{1/2}}{1+l_{2k}^2-l_{2k-1}^2-2l_{2k}}\right|^{l_{2k}}, \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{|z|\to\infty} I_3(z) &= |(\sqrt{a^2-4b})/(2-a-2\sqrt{1-a+b})|^{\sqrt{1-a+b}/2} \\
&= \left|\frac{(1+l_{2k}^4+l_{2k-1}^4-2(l_{2k}^2+l_{2k-1}^2+l_{2k}^2l_{2k-1}^2))^{1/2}}{1-l_{2k}^2+l_{2k-1}^2-2l_{2k-1}}\right|^{l_{2k-1}},
\end{aligned}
$$

we have that

$$
\begin{aligned}
|A_k| &= \lim_{|z|\to\infty}|z|\,|W^k(z)| \\
&= \frac{|1+l_{2k}^4+l_{2k-1}^4-2(l_{2k}^2+l_{2k-1}^2+l_{2k}^2l_{2k-1}^2)|^{(1+l_{2k}+l_{2k-1})/2}}{2|1+l_{2k}^2-l_{2k-1}^2-2l_{2k}|^{l_{2k}}|1-l_{2k}^2+l_{2k-1}^2-2l_{2k-1}|^{l_{2k-1}}}.
\end{aligned}
$$

Since for $x, y \in \boldsymbol{R}$, $1+x^4+y^4-2(x^2+y^2+x^2y^2) = (x-y-1)(x+y-1)(x-y+1)(x+y+1)$ and $1+x^2-y^2-2x = (x-y-1)(x+y-1)$, we have $|A_k| = 1/2D(l_{2k}, l_{2k-1})$. □

3.4. Let us prove Theorem 1. Fix non-negative numbers l_1, l_2, l_3, and l_4. We put $\ell = (l_1, \ldots, l_4)$. Let $T = \Psi^{-1}\circ\Phi_\ell$. Then $T(0) = 0$ and $|T(t)| < 1$ on Δ. By Proposition 2 and Lemma 6, we have

$$
\left|\frac{dT}{dt}(0)\right| = D(l_1, l_2)D(l_3, l_4).
$$

Therefore, by Lemma 5 and Theorem 1 in [6], the image of Δ under T contains a disc

$$B := \{w \in \boldsymbol{C} \mid |w| < \exp(-\mathscr{M}(\ell))\}. \tag{11}$$

Let $R = \hat{\boldsymbol{C}} \setminus \{P_1, \dots, P_4\}$ and γ a simple closed curve which separates $\{P_1, P_2\}$ and $\{P_3, P_4\}$. We denote by f a deformation from R to S_0 such that $f(\gamma) = N_0$ and that $f(P_m) = P_0^m$ for $m = 1, \dots, 4$. We assume that $\mathscr{M}(\ell) < 2\pi \operatorname{mod}_R(\gamma)$. Then, by (4) and (11), we have $\Psi^{-1}([f]) \in B$. Thus, $[f]$ is contained in the image under Φ_ℓ. Namely, there exists $t \in \Delta$ such that $[f] = [g_t]$. Therefore there exists an isomorphism h from M_t to R such that $h(\gamma_t)$ is homotopic to γ and that $h(Q_t^m) = P_m$ for $m = 1, \dots, 4$.

Finally, let $\varphi = h^*(\varphi_t)$. Then, by Lemma 4, φ is the quadratic differential on R with conditions in Theorem 1 for γ, ℓ, and $L = 1$. We have thus proved the theorem. $\square$

4. The deformation space of Riemann surfaces of type $(1, 1)$ with a node

4.1. As in Section 2.1, we construct a global coordinate of the deformation space of Riemann surfaces of type $(1, 1)$ with a node.

We first construct a Riemann surface S_0 of type $(1, 1)$ with a node. Let $\Sigma = \boldsymbol{C} \setminus \{-1, 1\}$. Then S_0 is obtained by identifying -1 and 1 in Σ. We denote by N_0 the node of S_0.

