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TOPOLOGY OF COMPLEX POLYNOMIALS VIA POLAR CURVES

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1. The main results

The use of the local polar varieties in the study of singular spaces is already a classical subject, see Lê-Teissier [LT] and the references therein.

In this note we consider the global polar curves associated with an affine smooth hypersurface F in $\boldsymbol{C}^n$. Instead of considering the higher dimensional polar varieties associated with F, we choose to look at the polar curves for the various generic linear sections of F. This approach is motivated by our use of classical dual varieties and also by our main interest in numerical invariants describing the topology of F in terms of these family of polar curves.

More precisely, let $f \in \boldsymbol{C}[x_1, \dots, x_n]$ be a polynomial and assume that the fiber $F_t = f^{-1}(t)$ is smooth and connected. Our main result computes the Euler characteristic $\chi(F_t)$ of the hypersurface F_t in terms of the polar invariants of the intersections $F_t \cap E^k$, where E^k is a general linear subspace in $\boldsymbol{C}^n$ of codimension k, for $k = 0, 1, \dots, n-1$.

First we define these polar invariants. For any hyperplane

$$H : h = 0 \textit{ where } h(x) = h_0 + h_1x_1 + \cdots + h_nx_n$$

we define the corresponding polar variety Γ_H to be the union of the irreducible components of the variety

$$\{x \in \boldsymbol{C}^n \mid \operatorname{rank}(df(x), dh(x)) = 1\}$$

which are not contained in the critical set $S(f) = \{x \in \boldsymbol{C}^n \mid df(x) = 0\}$ of f.

Note that Γ_H depends only on the direction $H^d = (h_1 : \cdots : h_n) \in \boldsymbol{P}^{n-1}$ of the hyperplane H.

LEMMA 1. *For a generic hyperplane H we have the following properties.*

(i) *The polar variety Γ_H is either empty or a curve, i.e. each irreducible component of Γ_H has dimension* 1.

(ii) $\dim(F_t \cap \Gamma_H) \leq 0$ *and the intersection multiplicity (F_t, Γ_H) is independent of H.*

(iii) *The multiplicity (F_t, Γ_H) is equal to the number of tangent hyperplanes to F_t parallel to the hyperplane H. For each such tangent hyperplane H_a, the*

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intersection $F_t \cap H_a$ has precisely one singularity, which is an ordinary double point.

Note that (i) and (ii) above are well-known, see for instance Tibar [Ti]. On the other hand, the last property (iii) is exactly the analog of the defining property of a projective Lefschetz pencil, see [L].

DEFINITION 2. The non-negative integer (F_t, Γ_H) is called the polar invariant of the hypersurface F_t or of the polynomial f at the point t, and is denoted by $P(F_t)$ or by $P(f, t)$, depending on the point of view we want to emphasize.

Note that $P(F_t)$ corresponds exactly to the classical notion of class of a projective hypersurface.

Our main result is the following.

THEOREM 3. *For a generic hyperplane H, the homotopy type of the fiber F_t is obtained from the homotopy type of the section $F_t \cap H$ by attaching $P(F_t)$ cells of dimension $n-1$.*

In particular

$$P(F_t) = (-1)^{n-1}(\chi(F_t) - \chi(F_t \cap H))$$

In the next section we describe geometrically what is meant by a generic hyperplane is these statements, see Theorem 3'. We note here just that the Zariski open set of hyperplanes H for which Lemma 1 (iii) holds is smaller than the open sets corresponding to the claims in Lemma 1 (i) and (ii) and in Theorem 3.

COROLLARY 4.

$$\chi(F_t) = \sum_{h=0,n-1} (-1)^{n-1-k} P(F_t \cap E^k)$$

where E^k is a generic linear subspace in $\boldsymbol{C}^n$ of codimension k, for $k = 0, 1, \ldots, n-1$.

The last term in this sum $P(F_t \cap E^{n-1})$ is set by convention to be the degree of the polynomial f.

