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THE GROUP OF HOMOTOPY SELF-EQUIVALENCES OF A UNION OF $(n-1)$-CONNECTED $2n$-MANIFOLDS

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Abstract

In this paper we determine the group $\mathscr{E}(X \vee Y)$ of pointed homotopy self-equivalence classes as the quotient of an iterated semi-direct product involving $\mathscr{E}(X)$, $\mathscr{E}(Y)$ and the $2n$-th homotopy groups of X and Y, in the case where X and Y are $(n-1)$-connected $2n$-manifolds or, more generally, are CW-complexes obtained by attaching a $2n$-cell to a one-point union $\bigvee^m S^n$ of m copies of the n-sphere for which a certain quadratic form has non-zero determinant ($n \geq 3$). In the case of manifolds this determinant is ± 1. We include some examples, in particular one in which $\mathscr{E}(X \vee Y)$ does not itself inherit a semi-direct product structure.

§0. Introduction

A method was given in 1958 by Barcus and Barratt [1] for calculating the group $\mathscr{E}(X)$ of (pointed) homotopy self-equivalence classes of simply-connected CW complexes of the form $X = K \cup_\alpha e^{q+1}$ obtained by attaching a $(q+1)$-cell to a complex K of dimension $\leq q-1$: this method was extended by Rutter [13] to general simply-connected complexes. Since 1958 general results about the group $\mathscr{E}(X)$, such as conditions for finite presentability, have been obtained and many calculations have been made.

P. J. Kahn [6] made calculations of $\mathscr{E}(X)$ for $X = (S^n \vee \cdots \vee S^n) \cup_\alpha e^{2n}$ and, in particular, for $(n-1)$-connected $2n$-manifolds. In this note we calculate $\mathscr{E}(X \vee Y)$ in the case where X and Y are $(n-1)$-connected $2n$-manifolds ($n \geq 3$) or, more generally, are spaces obtained by attaching a $2n$-cell to a union of n-spheres for which a certain quadratic form has non-zero determinant. Our main result stated in §1 is that, for such spaces, $\mathscr{E}(X \vee Y)$ is a quotient of a certain iterated semi-direct product in case $X \not\simeq Y$, and involves a further semi-direct product in case $X = Y$. We also give criteria for which this quotient is not itself a semi-direct product: in previous cases calculations have been completed in general only in cases where a corresponding extension is a semi-direct product.

Previous calculations of $\mathscr{E}(X \vee Y)$ for a one-point union have been made in cases where either X or Y is an h-cogroup (see for example Maruyama–Mimura

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[8], Oka–Sawashita–Sugawara [9], Rutter [14] and [15], Sieradski [17] and Yamaguchi [20]). In our case the spaces are not in general h-cogroups. Proofs and other results are given in §2 and §3, and some examples, including one which involves a non-trivial extension are given in §4.

§1. Main results

We consider complexes $X_\alpha = \bigvee^m S^n \cup_\alpha e^{2n}$ obtained by attaching one $2n$-cell to a union of n-cells ($n \geq 3$). By the Hilton–Milnor theorem, the attaching map α has the form

$$\alpha = \sum_{i=1}^{m} \iota_i \circ \alpha^i + \sum_{i<j} [\iota_i, \iota_j] \circ \alpha^{ij}.$$

Here $\alpha^i \in \pi_{2n-1}(S^n)$, $\alpha^{ij} \in \pi_{2n-1}(S^{2n-1})$, and $\iota_i : S^n \to \bigvee^m S^n$ is the canonical inclusion of the i-th sphere S^n in $\bigvee^m S^n$. We define an integer matrix $Q(\alpha) = (a_{ij})$ by

$$a_{ij} = \begin{cases} \deg \alpha^{ij}, & \text{for } i < j \\ (-1)^n \deg \alpha^{ji}, & \text{for } i > j \\ H(\alpha^i), & \text{for } i = j, \end{cases}$$

where $H(\alpha^i)$ is the Hopf invariant of α^i: in case n is odd, we have $a_{ii} = 0$. Therefore $Q(\alpha)$ is symmetric in case n is even, and is skew-symmetric in case n is odd. The matrix $Q(\alpha)$ can also be defined as the matrix of the cup product form on $H^n(X)$ (compare [19] and [3]). In what follows we consider only those complexes X_α for which the matrix $Q(\alpha)$ has non-zero determinant. Any $(n-1)$-connected $2n$-manifold has the homotopy type of a space X_α as above, and its associated matrix $Q(\alpha)$ is unimodular (see [19, page 169]): in this case the matrix $Q(\alpha)$ is, up to sign, the inverse of the matrix of the n-symmetric bilinear form determined by linking numbers on $X \backslash \operatorname{int} E^2$ (see [19, pages 164 and 182]).

