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ON HASSE ZETA FUNCTIONS OF ENVELOPING ALGEBRAS OF SOLVABLE LIE ALGEBRAS

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1. Introduction

In the paper [F1], we generalized the Hasse zeta functions $\zeta_A(s)$ of commutative finitely generated rings A over the ring $\boldsymbol{Z}$ of integers, to non-commutative rings.

The aim of this paper is to prove

THEOREM 1.1. *Let R be a finitely generated commutative ring over $\boldsymbol{Z}$, let $\mathfrak{g}$ be a solvable Lie algebra over R which is free of finite rank n as an R-module, and let A be the universal enveloping algebra of $\mathfrak{g}$ over R. Then*

$$\zeta_A(s) = \zeta_R(s-n).$$

1.2. We review the definition of the function $\zeta_A(s)$. For a (not necessarily commutative) finitely generated ring A over $\boldsymbol{Z}$, the Hasse zeta function $\zeta_A(s)$ of A is defined by

$$\zeta_A(s) = \prod_{r\geq 1} \zeta_{A,r}(s)$$

where r runs over integers ≥ 1, and

$$\zeta_{A,r}(s) = \prod_p \exp \sum_{n=1}^{\infty} \frac{\sharp \mathfrak{S}_{A,r}(\boldsymbol{F}_{p^n})}{n} (p^{-s})^n$$

where $\mathfrak{S}_{A,r}$ is a certain scheme of finite type over $\boldsymbol{Z}$, p runs over prime numbers, and $\boldsymbol{F}_{p^n}$ is a finite field with p^n elements, so the function $\zeta_{A,r}(s)$ coincides with the product of Weil's zeta functions of $\mathfrak{S}_{A,r} \otimes_{\boldsymbol{Z}} \boldsymbol{F}_p$ [We] for all prime numbers p. For the algebraic closure K of $\boldsymbol{F}_p$, $\mathfrak{S}_{A,r}(K)$ is identified with the set of the isomorphism classes of all r-dimensional irreducible representations of A over K, and $\mathfrak{S}_{A,r}(\boldsymbol{F}_{p^n})$ is identified with the $\mathrm{Gal}(K/\boldsymbol{F}_{p^n})$-fixed part of $\mathfrak{S}_{A,r}(K)$.

Theorem 1.1 is deduced from the following Theorem 1.3.

THEOREM 1.3. *Let B be a finitely generated algebra over $\boldsymbol{Z}$, let δ be a derivation of B, and let A be the ring $\{\sum_{i=0}^{N} b_i t^i; N \geq 0, b_i \in B\}$ in which t is an indeterminate and the multiplication is expressed as $tb - bt = \delta(b)$ $(b \in B)$. Then*

$$\zeta_A(s) = \zeta_B(s-1).$$

We show that Theorem 1.1 follows from Theorem 1.3.

We may assume that R is a finite field of characteristic $p > 0$, for $\zeta_R(s)$, $\zeta_A(s)$ are products of $\zeta_{R/\mathfrak{m}}(s), \zeta_{A/\mathfrak{m}A}(s)$ over all maximal ideals $\mathfrak{m}$ of R, respectively. So assume R is a finite field k.

Since $\mathfrak{g}$ is a solvable Lie algebra, there exists a sequence of subalgebras of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = \{0\}$$

where $\mathfrak{g}_i$ is of dimension $n - i$ as a k-vector space, and $[\mathfrak{g}_{i-1}, \mathfrak{g}_i] \subset \mathfrak{g}_i$ for $1 \leq i \leq n$. Take the universal enveloping algebras of $\mathfrak{g}_{i-1}$ and $\mathfrak{g}_i$ as A and B, respectively, and apply Theorem 1.3 inductively, then we obtain Theorem 1.1.

In section 2, we prove Theorem 1.3.

A proof of a special case of Theorem 1.1 and a proof of Theorem 1.3 in the case B is commutative are given in our previous papers [F2], [F3], respectively.

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2. Proof of Theorem 1.3

2.1. Let A, B and δ be as in Theorem 1.3. Since $\zeta_A(s) = \prod_p \zeta_{A/pA}(s)$, $\zeta_B(s) = \prod_p \zeta_{B/pB}(s)$ where p ranges over all prime numbers, we may assume that B is an $\boldsymbol{F}_p$-algebra. Let $k = \boldsymbol{F}_p$, and let K be the algebraic closure of k. Let $\mathfrak{S}_A = \coprod_{r \geq 1} \mathfrak{S}_{A,r}$, and for an extension k' of k, let $\mathfrak{S}_A(k')$ be the set of k'-rational points of $\mathfrak{S}_A$. We define $\mathfrak{S}_B$ and $\mathfrak{S}_B(k')$ as in the case of A. Let $B_K = B \otimes_k K$.