To explicitly compute the polar invariant $P(f, t)$ one may proceed as follows. Let $(\gamma_i)_i$ be the finite set of fixed parametrisations, one for each branch at infinity of the polar curve Γ_H. Each such parametrisation is given by a Laurent series (convergent in a punctured disc at the origin)

$$\gamma(s) = a_k s^k + a_{k+1} s^{k+1} + \cdots$$

where $k \in \boldsymbol{Z}$, $k < 0$ and $a_j \in \boldsymbol{C}^n$ for all $j \geq k$. For such a Laurent series (even for series with $k \geq 0$), we set $\operatorname{ord} \gamma = k$ if $a_k \neq 0$. With this notation we have the following result.

PROPOSITION 5. (i) $P(f,t) = -\sum_i \operatorname{ord}(f(\gamma_i) - t)$;

(ii) *The function* $P(f,-)$ *takes its maximal value on an open set* $U = \boldsymbol{C}\backslash\{c_1,\dots,c_m\}$.

The function $P(f,-)$ has a jump at a value $c \in \boldsymbol{C}$, say $c = c_j$, if and only if there is a parametrisation γ' for a branch at infinity of the polar curve of F_c such that $\lim_{s\to 0} f(\gamma'(s)) = c$. When this is the case, then $P(f,u) - P(f,c) = \sum \operatorname{ord}(f(\gamma') - c)$ where the sum is extended to all the parametrisations γ' with the above property.

Assume now that the polynomial f has only isolated singularities. Then Artal-Bartolo, Luengo-Velasco and Melle-Hernández [ALM] have introduced some (possibly negative) integers $\lambda(f)$ and $\lambda(f,t)$ called Milnor numbers at infinity such that

(i) the Euler characteristic of the generic fiber F_{gen} of f is given by

$$\chi(F_{gen}) = 1 + (-1)^{n-1}(\mu(f) + \lambda(f))$$

where $\mu(f)$ is the total Milnor number of f, and

(ii) if F_t is any smooth fiber of f, then

$$\chi(F_t) = 1 + (-1)^{n-1}(\mu(f) + \lambda(f) - \lambda(f,t))$$

In this case we have the following result.

PROPOSITION 6. (i) *For any polynomial f and a generic hyperplane H, the critical set of the restriction $f_H : H \to \boldsymbol{C}$ of f to H satisfies* $\dim S(f_H) \leq \max(\dim S(f) - 1, 0)$.

(ii) *Assume that f has only isolated singularities. Then for a generic hyperplane H, the restriction $f_H : H \to \boldsymbol{C}$ of f to H has only isolated singularities and*

$$P(f,t) = \mu(f) + \mu(f_H) + \lambda(f) + \lambda(f_H) - \lambda(f,t) - \lambda(f_H,t)$$

In particular, for all $t \in \boldsymbol{C}$ one has

$$\lambda(f,t) + \lambda(f_H,t) \geq 0$$

Remark 7. (i) When $\dim S(f) \leq 0$, it is not true in general that $\dim S(f) > \dim S(f_H)$ for a generic hyperplane. To see this, it is enough to consider the polynomial $f = x_1^d + \cdots + x_n^d$ for which $\dim S(f) = \dim S(f_H) = 0$ or the polynomial in our Example 11 below where $S(f) = \emptyset$ and $\dim S(f_H) = 0$.

(ii) When f has isolated singularities on $\boldsymbol{C}^n$ and at infinity (in the sense that the projective closure of F_t has only isolated singularities), then for a generic hyperplane H the restriction f_H has no singularities at infinity, in particular $\lambda(f_H) = 0$.

It is likely that a similar property holds for the polynomials with isolated singularities at infinity relative to an arbitrary compactification of $\boldsymbol{C}^n$, see [ST] for more on this class of polynomials and also Note 3.8 in [Ti].

Moreover, if f and f_H are such polynomials, then by our Proposition 6 (ii) the function $P(f,-)$ is constant on a neighborhood U of t_0 if and only if f and f_H are locally trivial over U.

A general discussion on polar curves from the point of view of their relations to topological triviality at infinity can be found in [Ti], see our Acknowledgements (ii) at the end of this paper. Some results for $n=2$ can be found in Assi [A].