We shall in general use the same symbol to denote a map and its homotopy class.

Let $X = X_\alpha = (\bigvee^{m_1} S^n) \cup_\alpha e^{2n}$ and $Y = X_\beta = (\bigvee^{m_2} S^n) \cup_\beta e^{2n}$ ($n \geq 3$), where $\bigvee^m S^n$ denotes a one point union of m copies of the n-sphere. A map $h : X \to Y$ induces a homotopy commutative diagram

$$\begin{array}{ccccccccc} S^{2n-1} & \xrightarrow{\alpha} & \bigvee^{m_1} S^n & \xrightarrow{\iota} & X & \xrightarrow{p} & S^{2n} & \xrightarrow{S\alpha} & \bigvee^{m_1} S^{n+1} \\ \downarrow{\scriptstyle \tilde{h}'} & & \downarrow{\scriptstyle \hat{h}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \tilde{h}} & & \downarrow{\scriptstyle S\hat{h}} \\ S^{2n-1} & \xrightarrow{\beta} & \bigvee^{m_2} S^n & \xrightarrow{\iota'} & Y & \xrightarrow{p'} & S^{2n} & \xrightarrow{S\beta} & \bigvee^{m_2} S^{n+1} \end{array}$$

of cofibre sequences, where the vertical maps are unique up to homotopy, and where $\tilde{h} \simeq S\tilde{h}'$. If h is cellular, $\hat{h}$ and $\tilde{h}$ can be chosen so that the two middle squares are strictly commutative. We shall always assume that the three maps are chosen in this way.

The fibre sequence $\Omega X * \Omega Y \xrightarrow{\imath} X \vee Y \xrightarrow{\jmath} X \times Y$ induces the exact sequence of pointed sets

$$[X \vee Y, \Omega X * \Omega Y] \xrightarrow{\imath_*} [X \vee Y, X \vee Y] \xrightarrow{\jmath_*} [X \vee Y, X \times Y],$$

where the preferred element for exactness is the class of the trivial map. In this paper we prove that j_* induces a faithful representation of $\mathscr{E}(X \vee Y)$ onto the quotient of an iterated semidirect product. This representation involves, besides $\mathscr{E}(X)$ and $\mathscr{E}(Y)$, some groups related to the homotopy groups of X and Y. One of these is

$$G = \frac{i_*[S^{2n} \vee S^{2n}, \Omega X * \Omega Y]}{\operatorname{im} \Gamma(\imath \vee \imath', \alpha \vee \beta) \cap i_*[S^{2n} \vee S^{2n}, \Omega X * \Omega Y]},$$

where $\Gamma(\imath \vee \imath', \alpha \vee \beta) : [\bigvee^{m_1} S^{n+1} \vee \bigvee^{m_2} S^{n+1}, X \vee Y] \to [S^{2n} \vee S^{2n}, X \vee Y]$ is the homomorphism defined in [10, §3.2]. We recall the definition of this homomorphism in §3. For $n \geq 3$, the group $i_*[S^{2n} \vee S^{2n}, \Omega X * \Omega Y]$, and hence G, is a finitely generated free $\boldsymbol{Z}/2$-module (see §2 and §3). We also define (for $n \geq 3$)

$$\begin{aligned} R_{\alpha,\beta} &= (S\alpha)^*[\textstyle\bigvee^{m_1} S^{n+1}, Y] \\ &\cong \frac{(S\alpha)^*[\bigvee^{m_1} S^{n+1}, \bigvee^{m_2} S^n]}{\beta_*\pi_{2n}(S^{2n-1}) \cap (S\alpha)^*[\bigvee^{m_1} S^{n+1}, \bigvee^{m_2} S^n]}, \end{aligned}$$

and similarly $R_{\beta,\alpha} = (S\beta)^*[\bigvee^{m_2} S^{n+1}, X]$: each of these is also a finitely generated free $\boldsymbol{Z}/2$-module for $n \geq 3$. Our main result is the following theorem.