Next, for $t \in \Delta^*$, we construct a Riemann surface S_t of type $(1, 1)$ by a plumbing procedure. Let U^1 (resp. U^2) denote the right (resp. left) half-plane in Σ. Let $Z^1(z) = (z-1)/(z+1)$ and $Z^2(z) = (1+z)/(1-z)$. For $0 < |t| < 1$ and $k = 1, 2$, let $U_t^k = \{P \in U^k \mid |Z^k(P)| > |t|\}$. Identifying U_t^1 and U_t^2 by the mapping $Z^1 Z^2 = t$, we obtain a Riemann surface S_t of type $(1, 1)$. This identification also gives a ring domain U_t in S_t. We can observe that U_t is the characteristic ring domain of the J-S differential with respect to the central curve γ_t of U_t. Let f_t denote a deformation from S_t to S_0 such that $f_t(\gamma_t) = N_0$. We define a holomorphic mapping Ψ from Δ to $D(S_0)$ by

$$\Psi(t) = [f_t].$$

Then, as in Section 2.1, Ψ becomes a global coordinate of $D(S_0)$.

4.2. We denote by J the involution of Σ defined by $J(z) = -z$. Let V^1 (resp V^2) be an open neighborhood of 1 (resp. -1) in $\boldsymbol{C}$ such that $J(V^1) = V^2$ and that $V^1 \cap V^2 = \emptyset$. Let W^1 be a conformal mapping from V^1 to Δ such that $W^1(1) = 0$. We set $W^2 = W^1 \circ J$ on V^2. For $0 < |t| < 1$, we denote by M_t a Riemann surface obtained by a plumbing procedure by using coordinates $\{(V^k, W^k)\}_{k=1,2}$. We define a deformation g_t from M_t to S_0 as in the previous section. By the representation Φ of the plumbing constructions using $\{(V^k, W^k)\}_{k=1,2}$ we mean a holomorphic mapping

$$\Phi(t) = [g_t].$$

4.3. As in Section 2.4, we compute the derivative $T := \Psi^{-1} \circ \Phi$ at the origin.

PROPOSITION 7. *Let $\{(V^k, W^k)\}_{k=1,2}$ be as in Section 4.2. Suppose that for $k = 1,2$, the Laurent development of W^k near the puncture has the form*

$$W^1(z) = A(z-1) + \cdots \quad \text{and} \quad W^2(z) = -A(z+1) + \cdots,$$

where z is the global coordinate of Σ. Then

$$\frac{dT}{dt}(0) = \frac{1}{4A^2}.$$

The proof is completely analogous to that of Proposition 2. Thus, to obtain Proposition 7, it suffice to show the following lemma.

LEMMA 8. *Let A_t be the characteristic ring domain of the J-S differential on M_t with respect to γ_t. Then there exists $\delta_0 > 0$ such that for $|t| < \delta_0$, it holds that $\gamma_t \subset A_t$.*

Proof. For $k = 1,2$, let $V_t^k = \{P \in V^k \,|\, |W^k(P)| > |t|\}$ and $M_t' = \Sigma \setminus \bigcup_{k=1}^2 \{P \in V^k \,|\, |W^k(P)| \leq |t|\}$. We denote by j_t the canonical surjection from M_t' to M_t. Let V_t be an image of V_t^k under j_t. We define a conformal mapping W_t from V_t to $A_{|t|}$ by $W_t(j_t(P)) = W_t^1(P)$ for $P \in U_t^1$. A central curve γ_t of V_t is a closed curve in M_t defined by $\{P \in V_t \,|\, |W_t(P)| = |t|^{1/2}\}$.

By the definition of M_t, there exists an involution J_t of M_t such that $j_t \circ J = J_t \circ j_t$. Moreover, J_t is satisfies that $J_t(\gamma_t) = \gamma_t$. Hence it is easy to see that there exist two fixed points of J_t on γ_t.

Let η_t denote the J-S differential on M_t with respect to γ_t. We denote by σ_t a central trajectory in A_t. Then, as in case γ_t, we have $J_t(\sigma_t) = \sigma_t$, and hence there exist two fixed points of J_t on σ_t. Here, we can check that the cardinality of the set of fixed points of J_t is equal to three. Therefore σ_t intersects γ_t.