2. Proofs, examples and some further results

Proof of Lemma 1 (i). Let $\phi : \boldsymbol{C}^n \to \boldsymbol{C}^n$ be the gradient map associated to f. Then there are algebraic subsets Y_j in $\boldsymbol{C}^n$ for $j = 0, 1, \dots, n$ such that

(i) $\operatorname{codim} Y_j \geq j$ and

(ii) $\dim \phi^{-1}(y) \geq j$ implies that $y \in Y_j$.

If we choose a line $L : y_1 - h_1 y_n = \cdots = y_{n-1} - h_{n-1} y_n = 0$ such that $L \cap Y_j \subset \{0\}$ for all $j > 1$ and that L is not contained in Y_1, then for the corresponding hyperplane $H : h_1 x_1 + \cdots + h_{n-1} x_{n-1} + x_n = 0$ the variety $\Gamma_H = closure(\phi^{-1}(L \backslash \{0\})$ is a curve.

This curve has no isolated points as components, since such a point would be defined locally by $(n-1)$ equations, a contradiction.

Proof of Lemma 1 (ii), (iii) *and of Theorem* 3. This proof is based on the results of Némethi on the Lefschetz theory for affine varieties, see [N1] and, for more details, [N2].

To simplify the notation, in this proof we omit the subscript t, e.g. we write F instead of F_t.

Let V be the projective closure of F in $\boldsymbol{P}^n$. Let $H_\infty : x_0 = 0$ be the hyperplane at infinity in this projective space $\boldsymbol{P}^n$.

In the dual projective space $\hat{\boldsymbol{P}}^n$ we use $(h_0 : \cdots : h_1)$ as homogeneous coordinates. Let $\hat{V} \subset \hat{\boldsymbol{P}}^n$ be the dual of the hypersurface V.

We introduce the following sets to describe the bad directions of hyperplanes in $\boldsymbol{C}^n$. The affine hyperplanes in $\boldsymbol{C}^n$ are parametrized by the open set $A = \hat{\boldsymbol{P}}^n \backslash \{\infty\}$, where $\infty = (1 : 0 : \cdots : 0)$ is the point corresponding to H_∞. Consider the projection $p : A \to \boldsymbol{P}^{n-1}$ associating to a hyperplane H its direction H^d. Note that the fibers of p are precisely the pencils of parallel hyperplanes in $\boldsymbol{C}^n$. We define $D(F) = \{(x, H) \in F \times A \mid T_x F = H\}$ and $\hat{F} = pr_2(D(F))$, the dual of the affine variety F. Let $D(V)$ be the closure of $D(F)$ in $\boldsymbol{P}^n \times \hat{\boldsymbol{P}}^n$ and

$$D(V)_\infty = \{(x, H) \in D(V) \mid x_0 = 0\}$$

Next let $\hat{V}_\infty = pr_2(D(V)_\infty)$, $W_1 = p(\hat{V}_\infty \backslash \{\infty\})$.

Let $C = C_\infty(\hat{V})$ be the projective tangent cone to the dual variety at the point ∞ and let $W_2 = p(C \backslash \{\infty\})$. When $\infty \notin \hat{V}$, we set $C = W_2 = \emptyset$.

EXAMPLE 8. The case $(n = 2)$. In this case it is very easy to describe the sets W_1 and W_2.

Note first that the directions H^d in $\boldsymbol{P}^1$ can be identified to the points on the line at infinity H_∞, i.e. a direction $(a : b)$ corresponds to the point $(0 : b : -a)$.

Under this identification, the directions in W_1 (resp. W_2) correspond to the points $p \in V \cap H_\infty$ such that the germ (V, p) has a tangent direction different from (resp. equal to) H_∞.

Moreover, in this case we have the equivalences: $\dim(\hat{V}) < 1 \Leftrightarrow \dim \phi(\boldsymbol{C}^2) < 2 \Leftrightarrow \mathrm{Hess}(f) = 0 \Leftrightarrow f$ is a linear form $\Leftrightarrow$ the generic fiber of f has no tangents parallel to a generic direction (note that $\hat{V}$ cannot be a line!).