Let M be a finite dimensional irreducible representation of A over K, and let N be an irreducible representation of B over K which is a subrepresentation of M. Let $\chi_N : B_K \to \mathrm{End}_K(N)$ be the action of B_K on N.

Definition 2.2. *Let $\{\delta_j; 1 \leq j \leq m\}$ $(m \in \boldsymbol{Z}, m \geq 1)$ be a family of k-derivations of B. We say "$\{\chi_N \circ \delta_j; 1 \leq j \leq m\}$ are linearly independent (resp. dependent) modulo inner derivation" if the canonical images of $\{\chi_N \circ \delta_j; 1 \leq j \leq m\}$ in the space*

$$\{B_K \to \mathrm{End}_K(N); K\text{-linear}\}/\{B_K \to \mathrm{End}_K(N); b \mapsto \chi_N([b, b_0]) \textit{ for some } b_0 \in B_K\}$$

are linearly independent (resp. dependent) over K.

Lemma 2.3. *There exists an integer $l \geq 0$ such that $\{\chi_N \circ \delta^{p^i}; 0 \leq i \leq l-1\}$ are linearly independent modulo inner derivation and $\{\chi_N \circ \delta^{p^i}; 0 \leq i \leq l\}$ are linearly dependent modulo inner derivation.*

Proof. Note that $\delta^{p^i}(b)$ is a k-derivation for any $i \in \boldsymbol{Z}, i \geq 0$ ([S-F] Chapter 1, Proposition 2.3.2). In the K-linear space

$$\{B_K \to \mathrm{End}_K(N); K\text{-linear}\}/\{B_K \to \mathrm{End}_K(N); b \mapsto \chi_N([b, b_0]) \text{ for some } b_0 \in B_K\},$$

the images of $\{\chi_N \circ \delta^{p^m}; m \in \boldsymbol{Z}, m \geq 0\}$ are contained in the following K-linear subspace: $\{h : B_K \to \mathrm{End}_K(N); K\text{-linear}, h(ab) = \chi_N(a)h(b) + h(a)\chi_N(b)\}/\{B_K \to \mathrm{End}_K(N); b \mapsto \chi_N([b, b_0]) \text{ for some } b_0 \in B_K\}$, which is finite dimensional over K since the maps h satisfying the condition are determined by the values of h at the generators of B over k. □

We use the same notation δ for the K-derivation of B_K which is induced from δ in Theorem 1.3.

We have the following proposition.

PROPOSITION 2.4. *Let l be as in Lemma 2.3. Then the map*

$$N^{\oplus p^l} \to M; \ (x_\iota)_{0 \leq \iota \leq p^l - 1} \ \mapsto \ \sum_{\iota=0}^{p^l-1} t^\iota x_\iota$$

is bijective.

We prove Proposition 2.4 by using the following Lemmas 2.5, 2.6, and 2.7.

LEMMA 2.5. *For $b \in B_K$ and for $m \in \boldsymbol{Z}, m \geq 0$,*

$$bt^m = \sum_{j=0}^{m} (-1)^j \binom{m}{j} t^{m-j} \delta^j(b).$$

Especially, let $f : \{a \in \boldsymbol{Z}; a > 0\} \to \{a \in \boldsymbol{Z}; a > 0\}$ be the function defined by $f(a) = a - p^r$ where $p^r \| a$. For an integer $m > 0$, for any $b \in B_K$, and for $x \in N$,

$$(t^m b - bt^m)x = \alpha t^{f(m)} \delta^{m-f(m)}(b)x$$
$$+ \text{(a linear combination of the elements } t^\iota \delta^{m-\iota}(b)x \ (0 \leq i < f(m)))$$

where $\alpha \in \boldsymbol{F}_p$, $\alpha \neq 0$.

Proof. See [S-F] Chapter 1, Proposition 1.3. □

LEMMA 2.6. *Assume that $\{\chi_N \circ \delta^{p^i}; 0 \leq i \leq l - 1\}$ are linearly independent over K modulo inner derivation. Then the map*

$$N^{\oplus p^l} \to M; \quad (x_\iota)_{0 \leq \iota \leq p^l - 1} \ \mapsto \ \sum_{\iota=0}^{p^l-1} t^\iota x_\iota$$

is injective.