THEOREM A. *Let $X \not\simeq Y$, let $n \geq 3$, and let $Q(\alpha)$ and $Q(\beta)$ be non-singular matrices. Then the map $j_* : [X \vee Y, X \vee Y] \to [X \vee Y, X \times Y]$ induces a faithful representation of $\mathscr{E}(X \vee Y)$ onto the quotient of an iterated semi-direct product:*

$$\mathscr{E}(X \vee Y) \cong (G \rtimes \bar{U})/(R_{\beta,\alpha} \times R_{\alpha,\beta}),$$

where $\bar{U} = (\imath_*\pi_{2n}(\bigvee^{m_1} S^n) \times \imath'_*\pi_{2n}(\bigvee^{m_2} S^n)) \rtimes (\mathscr{E}(X) \times \mathscr{E}(Y))$. Furthermore G, $R_{\beta,\alpha}$ and $R_{\alpha,\beta}$ are finitely generated free $\boldsymbol{Z}/2$-modules.

The proof of Theorem A is given in §2. In Proposition 6 we describe the action of $\bar{U}$ on G (see Proposition 1) for the semi-direct product $G \rtimes \bar{U}$. In Proposition 5 we describe $R_{\beta,\alpha} \times R_{\alpha,\beta}$ as a subgroup of the semi-direct product structure $G \rtimes \bar{U}$. In Proposition 10 we compute G. We also give, in Proposition 7, precise conditions under which the structure on $\mathscr{E}(X \vee Y)$ as the quotient $(G \rtimes \bar{U})/(R_{\alpha\beta} \times R_{\beta\alpha})$ of a semi-direct product induces on $\mathscr{E}(X \vee Y)$ the structure of a semi-direct product of the form $G \rtimes U$.

Where $X \simeq Y$ we may assume $X = Y$ without loss of generality. In this case we denote by $S(X \vee X)$ the subgroup of $\mathscr{E}(X \vee X)$ corresponding to the group obtained by putting $Y = X$ in the quotient of the iterated semi-direct product given in Theorem A. Thus (see §2)

$$\mathscr{S}(X \vee X) = \left\{\sigma \in \mathscr{E}(X \vee X) : j_*(\sigma) = \begin{pmatrix} f & g \\ h & k \end{pmatrix}, f, k \in \mathscr{E}(X), \tilde{H}_*(g) = 0 = \tilde{H}_*(h)\right\}.$$

The group $\mathscr{E}(X \vee X)$ is determined as a further split extension in the following way.

Theorem B. *Let $n \geq 3$, and let $Q(\alpha)$ be a non-singular matrix. Then there is a split exact sequence of groups and homomorphisms*

$$\mathscr{S}(X \vee X) \rightarrowtail \mathscr{E}(X \vee X) \twoheadrightarrow \boldsymbol{Z}/2.$$

The splitting is given by $\{1, -1\} \to \mathscr{E}(X \vee X)$ where (-1) maps to the homeomorphism $T : X \vee X \to X \vee X$ given by $(x, y) \mapsto (y, x)$.

The proof Theorem B is given in §2. In Proposition 8 we note the action of $\boldsymbol{Z}/2$ on $(G \rtimes \bar{U})/(R_{\alpha\beta} \times R_{\beta\alpha})$ in the split extension $(G \rtimes \bar{U})/(R_{\alpha\beta} \times R_{\beta\alpha}) \rightarrowtail \mathscr{E}(X \vee X) \twoheadrightarrow \boldsymbol{Z}/2$ of Theorem B.

In §4 we give some examples.

§2. Proofs and further results

Each element of the set $[X \vee Y, X \times Y]$ can be written as a matrix

$$\begin{pmatrix} f & g \\ h & k \end{pmatrix} \in \begin{pmatrix} [X, X] & [Y, X] \\ [X, Y] & [Y, Y] \end{pmatrix}.$$

The following result characterises the elements in the image of $j_* : \mathscr{E}(X \vee Y) \to [X \vee Y, X \times Y]$. Its proof is given, for $m_1 = m_2$, in [2] for n even, and in [7] for n odd. The same proofs yield the case $m_1 \neq m_2$.