If we take any polynomial $f \in \boldsymbol{C}[x, y]$, we know that $f = g(h)$ where $g \in \boldsymbol{C}[t]$ and $h \in \boldsymbol{C}[x, y]$ is such that the generic fiber F_h of h is connected. It follows that the generic fiber of f is the disjoint union of $k = \deg(g)$ copies of F_h and hence we have $\mathrm{Hess}(f) = 0 \Leftrightarrow \dim \phi(\boldsymbol{C}^2) < 2 \Leftrightarrow$ there is a linear coordinate change of $\boldsymbol{C}^2$ such that $f(x, y) = g(x)$.

Note that for f homogeneous, this last statement is a well known fact in classical invariant theory, see [GY], p. 235.

To come back to the proofs, we know by [N1], [N2] that $\dim W_i \le n - 2$ for $i = 1, 2$. We will show the following more precise version of Lemma 1 (ii) and Theorem 3.

THEOREM 3′. *For a hyperplane H whose direction H^d is not in $W_1 \cup W_2$ the claims of Lemma* 1 (ii) *and Theorem* 3 *hold.*

First assume that there is a component D in $\Gamma_H \cap F$ with $\dim D > 0$. At the points of D, the tangent hyperplane $T_x F$ has always the fixed direction H^d. But such a D is unbounded and this implies that $H^d \in W_1$, a contradiction.

Note that the set U of directions H^d in $P^{n-1} \setminus (W_1 \cup W_2)$ for which the corresponding projective pencil $L_1^c = closure(p^{-1}(H^d))$ is transverse to the dual hypersurface $\hat{V}$ is open and dense (for the exact meaning of transverse in this context, see the discussion of the Cases 1 and 2 below).

It follows that for any direction $H^d \in P^{n-1} \setminus (W_1 \cup W_2)$ we can find a small 1-parameter deformation H_s^d such that $H_0^d = H^d$ and $H_s^d \in U$ for all $s \in (0, \varepsilon)$.

Let Γ_s be the polar variety of F corresponding to the direction H_s^d. Note that all the intersections $\Gamma_s \cap F$ are finite, by the first part in Lemma 1 (ii), that was proved above.

We will show below that $(\Gamma_s, F) = |\Gamma_s \cap F|$ for $s \neq 0$. The only way in which (Γ_0, F) can have a different value is when some point in the intersection $\Gamma_s \cap F$ tend to infinity when s tends to 0. But this would imply that $H^d = H_0^d \in W_1$, a contradiction.

It follows that $(\Gamma_0, F) = (\Gamma_s, F)$ for $s \neq 0$, and therefore in computing this polar invariant we may assume that the pencil L_1^c is transversal to $\hat{V}$.

To continue the proof, we note that there are two different cases to discuss.

CASE 1. $\deg(\hat{V}) = 0 \Leftrightarrow \dim(\hat{V}) < n-1 \Leftrightarrow \dim \phi(C^n) < n$.

In this case we can choose H^d such that the associated affine pencil $L = p^{-1}(H^d)$ is disjoint from $\hat{V}$. It follows then from [N1], Theorem 2 that this pencil induces a regular function $g : F \to C$ whose fibers are precisely the hyperplane sections of F by the hyperplanes in L and which is a locally trivial fibration. In particular, the inclusion of $F \cap H$ in F is a homotopy equivalence for any $H \in L$.

In this case it is also clear that $\Gamma_H \cap F = \emptyset$, i.e. $P(F) = 0$.

CASE 2. $\deg(\hat{V}) > 0 \Leftrightarrow \dim(\hat{V}) = n-1 \Leftrightarrow \dim \phi(C^n) = n$.

For H^d generic, the corresponding projective pencil L^c given by the closure of L will meet the dual $\hat{V}$ at ∞ (if $\infty \in \hat{V}$) and at some simple points $a_1, \dots, a_m$ on $\hat{V}$, all the intersections being transverse. Then we have

$$\deg(\hat{V}) = (\hat{V}, L^c)_\infty + \sum_i (\hat{V}, L^c)_{a_i} = \text{mult}_\infty \hat{V} + m$$

For each a_i, the corresponding hyperplane H_{a_i} is tangent to V at points in F, since $H^d \notin W_1$.