Proof. We prove this by induction. Let i be an integer such that $1 \leq i \leq p^l - 1$, and assume that

$$N + tN + \cdots + t^{\iota-1}N \cong N^{\oplus \iota}$$

as K-linear spaces by the map defined above. We show that

(i) $$N + tN + \cdots + t^{\iota-1}N + t^\iota N \cong N^{\oplus (i+1)}.$$

This (i) is equivalent to the fact that the map

$$t^i : N \to M/(N + tN + \cdots + t^{i-1}N); x \mapsto t^i x \bmod N + tN + \cdots + t^{i-1}N$$

is injective. □

Now we have a lemma.

LEMMA 2.6.1. *The above map t^i is a B-homomorphism.*

Proof. This follows from Lemma 2.5. □

By Lemma 2.6.1, if the map t^i is not injective, it is the 0-map (since N is irreducible). We assume that t^i is the 0-map, and will get a contradiction.

The fact that t^i is the 0-map is equivalent to

$$t^i N \subset N + tN + \cdots + t^{i-1}N.$$

Then for $x \in N$, it can be expressed as

$$t^i x = g_0(x) + \cdots + t^{i-1} g_{i-1}(x)$$

where $g_j : N \to N$ is a K-linear map for $0 \le j \le i-1$. For $b \in B_K$, by Lemma 2.5,

$$\text{(ii)} \qquad bt^i x = \sum_{j=0}^{i} (-1)^j \binom{i}{j} t^{i-j} \delta^j(b) x = t^i bx + \sum_{j=1}^{i} (-1)^j \binom{i}{j} t^{i-j} \delta^j(b) x$$

$$= g_0(bx) + \cdots + t^{i-1} g_{i-1}(bx) + \sum_{j=1}^{i} (-1)^j \binom{i}{j} t^{i-j} \delta^j(b) x.$$

Moreover,

$$\text{(iii)} \qquad bt^i x = bg_0(x) + \cdots + bt^{i-1} g_{i-1}(x).$$

We compare the two equations (ii) and (iii). The most important parts in (ii) and (iii) are the $t^{f(i)}N$-components where f is as in Lemma 2.5.

To prepare to compare the equations, we have some lemmas.

LEMMA 2.6.2. *We have the following equation. For $b \in B_K$ and for $m \in \mathbf{Z}$, $0 \le m \le i-1$,*

$$\text{(iv)} \quad g_m(bx) + (-1)^{i-m} \binom{i}{m} \delta^{i-m}(b) x = bg_m(x) + \sum_{j=1}^{i-1-m} (-1)^j \binom{m+j}{m} \delta^j(b) g_{m+j}(x).$$

Proof. The left hand side is the $t^m N$-component in $bt^i x$ in the equation (ii), and the right hand side is that in (iii). □

LEMMA 2.6.3. (1) *For* j $(0 \leq j \leq i-1)$ *such that* $f(j) \geq f(i)$, g_j *is a scalar map. That is,* $g_j(x) = C_j x$ $(x \in N)$ *for some* $C_j \in K$.

(2) *For* j $(0 \leq j \leq i-1)$ *such that* $f(j) > f(i)$, g_j *is the* 0*-map*.

Proof. We fix an integer m such that $f(i) < m \leq i-1$. Assume that for j $(0 \leq j \leq i-1)$ such that $j > m$, g_j is a scalar map C_j $(C_j \in K)$, and for j $(0 \leq j \leq i-1)$ such that $f(j) > m$, g_j is the 0-map.

We show that g_m is a scalar map C_m $(C_m \in K)$, and for j $(0 \leq j \leq i-1)$ such that $f(j) = m$, g_j is the 0-map.

Remark that $j > f(j)$, so Lemma 2.6.3 follows from this by downward induction on m.