Finally, we can prove Lemma 8 by an argument similar to that of the proof of (iii) of Lemma 1. □

4.4. As in Section 2.5, we obtain the following corollary.

COROLLARY 3. *Let $\{(V^k, W^k)\}_{k=1,2}$ be as in Proposition 7. Let R be a Riemann surface of type $(1,1)$. Suppose that there exists an essential curve γ on R such that*

$$\sinh(2\pi \operatorname{mod}_R(\gamma))/2\pi \operatorname{mod}_R(\gamma) > 4|A|^2.$$

Then R is obtained by the plumbing construction using $\{(V^k, W^k)\}_{k=1,2}$.

Since every Riemann surface of type $(1,1)$ has an essential curve whose modulus is more than or equal to $\sqrt{3}/2$ (see (i) of Lemma 12), we have

COROLLARY 4. *Let $\{(V^k, W^k)\}_{k=1,2}$ be as in Proposition 7. Assume that*

$$|A| \leq \{\sinh(\sqrt{3}\pi)/4\sqrt{3}\pi\}^{1/2} = 2.302\ldots,$$

then every once punctured torus is obtained by the plumbing construction using $\{(V^k, W^k)\}_{k=1,2}$.

5. A Solution to the problem in case of once punctured tori

In this section, we give an answer of our problem in case of once punctured tori. Almost all results in this section are obtained by arguments similar to those of Section 3. Hence we omit proofs for several results.

5.1.

THEOREM 2. *Let R be a once punctured torus. Let γ be an essential curve on R. Assume that a non-negative number l and a positive number L has the condition that*

$$\mathcal{N}(l/L) < 2\pi \operatorname{mod}_R(\gamma).$$

Then there exists a unique quadratic differential φ with closed trajectories on R such that

(a) *φ has the closed trajectory homotopic to γ and the φ-length of this trajectory is equal to L.*

(b) *For a coordinate w at the puncture, the respective development of $\varphi = \varphi(w)\,dw^2$ is*

$$\varphi(w)\,dw^2 = \left(-\frac{l^2}{4\pi^2}\frac{1}{w^2} + \cdots\right) dw^2,$$

where $\mathcal{N}(x) := m(16D(1,x)^2)$ and the functions $D(-,-)$ and $m(-)$ are given in Theorem 1.

As in the case of type $(0,4)$, we may assume that $L = 1$.

5.2. We define the quadratic differential φ on Σ by

$$\varphi = \varphi(z)\,dz^2 := -\frac{1}{4\pi^2}\frac{l^2z^2 + (4 - l^2)}{(z-1)^2(z+1)^2}\,dz^2.$$

φ is a differential with closed trajectories and has the closed trajectories around either -1 or 1 with φ-length one. Moreover, if l is positive, then φ has the closed trajectory around ∞ with the φ-length l.

Let V^1 and V^2 be the characteristic ring domains of φ with respect to -1 and 1 respectively and W^k be a conformal mapping from V^k to Δ^* such that $W^2 \circ J = W^1$ and that W^k maps a puncture to the origin. We define M_t, g_t and γ_t as the previous section. The representation of the plumbing construction using $\{(V^k, W^k)\}_{k=1,2}$ is denoted by Φ.

By definition, we have the following.

LEMMA 9. *There exists a quadratic differential φ_t on M_t with be conditions in Theorem 2 for γ_t, l, and $L = 1$.*

By the same argument as that in the proof of Lemma 5, we have the following lemma.

LEMMA 10. *Φ is injective.*

Here, we show the following.

LEMMA 11. *Suppose that W^1 and W^2 have the expansion as in Proposition 7. Then*

$$|A| = 1/8D(1, l),$$

where $D(-,-)$ is defined as in Theorem 1.

Proof. We only show the case $k = 1$. The case where $k = 2$ is obtained by the similar way.

Let $r(z) = 1 + 4/(z-1)$. Then r is an automorphism of Σ such that $r(1) = \infty$, $r(\infty) = 1$, and $r(-1) = -1$. By assumpution, we have $W^1 \circ r(z) = 4A/z + \cdots$ for $|z| > M$, where $M > 0$ is taken to be sufficiently large.