By [L] or [D1], the section $F \cap H_{a_i}$, has exactly one singularity, say b_i, which is an ordinary double point, i.e. $\mu(F \cap H_{a_i}, b_i) = 1$.

Then it is easy to see that we have $\Gamma_H \cap F = \{b_1, \dots, b_m\}$ for any $H \in L$. Moreover, an easy local computation using $f - t$ as a local coordinate at b_i shows that $(\Gamma_H, F)_{b_i} = 1$. Hence $P(F) = m$ is independent of the choice of the generic hyperplane H.

Our Theorem 3 now follows from Theorem 9 (a) in [N1] and Remark 5.7 in [N2].

Remark 9. When the dual hypersurface $\hat{V}$ has only isolated singularities $\{a_1, \dots, a_m\}$, then one can use the formula for the degree of $\hat{V}$, see for instance Kleiman [K], in order to compute the multiplicity $\text{mult}_\infty \hat{V}$.

$$\deg(\hat{V}) = (d-1)^{n-1} d - \sum_i \mu(V, a_i) - \sum_i \mu^{n-1}(V, a_i)$$

where d is the degree of the polynomial f and $\mu^{n-1}(V, a)$ is the Milnor number of a generic local hyperplane section of the singularity (V, a) as in Teissier [T].

When the part at infinity V_∞ has also isolated singularities, then we get via a simple computation based on [D2], p. 159 and p. 162

$$\text{mult}_\infty \hat{V} = \sum_i (\mu(V_\infty, b_i) - \mu^{n-1}(V, b_i))$$

where $\{b_1, \dots, b_m\} = V_{\text{sing}} \cup V_{\infty, \text{sing}}$, i.e. $\text{mult}_\infty \hat{V}$ is a sum of local contributions measuring how far is the hyperplane H_∞ from a generic hyperplane at each of the points b_i.

In particular, it follows that if we look at the fibers F_t of the polynomial f, the corresponding dual varieties $\hat{V}_t$ may have different degrees and different multiplicities at ∞ as well.

Proof of Proposition 5. Let (C_γ, a) be the branch of the projective closure C of Γ_H at a point $a \in H_\infty$ corresponding to the parametrisation γ written down in the previous section.

We can assume that $a = [0 : \cdots : 0 : 1]$. Then the parametrisation γ corresponds to a mapping

$$s \mapsto [s^{-k} : g_1(s) : \cdots : g_{n-1}(s) : 1]$$

where all the function germs g_i are holomorphic.

We obviously have $(C_\gamma, H_\infty)_a = -k$. By taking the sum over all the branches of C at infinity we get $\deg C = (C, H_\infty) = -\sum_\iota \mathrm{ord} = \gamma_i$.

Let $\tilde{f}(x_0, \dots, x_n)$ be the homogeneisation of the polynomial $f - t$. Then we have $(C_\gamma, V)_a = \mathrm{ord}\,\tilde{f}(s^{-k}, g_1(s), \dots, 1) = -kd + \mathrm{ord}\,\tilde{f}(1, \gamma(s)) = -kd + \mathrm{ord}\, f(\gamma)$, where d is the degree of the polynomial f. Summing over all the branches of C at infinity, we get $\sum_{a \in C \cap H_\infty} (C, V)_a = \deg C \deg V + \sum_\iota \mathrm{ord}(f(\gamma_i) - t)$. By Bezout Theorem, this gives us the first claim in Proposition 5.

The second claim follows from the first, once we notice the following "stability" of a generic hyperplane: For any $t_0 \in \boldsymbol{C}$ there exist an $\varepsilon > 0$ and a hyperplane H_{t_0} satisfying the claims in Lemma 1 with respect to all the hypersurfaces F_t for $|t - t_0| < \varepsilon$. When $t_0 = c$, a special value, one should pick such a hyperplane H_c and the parametrisations γ' are associated to the branches at infinity of the polar curve Γ_{H_c}.

Remark 10. In [N1], [N2], A. Nemethi has introduced for any polynomial f a bad set Λ_f such that f is a locally trivial fibration over $\boldsymbol{C}\backslash\Lambda_f$. This set is not in general the minimal one with this property, see [NZ] for a polynomial f with $\Lambda_f = \boldsymbol{C}$.