We consider Lemma 2.6.2. Since $m > f(i)$, from the computation of the coefficient, the part $(-1)^{i-m}\binom{i}{m}\delta^{i-m}(b)x$ in the left hand side of the equation (iv) is 0. So the equation (iv) is

$$g_m(bx) = bg_m(x) + \sum_{j=1}^{i-1-m} (-1)^j \binom{m+j}{m} \delta^j(b) g_{m+j}(x). \tag{v}$$

By the theorem of Burnside ([F-D] Corollary 1.16), any K-linear map: $N \to N$ is obtained as an action of an element of B_K. So we write $g_j(x) = b_j x$ for $b_j \in B_K$ $(0 \leq j \leq i-1)$. By the hypothesis of this induction, the equation (v) is equivalent to the equation

$$[b_m, b]x = \sum_{\substack{j \in f^{-1}(m) \\ 0 \leq j \leq i-1}} \alpha_j C_j \delta^{j-m}(b)x$$

where $\alpha_j \in \boldsymbol{F}_p$, $\alpha_j \neq 0$ (We denote $m+j$ in (v) by j here). For $j \in f^{-1}(m)$ such that $0 \leq j \leq i-1, j-m = p^r$ for some $r \in \boldsymbol{Z}, 0 \leq r \leq l-1$. From the linear independence of $\{\chi_N \circ \delta^{p^i}; 0 \leq i \leq l-1\}$ modulo inner derivation, $C_j = g_j = 0$ for $j \in f^{-1}(m)$ such that $0 \leq j \leq i-1$. So $\chi_N(b_m b) - \chi_N(bb_m) = 0$. Hence $g_m = \chi_N(b_m)$ is B-linear. Since N is irreducible, g_m is a scalar map. □

Now we accomplish the proof of Lemma 2.6.

We compare the $t^{f(i)}N$-components in (ii) and (iii). We put $m = f(i)$ in (iv). The coefficient

$$(-1)^{i-f(i)}\binom{i}{f(i)}$$

which is on the left hand side of (iv) is not zero. By Lemma 2.6.3 and the argument in its proof,

$$\chi_N(b_{f(i)}b) - \chi_N(bb_{f(i)}) = \alpha_i \chi_N \circ \delta^{i-f(i)}(b) + \sum_{\substack{j \in f^{-1}(f(i)) \\ 0 \leq j \leq i-1}} \alpha_j C_j \chi_N \circ \delta^{j-f(i)}(b)$$

where $\alpha_j \in \boldsymbol{F}_p$, $\alpha_j \neq 0$ ($j \in f^{-1}(f(i))$, $0 \leq j \leq i$). For each $j \in f^{-1}(f(i))$ such that $0 \leq j \leq i$, there exists $r \in \boldsymbol{Z}, 0 \leq r \leq l-1$ such that $j - f(i) = p^r$. This contradicts the assumption that $\{\chi_N \circ \delta^{p^i}; 0 \leq i \leq l-1\}$ are linearly independent modulo inner derivation. □

LEMMA 2.7. *Assume that* $\{\chi_N \circ \delta^{p^i}; 0 \leq i \leq l-1\}$ *are linearly independent and* $\{\chi_N \circ \delta^{p^i}; 0 \leq i \leq l\}$ *are linearly dependent modulo inner derivation. Then there exists an irreducible representation* N' *of* B *over* K *which is a subrepresentation of* M *such that* $N' \cong N$ *as a* B_K*-module and*

$$\sum_{i=0}^{p^l-1} t^i N' = M.$$

Proof. Assume that

$$\chi_N \circ \delta^{p^l} = \sum_{i=0}^{l-1} \gamma_i \chi_N \circ \delta^{p^i} + \chi_N \circ [b_0, \quad]$$

where $\gamma_i \in K$, $b_0 \in B_K$. Put

$$t' = t^{p^l} - \sum_{i=0}^{l-1} \gamma_i t^{p^i} - b_0.$$

Since $[t^{p^i}, \quad] = \delta^{p^i}$, $\chi_N(bt' - t'b) = 0$ for all $b \in B_K$. Let $W = \{x \in M; bx = 0$ for any $b \in \mathrm{Ann}(N)\}$, where $\mathrm{Ann}(N) = \{b \in B_K; bN = 0\}$. Then W is stable under the actions of B_K and t'. Let N' be the irreducible representation of $B[t']$ over K which is contained in W. In N', the action of t' commutes with the actions of $B[t']$, and hence is a scalar map. So N' is an irreducible representation also of B_K. Since $\mathrm{Ann}(N)$ kills N', N' is isomorphic to N over B_K. The subrepresentation $\sum_{i=0}^{p^l-1} t^i N'$ of M is stable under the actions of the elements of B_K and t. So it coincides with M. Hence we obtain the result. □

LEMMA 2.8. *Let* N'' *be an irreducible representation of* B *over* K *which is contained in* M. *Then* $N'' = N$.