Theorem. *Let $X = \bigvee^{m_1} S^n \cup_\alpha e^{2n}$ and $Y = \bigvee^{m_2} S^n \cup_\beta e^{2n}$ such that $Q(\alpha)$ and $Q(\beta)$ are non-singular matrices, and let $j_*(\sigma) = \begin{pmatrix} f & g \\ h & k \end{pmatrix}$, where $\sigma \in [X \vee Y, X \vee Y]$. Then, $\sigma \in \mathscr{E}(X \vee Y)$ if, and only if, either*

(i) *f and k are homotopy equivalences and h and g are homologically trivial,*

or

(ii) *g and h are homotopy equivalences and f and k are homologically trivial.*

Using this result, Theorem B is an elementary consequence of Theorem A.

By obstruction theory, a map $h : X \to Y$ is homologically trivial if, and only if, $h \in p^* \iota'_* \pi_{2n}(\bigvee^{m_2} S^n)$. Also by obstruction theory the group structure on $\iota'_* \pi_{2n}(\bigvee^{m_2} S^n)$ induces a group structure on $p^* \iota'_* \pi_{2n}(\bigvee^{m_2} S^n)$ for which p^* is a homomorphism. We note the following isomorphisms:

$$\iota'_*\pi_{2n}(\bigvee^{m_2} S^n) \cong \frac{\pi_{2n}(\bigvee^{m_2} S^n)}{\beta_*\pi_{2n}(S^{2n-1})}, \quad \text{and}$$

$$p^*\iota'_*\pi_{2n}(\bigvee^{m_2} S^n) \cong \frac{\pi_{2n}(\bigvee^{m_2} S^n)}{\beta_*\pi_{2n}(S^{2n-1}) + (S\alpha)^*[\bigvee^{m_1} S^{n+1}, \bigvee^{m_2} S^n]}.$$

We consider the set of matrices

$$U = \left\{ \begin{pmatrix} f & g \\ h & k \end{pmatrix} \in \begin{pmatrix} \mathscr{E}(X) & p'^*\iota_*\pi_{2n}(\bigvee^{m_1} S^n) \\ p^*\iota'_*\pi_{2n}(\bigvee^{m_2} S^n) & \mathscr{E}(Y) \end{pmatrix} \right\}.$$

We shall often identify U with a subset of $[X \vee Y, X \times Y]$ as indicated above. We consider also the set of matrices

$$\bar{U} = \left\{ \begin{pmatrix} f & \bar{g} \\ \bar{h} & k \end{pmatrix} \in \begin{pmatrix} \mathscr{E}(X) & \iota_*\pi_{2n}(\bigvee^{m_1} S^n) \\ \iota'_*\pi_{2n}(\bigvee^{m_2} S^n) & \mathscr{E}(Y) \end{pmatrix} \right\}.$$

The set $\bar{U}$, endowed with the operation

$$\begin{pmatrix} f & \bar{g} \\ \bar{h} & k \end{pmatrix} \diamond \begin{pmatrix} f_1 & \bar{g}_1 \\ \bar{h}_1 & k_1 \end{pmatrix} = \begin{pmatrix} ff_1 & f\bar{g}_1 + \bar{g}\tilde{k}_1 \\ \bar{h}\tilde{f}_1 + k\bar{h}_1 & kk_1 \end{pmatrix},$$

is a group with identity $\begin{pmatrix} 1 & \bar{0} \\ \bar{0} & 1 \end{pmatrix}$. The maps $\tilde{f}_1$ and $\tilde{k}_1$ have been defined in §1.