We put $\psi := r^*\varphi$ and $V^* := r^{-1}(V^1)$. By the definition of φ, ψ has the closed trajectories around either -1 or ∞ whose ψ-length is equal to one. Moreover, if $l > 0$, then ψ has the closed trajectories around 1 whose ψ-lengths are equal to l. Since V^* is the characteristic ring domain of ψ with respect to ∞ and $W^1 \circ r$ is a conformal mapping from V^* to Δ such that $W^1 \circ r(\infty) = 0$, by Lemma 6, we have $4|A| = 1/2D(1, l)$. Hence we conclude that $|A| = 1/8D(1, l)$. □

Then we can prove Theorem 2 by an argument similar to that of Section 3.4.

6. Some results related to the problem

6.1. The aim of this Section is to prove the following theorem which will be proved in Section 6.3.

THEOREM 3. *There exists the best possible constants $L_{0,4}$, $L_{1,1} > 0$ such that for $l < L_{0,4}$ (resp. $l < L_{1,1}$) and a Riemann surface R of type $(0,4)$ (resp. of type $(1,1)$), there exists a simple closed curve γ on R such that R has the quadratic differential with the conditions in Theorem 1 for $\gamma, l_i = l$ for $i = 1, \dots, 4$ and $L = 1$. (resp. in Theorem 2 for γ, l, and $L = 1$). Furthermore, it holds that $0.506 < L_{0,4} < 0.835$ and that $2.379 < L_{1,1} < 3.338$.*

6.2. To prove Theorem 3, we show several lemmas.

LEMMA 12. (i) *Every Riemann surface of type $(0,4)$ (resp. of type $(1,1)$) has an essential curve such that its modulus is more than or equal to $\sqrt{3}/4$ (resp. $\sqrt{3}/2$).*

(ii) *There exists a Riemann surface S of type $(0,4)$ (resp. of type $(1,1)$) such that every essential curve σ satisfies $\mathrm{mod}_S(\sigma) \leq \sqrt{3}/4$ (resp. $\sqrt{3}/2$).*

Proof. We only show the case of Riemann surfaces of type $(0,4)$. The case of Riemann surfaces of type $(1,1)$ is proved in a similar fashion.

(i) Let $R = \hat{\boldsymbol{C}} \setminus \{\pm e^{\pi i/4}, \pm e^{-\pi i/4}\}$. We denote by $T(R)$ the Teichmüller space of R (cf. [7, Chapter 5] and [14, Chapter two]). Let γ be a simple closed curve $\boldsymbol{R} \cup \{\infty\}$ on R. Then we have $\mathrm{mod}_R(\gamma) = 1/2$ (see Section 6.4, Example).

Let φ be the J-S differential on R with respect to γ. For $t \in \mathscr{H} := \{t \in \boldsymbol{C} \mid \mathrm{Im}\, t > 0\}$, we denote by f_t the quasiconformal mapping from R to a Riemann surface R_t of type $(0,4)$ which has the Beltrami coefficient

$$\mu_t := \frac{t-i}{t+i}\frac{|\varphi|}{\varphi}.$$

It is known that $\chi : \mathscr{H} \ni t \to [R_t, f_t] \in T(R)$ is biholomorphic (cf. [14, Section 2.6.5]). A direct computation gives (cf. [13, Section 1.3])

$$\mathrm{mod}_{R_t}(f_t(\gamma)) = \mathrm{Im}\, t \cdot \mathrm{mod}_R(\gamma) = \mathrm{Im}\, t/2. \tag{12}$$

We denote by $\mathrm{Mod}(R)$ the Teichmüller modular group of R (cf. [14, Section 2.3]). It is easy to observe that (see [5, p. 165]).

$$\chi^{-1}\mathrm{Mod}(R)\chi = PSL_2(\boldsymbol{Z}).$$

This implies that for any Riemann surface S of type $(0,4)$, there exists $t \in \{t \in \mathscr{H} \mid \mathrm{Im}\, t \geq \sqrt{3}/2\}$ such that S is biholomorphic to R_t. By (12), $f_t(\gamma)$ is a simple closed curve on R_t whose modulus is more than or equal to $\sqrt{3}/4$. We have thus proved (i).

(ii) Let $S = R_{1/2+\sqrt{3}i/2}$ and let σ be an essential curve on S. Then we can check that there exists a quasiconformal mapping g from $R = R_i$ to S such that $g(\gamma)$ is freely homotopic to σ.