Proof. As a B_K-module, M has a composition series whose all quotients are isomorphic to N. Hence $N'' \cong N$ over B_K.

To prove $N'' = N$, it is sufficient to prove that the image of any B_K-homomorphism

$$h : N \to M = \sum_{i=0}^{p^l-1} t^i N,$$

is contained in N. Write $h(x) = \sum_{i=0}^{p^l-1} t^i h_i(x)$ $(x \in N)$ where h_i $(0 \le i \le p^l - 1)$ are K-linear maps $N \to N$. For any $x \in N$ and $b \in B_K$,

$$h(bx) = h_0(bx) + th_1(bx) + \cdots + t^{p^l-1}h_{p^l-1}(bx),$$

and

$$h(bx) = bh_0(x) + bth_1(x) + \cdots + bt^{p^l-1}h_{p^l-1}(x).$$

We compare the t^mN-components $(0 \le m \le p^l - 1)$ of the above two equations, then we have

$$h_m(bx) = bh_m(x) + \sum_{j=1}^{p^l-1-m} (-1)^j \binom{m+j}{m} \delta^j(b)h_{m+j}(x).$$

This equation has the same form as (v). So from the argument in the proof of Lemma 2.6.3, $h_i = 0$ for $1 \le i \le p^l - 1$. So $h(N) \subset N$. □

From Lemma 2.8, we obtain

COROLLARY 2.8.1. *There exists a surjective map*

$$\pi : \mathfrak{S}_A(K) \to \mathfrak{S}_B(K);\ \textit{the class of } M \mapsto \text{ the class of } N.$$

This map π *commutes with the action of the Galois group* $\mathrm{Gal}(K/k)$.

By Lemmas 2.6, 2.7, and 2.8, we obtain Proposition 2.4.

2.9. Let l be as in Lemma 2.3.

From the above argument, we have that the irreducible representation M of A over K is determined by χ_N and the action of t^{p^l}. Write

$$\chi_N \circ \delta^{p^l} = \sum_{i=0}^{l-1} \gamma_i \chi_N \circ \delta^{p^i} + [b_0, \quad]$$

where $\gamma_i \in K$ and $b_0 \in B_K$. Put

$$t' = t^{p^l} - \sum_{i=0}^{l-1} \gamma_i t^{p^i} - b_0.$$

By Proposition 2.4, the action of t' on M is completely determined by its action on N, and from the argument of the proof of Lemma 2.7, t' acts on N as a scalar. Then we have for $x \in N$,

$$t'x = cx$$

for some $c \in K$. Hence

$$t^{p^l}x = \sum_{i=0}^{l-1} \gamma_i t^{p^i}x - (b_0 + c)x.$$

We can take $c \in K$ arbitrarily.

From this and Corollary 2.8.1, for each finite extension $\boldsymbol{F}_q$ of k which has q elements and $x \in \mathfrak{S}_B(\boldsymbol{F}_q)$, the $\mathrm{Gal}(K/\boldsymbol{F}_q)$-set $\pi^{-1}(x)$ is a K-principal homogeneous space. Since $H^1(\mathrm{Gal}(K/\boldsymbol{F}_q), K) = \{0\}, \pi^{-1}(x)$ is isomorphic to K as a Gal $(K/\boldsymbol{F}_q)$-set. Hence we have

$$\sharp\mathfrak{S}_A(\boldsymbol{F}_q) = \sharp\mathfrak{S}_B(\boldsymbol{F}_q) \cdot q.$$

This proves Theorem 1.3.

3. Remark

For a solvable Lie algebra $\mathfrak{g}$ over R where R is a finitely generated commutative ring over $\boldsymbol{Z}$, we have Theorem 1.1 which says that the Hasse zeta function of the universal enveloping algebra of $\mathfrak{g}$ over R is determined only by its rank over R.

But when Lie algebra $\mathfrak{g}$ is not solvable, we cannot say such things. For example, if A is the universal enveloping algebra of $sl_2(\boldsymbol{Z})$, we have

$$\zeta_A(s) = \zeta(s-3) \prod_{p:\text{odd prime}} (1 - p^{-(s-1)})^{(p-1)/2} \prod_{p:\text{odd prime}} (1 - p^{-s})^{-(p-1)/2}$$

(see [F1]).

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