We conjecture that all the special values c_j from Proposition 5 (ii) are contained in this set Λ_f.

A much harder question is to compare the set $C_f = \{c_i, \dots, c_m\}$ to the minimal set B_f such that f is locally trivial over $\boldsymbol{C}\backslash B_f$. For $n = 2$, we have $C_f = B_f$, since in this case $\chi(F_t \cap H) = d$, the degree of the polynomial f and the set B_f is detected by the jumps in the Euler number of F_t by Hà-Lê [HL].

Proof of Proposition 6. In fact, we have to prove only the first claim (i), since (ii) follows directly from Theorem 3 and Proposition 5 (ii) above.

Choose a point $y \in \boldsymbol{C}^n$ such that $y \neq 0$ and $\dim \phi^{-1}(y) \leq 0$ (note that this choice is generic!). Let H be any hyperplane whose direction H^d is defined by y and such that $\dim(S(f) \cap H) \leq \dim S(f) - 1$. We will show that the corresponding restriction f_H satisfies (i).

Let $e = \dim S(f)$ and let D be a component of the singular locus $S(f_H)$ with $\dim D > \max(e - 1, 0)$. It follows then that f_H is constant along D, i.e. there is a value t such that $D \subset F_t$. This means that at any point $x \in D\backslash F_{t,\mathrm{sing}}$, the tangent hyperplane T_xF_t has the same direction H^d. This is a contradiction since

$$\dim(D \cap F_{t,\mathrm{sing}}) \leq \dim(S(f) \cap H) \leq e - 1$$

EXAMPLE 11. *Consider the polynomial* $f = x - 3x^3y^2 + 2x^4y^3 + yz$, which is the simplest of the polynomials $f_{n,q}$ introduced in [PZ], i.e. $n = q = 1$.

This polynomial is equivalent to a linear form via an automorphism of $\mathbf{C}^3$, in particular $S(f) = \emptyset$, $\mu(f) = 0$, $\lambda(f) = 0$.

One can look at the hyperplane $H : ax + by + z + c = 0$ and note that for $a \neq 0$ the restriction $f_H(x, y) = f(x, y, -ax - by - c)$ has isolated singularities at infinity such that $\lambda(f_H) = \lambda(f_H, -a) = 1$. By Proposition 6 in which we choose the hyperplane H such that $a \neq -t$, we have the following.

$$P(f, t) = \mu(f_H) + 1$$

The polar curve Γ_H is given by the equations

$$1 - 9x^2y^2 + 8x^3y^3 - ay = 0 \quad \text{and} \quad -6x^3y + 6x^4y^2 + z - by = 0$$

To compute the parametrisations for the branches at infinity of the polar curve, it is enough to do this for the plane curve given by the first equation. A direct computation using Proposition 5 (i) yields $P(f, t) = 8$ for all $t \in \mathbf{C}$. This gives in particular $\mu(f_H) = 7 > 0$ as claimed in our Remark 7 (i) above.

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(ii) Upon the receipt of our manuscript, M. Tibar has sent us his preprint [Ti].

Our Theorem 3 is proved there for families of affine hypersurfaces (which is more general) but under the more restrictive hypothesis that all these hypersurfaces have isolated singularities. Note also that in our Theorem 3′ we make precise the meaning of generic hyperplane in this context by using the constructions by Nemethi [N1], [N2].

We also relate the polar invariants to the Milnor numbers at infinity introduced by Artal-Bartolo, Luengo-Velasco and Melle-Hernández, showing that the sum of these invariants for f and for the restriction to a generic hyperplane f_H is positive.

The main result in [Ti] is that the constancy of the invariants $P(F_t \cap E^k)$ for $k = 0, 1, \ldots, n-1$ (in the notation of Corollary 4 above) for $t \in U \subset \mathbf{C} \setminus f(S(f))$ implies that f is smoothly locally trivial over U.

Since most of our results as well as our methods of proof are quite different, the interested reader will find useful to compare the two approaches.

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