We take $t \in \mathscr{H}$ such that $[R_t, f_t] = [S, g]$. Then there exists $\tau \in \mathrm{mod}(R)$ such that

$$\tau([S, f_{1/2+\sqrt{3}i/2}]) = [R_t, f_t] = [S, g].$$

By (i), there exists $A \in PSL_2(\boldsymbol{Z})$ such that $t = A(1/2 + \sqrt{3}i/2)$. This implies

$$\mathrm{Im}\, t = \mathrm{Im} A(1/2 + \sqrt{3}i/2) \leq \sqrt{3}/2. \tag{13}$$

By (12) and (13), we have

$$\mathrm{mod}_S(\sigma) = \mathrm{mod}_S(g(\gamma)) = \mathrm{mod}_{R_t}(f_t(\gamma)) = \mathrm{Im}\, t/2 \leq \sqrt{3}/4. \qquad \square$$

LEMMA 13. *Let $L > 0$ and $l > L/2$. Let $R = \hat{\boldsymbol{C}} \setminus \{P_1, \ldots, P_4\}$ and γ a simple closed curve which separates $\{P_1, P_2\}$ and $\{P_3, P_4\}$. Suppose that there exists the quadratic differential φ on R with the conditions in Theorem* 1 *for γ, $l_i = l$ for $i = 1, \ldots, 4$ and L. Then*

$$\mathrm{mod}_R(\gamma) > K(\cos(\pi L/4l))/2K(\sin(\pi L/4l)),$$

where $K(k)$ is the complete elliptic integral of the first kind for a modulus k.

Proof. We may assume that $L = 1$. We first recall the geometry of certain quadratic differentials on the three punctured sphere $\Sigma = \boldsymbol{C}\backslash\{-1, 1\}$. Fix $l > 1/2$. We define a quadratic differential ψ on Σ by

$$\psi = \psi(z)\,dz^2 = -\frac{1}{4\pi^2}\frac{z^2 + (4l^2 - 1)}{(z-1)^2(z+1)^2}\,dz^2.$$

Then ψ is the quadratic differential with closed trajectories on Σ that has the closed trajectories of length one around ∞ and of length l around either -1 or 1. Let A be the characteristic ring domain of ψ with respect to ∞. Then it is clear that the interior D_l of $\Sigma\backslash A \cup \{t \in \boldsymbol{R} \mid -1 < t < 1\}$ is a doubly connected domain. Here, we assume the following equation which is proved later.

$$\text{(14)} \qquad \operatorname{mod}(D_l) = K(\cos(\pi/4l))/4K(\sin(\pi/4l)).$$

Take R, γ and φ as in the assertion of this lemma. Let $\mathscr{A}$ be the characteristic ring domain of φ with respect to γ. By the structure of trajectories of φ, there exist injective holomorphic mappings h_1 and h_2 from D_l to R such that $h_1(D_l) \cap h_2(D_l) = \emptyset$, $h_k(D_l) \cap \mathscr{A} = \emptyset$ for $k = 1, 2$, and that the core curve of $h_k(D_l)$ is freely homotopic to γ. By (2) in [17, Proposition 1.5], we have

$$\begin{aligned}\operatorname{mod}_R(\gamma) &\geq \operatorname{mod}(h_1(D_l)) + \operatorname{mod}(h_2(D_l)) + \operatorname{mod}(\mathscr{A})\\ &> K(\cos(\pi/4l))/2K(\sin(\pi/4l)).\end{aligned}$$

□

To prove Lemma 13, it remains to show the following lemma.

LEMMA 14. *The equation* (14) *holds.*

Proof. We use the notation defined in Lemma 13 frequently. Let D_1 and D_{-1} be characteristic ring domains of ψ with respect to 1 and -1 respectively. We denote by α ($\operatorname{Im}\alpha > 0$) a zero of ψ. We note that $\bar{\alpha}$ is also that of ψ. Since the interval $\{t \in \boldsymbol{R} \mid |t| < 1\}$ is a (singular) vertical trajectory of ψ, there exists a conformal mapping f_1 (resp. f_{-1}) from D_1 (resp. D_{-1}) to Δ such that $f_1(1) = 0$ (resp. $f_{-1}(-1) = 0$) and that

$$f_1(D_1\backslash\{t \in \boldsymbol{R} \mid -1 < t < 1\}) = E_1' := \Delta\backslash\{s \in \boldsymbol{R} \mid -1 < s < 0\}$$

$$(\text{resp. } f_{-1}(D_{-1}\backslash\{t \in \boldsymbol{R} \mid -1 < t < 1\}) = E_{-1}' := \Delta\backslash\{s \in \boldsymbol{R} \mid 0 < s < 1\}).$$

We note that the domains of f_1 and f_{-1}, are extended to the closure of D_1 and that of D_{-1} respectively and that

$$\text{(15)} \qquad \begin{aligned} &f_1(\alpha) = -e^{\pi i(1-2l)/2l}, \quad f_1(\bar{\alpha}) = -e^{\pi i(2l-1)/2l},\\ &f_{-1}(\alpha) = e^{\pi i(2l-1)/2l}, \quad \text{and} \quad f_1(\bar{\alpha}) = e^{\pi i(1-2l)/2l}.\end{aligned}$$

Indeed, the extendabilities of f_1 and f_{-1} are trivial, since D_1 and D_{-1} have piecewise real analytic boundaries.

Here, we will only show that $f_1(\alpha) = -e^{\pi i(1-2l)/2l}$. The another equations in (15) are obtained by the similar arguments. By the definition of D_1, we have

$$f_1^*\left(-\frac{l^2}{4\pi^2}\frac{dz^2}{z^2}\right) = \psi. \tag{16}$$

Let $\beta := \{t\alpha \mid 0 < t < 1\}$. Then β is a horizontal trajectory segement of ψ. Moreover the ψ-length of β is equal to $(2l-1)/4$. By (16) and $f_1(0) = -1$, we have that $f_1(\beta') = \{e^{i\theta} \mid \pi/2l < \theta < \pi\}$. Since $f_1(\alpha)$ and $f_1(0)$ are end-points of $f_1(\beta')$, we have $f_1(\alpha) = e^{\pi i/2l} = -e^{\pi i(1-2l)/2l}$.

Let $E_1 = E_1' \cup \{e^{i\theta} \mid 0 < |\theta - \pi| < \pi(2l-1)/2l\}$ and $E_{-1} = E_{-1}' \cup \{e^{i\theta} \mid 0 < |\theta| < \pi(2l-1)/2l\}$. We construct an annulus E_l by introducing an equivalence relation on the disjoint union $E_1 \cup E_{-1}$. A point $P \in E_1$ is identified with a point Q on E_{-1} if and only if $|P| = |Q| = 1$ and $Q = -\bar{P}$. Then E_l is canonically biholomorphic to D_l.

Finally, since E_l is biholomorphic to the ring domain

$$\boldsymbol{C}\backslash\{t \in \boldsymbol{R} \mid |t| \le 1 \text{ or } |t| \ge 1/\sin(\pi/4l)\},$$

by the mappings

$$E_1 \ni z \mapsto 2i\sqrt{z}/(1-z), \quad \text{and}$$

$$E_{-1} \ni z \mapsto -2i\sqrt{-z}/(1+z),$$

where the branch of square root is taken as $\sqrt{1} = 1$, we conclude the assertion of this lemma. □

Lemma 15. *Let $L > 0$ and $l > 2L$. Let R be a once punctured torus and γ an essential curve on R. Suppose that there exists the quadratic differential on R with the conditions in Theorem 2 for γ, l and L. Then*

$$\mathrm{mod}_R(\gamma) > K(\cos(\pi L/l))/K(\sin(\pi L/l)).$$

Proof. We may assume that $L = 1$. As in the proof of Lemma 13, for $l > 2$, we define a quadratic differential ψ on Σ by

$$\psi = \psi(z)\,dz^2 = -\frac{1}{4\pi^2}\frac{l^2z^2 + (4-l^2)}{(z-1)^2(z+1)^2}\,dz^2.$$

Then ψ is a differential with the closed trajectories. Moreover ψ has the closed trajectories around either -1 or 1 with ψ-length one and around ∞ with the ψ-length l. Let $\mathscr{A}_{-1}$ and $\mathscr{A}_1$ denote characteristic ring domains of ψ with respect to -1 and 1 respectively. Let I be the imaginary axis on Σ. Notice that I is a vertical trajectory of ψ, since we now assume $l > 2$ and hence the zeros of ψ, $\pm\sqrt{l^2-4}/l$, is real and not equal to the origin. Since $J(\mathscr{A}_1) = \mathscr{A}_{-1}$ where $J(z) = -z$, each connected components $\mathscr{A}, \mathscr{A}'$ of $\Sigma\backslash\mathscr{A}_{-1} \cup \mathscr{A}_1 \cup I$ are doubly connected domains. Further, it holds that

$$\mathrm{mod}(\mathscr{A}) = \mathrm{mod}(\mathscr{A}') = K(\cos(\pi/l))/2K(\sin(\pi/l)). \tag{17}$$

We assume the equation (17) which will be proved later. Then, by an argument similar to that of the proof of Lemma 13, we can prove Lemma 15. □

To prove Lemma 15, we should show the following.

LEMMA 16. *The equation* (17) *holds.*

Proof. We may assume that $\mathscr{A}$ is the connected component of $\Sigma\backslash\mathscr{A}_{-1}\cup\mathscr{A}_1\cup I$ such that $\mathscr{A}\subset\{z\in\boldsymbol{C}\,|\,\mathrm{Re}\,z>0\}$. Since J is conformal and $J(\mathscr{A})=\mathscr{A}'$, it suffices to show that

$$\mathrm{mod}(\mathscr{A})=K(\cos(\pi/l))/2K(\sin(\pi/l)). \tag{18}$$

Let V be the characteristic ring domain of ψ with respect to ∞ and W the conformal mapping from V to Δ such that $W(\infty)=0$ and $W(I\backslash\{0\})=\{t\in\boldsymbol{R}\,|\,0<|t|<1\}$. We may assume that $W(\mathscr{A}\backslash\beta)\subset\{z\in\boldsymbol{C}\,|\,\mathrm{Im}z>0\}$, where $\beta:=\{t\in\boldsymbol{R}\,|\,0<t<\sqrt{l^2-4}/l\}$ ($\sqrt{l^2-4}/l$ is one of the zeros of ψ). We note that $\mathscr{A}$ has the piecewise real analytic boundaries and the ψ-length of β is equal to $(l-2)/4$. Hence, by an argument similar to that of the proof of Lemma 14, we obtain that the images by W of prime ends whose impressions are in β (cf. [16, p. 27]) are just the prime ends whose impressions are in $\{e^{i\theta}\,|\,0<\theta<\pi(l-2)/2l\}$ and $\{e^{i}\theta\,|\,\pi(l+2)/2l<\theta<\pi\}$. It is easy to see that if ξ_1 and ξ_2 are the impressions of the images of prime ends in $\mathscr{A}\backslash\beta$ whose impressions are $t\in\beta$ then $\xi_2=-\overline{\xi_1}$.

We now consider the mapping $\mathscr{F}$ on $\Delta\cap\{\mathrm{Im}z>0\}$ as follows:

$$\mathscr{F}(z):=\int_0^{2z/(1+z^2)}\frac{dz}{\sqrt{(1-z^2)(1-\sin^2(\pi/l)z^2)}},$$

where the branch of square root is taken as $\sqrt{1}=1$. Then $\mathscr{F}(1)=-\mathscr{F}(-1)=K(\sin(\pi/l))$, $\mathscr{F}(e^{i\pi(l-2)/2l})=-\overline{\mathscr{F}(e^{i\pi(l+2)/2l})}=K(\sin(\pi/l))+iK(\cos(\pi/l))$, and the image of $\Delta\cap\{z\in\boldsymbol{C}\,|\,\mathrm{Im}\,z>0\}$ by $\mathscr{F}$ is equal to the rectangle

$$\{(x,y)\in\boldsymbol{R}^2\,|\,|x|<K(\sin(\pi/l)),0<y<K(\cos(\pi/l))\}$$

(cf. [15, p. 280]). Hence $\mathscr{F}\circ W|_{\mathscr{A}\backslash\beta}$ maps conformally $\mathscr{A}\backslash\beta$ onto the rectangle above. Therefore, the mapping

$$\mathscr{A}\ni z\mapsto w=\exp\left\{\frac{\pi i}{K(\sin(\pi/l))}\mathscr{F}\circ W|_{\mathscr{A}}(z)\right\}$$

is well-defined and a conformal mapping from $\mathscr{A}$ to the ring domain

$$\left\{\exp\left\{-\pi\frac{K(\cos(\pi/l))}{K(\sin(\pi/l))}\right\}<|w|<1\right\}.$$

Thus, we conclude (18). □

6.3. Let us prove Theorem 3. We only observe the case $L_{0,4}$, since we obtain the case $L_{1,1}$ in the same way.

We can check that the function $\mathcal{M}(l,l,l,l)$ of l is monotone increasing on $\{l \in \boldsymbol{R} \mid l \geq 0\}$. A simple observation shows that

$$\mathcal{M}(l,l,l,l) < \sqrt{3}\pi/2 \text{ if } l \leq 0.506 \quad \text{and} \quad \mathcal{M}(l,l,l,l) > \sqrt{3}\pi/2 \text{ if } l \geq 0.507.$$

By Theorem 1 and (i) of Lemma 12, the constant $L_{0,4}$ exists and satisfies that $0.506 < L_{0,4}$. Since

$$K(\cos(\pi/4l))/2K(\sin(\pi/4l)) \begin{cases} < \sqrt{3}/4 & \text{if } l < 0.834 \quad \text{and,} \\ > \sqrt{3}/4 & \text{if } l \geq 0.835. \end{cases}$$

Therefore, by (ii) of Lemma 12 and Lemma 13, we obtain that $L_{0,4} < 0.835$. $\square$

6.4.

Remark. If $l = L > 0$, then the inequality in Lemma 13 gives a sharp condition. Indeed, we have the following example.

Example. For $0 < \theta < \pi/2$, let $R_\theta = \hat{\boldsymbol{C}} \setminus \{\pm e^{i\theta}, \pm e^{-i\theta}\}$. Let $\gamma = \boldsymbol{R} \cup \{\infty\}$. Then γ is a simple closed curve in R_θ that separates $\{e^{i\theta}, -e^{-i\theta}\}$ and $\{-e^{i\theta}, e^{-i\theta}\}$ and that satisfies

$$\operatorname{mod}_{R_\theta}(\gamma) = K(\cos\theta)/2K(\sin\theta).$$

These are known as the Teichmüller's module theorem (cf. [12, Chapter II, 1.2.]). Notice that $\operatorname{mod}_{R_\theta}(\gamma)$ is a strictly monotone increasing function on $0 < \theta < \pi/2$ and that $\operatorname{mod}_{R_{\pi/4}}(\gamma) = 1/2$.

Fix $\theta > \pi/4$. Then we can find a unique positive constant a_θ such that

$$\int_{-\infty}^{\infty} \frac{\sqrt{x^2 \sin^2 2\theta + a_\theta(x^4 - 2x^2\cos 2\theta + 1)}}{x^4 - 2x^2 \cos 2\theta + 1}\, dx = \frac{\pi}{2}.$$

Indeed, for $\pi/4 < \theta < \pi/2$, the function

$$a \mapsto \int_{-\infty}^{\infty} \frac{\sqrt{x^2 \sin^2 2\theta + a(x^4 - 2x^2\cos 2\theta + 1)}}{x^4 - 2x^2 \cos 2\theta + 1}\, dx$$

is a strictly monotone increasing, positive, and continuous function on $0 \leq a < \infty$. Moreover, the value at 0 of this function is less than $\pi/2$ and the value tends to $+\infty$ as $a \to +\infty$.

We define

$$\varphi_\theta = \varphi_\theta(z)\, dz^2 = \frac{4L^2}{\pi^2} \frac{z^2 \sin^2 2\theta + a_\theta(z^4 - 2z^2 \cos 2\theta + 1)}{(z^4 - 2z^2\cos 2\theta + 1)^2}\, dz^2.$$

Then we can observe that for $\pi/4 < \theta < \pi/2$, φ_θ is the quadratic differential on R_θ with the conditions in Theorem 1 for γ, $l_i = L$ for $i = 1, \dots, 4$ and L.

On the other hand, if $l = L$, then the right-hand side of the inequality in Lemma 13 is equal to $1/2$. Therefore the inequality is sharp when $l = L$.

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