The inverse of $\begin{pmatrix} f & \bar{g} \\ \bar{h} & k \end{pmatrix}$ is $\begin{pmatrix} f^{-1} & -f^{-1}\bar{g}\tilde{k}^{-1} \\ -k^{-1}\bar{h}\tilde{f}^{-1} & k^{-1} \end{pmatrix}$. Using the standard properties of the induced cofiber sequence, we can prove that $(\bar{h}\tilde{f}_1 + k\bar{h}_1)p'$ and $(f\bar{g}_1 + \bar{g}\tilde{k}_1)p'$ are independent of the choices of maps $\bar{h}$, $\bar{h}_1$, $\bar{g}$, $\bar{g}_1$ satisfying $\bar{h}p = h$, $\bar{g}p' = g$, $\bar{h}_1 p = h_1$, $\bar{g}_1 p' = g_1$. Therefore the group structure $(\bar{U}, \diamond)$ determines a group structure on the set U under the obvious projection $\pi : \bar{U} \to U$. We have the following Proposition.

PROPOSITION 1. *The projection $(\bar{U}, \diamond) \to \mathscr{E}(X) \times \mathscr{E}(Y)$ determines the semi-direct product*

$$\bar{U} = (\iota_*\pi_{2n}(\bigvee^{m_1} S^n) \times \iota'_*\pi_{2n}(\bigvee^{m_2} S^n)) \rtimes (\mathscr{E}(X) \times \mathscr{E}(Y))$$

with the (left) action given by $(f,k) \cdot (\bar{g}, \bar{h}) = (f\bar{g}\tilde{k}^{-1}, k\bar{h}\tilde{f}^{-1})$. The projection $(U, \diamond) \to \mathscr{E}(X) \times \mathscr{E}(Y)$ determines the semi-direct product

$$U = (p'^*\iota_*\pi_{2n}(\bigvee^{m_1} S^n) \times p^*\iota'_*\pi_{2n}(\bigvee^{m_2} S^n)) \rtimes (\mathscr{E}(X) \times \mathscr{E}(Y))$$

with a similar action. Also there is a group extension

$$R_{\beta,\alpha} \times R_{\alpha,\beta} \rightarrowtail \bar{U} \twoheadrightarrow U.$$

Now assume $X \not\simeq Y$ and denote by

$$\theta : \mathscr{E}(X \vee Y) \to U$$

the map induced by j_*. We prove that θ is an epimorphism and find a homomorphism $s : \bar{U} \to \mathscr{E}(X \vee Y)$ such that $\theta s = \pi$. If π has a right inverse, so does θ. Later, in Proposition 7, we study the general conditions under which θ has a right inverse.

First we recall some of the properties of the coaction in a principal cofibration. Let $C_\alpha = B \cup_\alpha CA$ be the mapping cone of a map $\alpha : A \to B$. There is a coaction $\varphi = \varphi_{C_\alpha} : C_\alpha \to SA \vee C_\alpha$ given by

$$\varphi(b) = b, \qquad \text{for } b \in B$$

$$\varphi(a,t) = \begin{cases} (a, 2t) \in SA, & \text{for } 0 \le t \le \frac{1}{2} \text{ and } a \in A \\ (a, 2t-1) \in C_\alpha, & \text{for } \frac{1}{2} \le t \le 1 \text{ and } a \in A. \end{cases}$$

Given $\zeta : SA \to Z$ and $\lambda : C_\alpha \to Z$, we define

$$\zeta \perp \lambda = (\zeta, \lambda)\varphi : C_\alpha \to Z.$$

If two maps $\lambda, \lambda_1 : C_\alpha \to Z$ coincide on B, then there is a difference map $d = d(\lambda, \lambda_1) : SA \to Z$, given by

$$d(a,t) = \begin{cases} \lambda(a, 2t), & 0 \le t \le \frac{1}{2} \\ \lambda_1(a, 2-2t), & \frac{1}{2} \le t \le 1. \end{cases}$$

The maps $d(\lambda, \lambda_1) \perp \lambda_1$ and λ are homotopic relatively to B, but the homotopy class of d is not uniquely determined by the homotopy class of λ and λ_1. In the sequel it is convenient to denote also by φ_X the composite $X \to S^{2n} \vee X \to X \vee S^{2n}$ of φ_X and the switching map.

Now we define

$$s : \bar{U} \to \mathscr{E}(X \vee Y) \quad \text{by} \quad \begin{pmatrix} f & \bar{g} \\ \bar{h} & k \end{pmatrix} \mapsto ((f \vee \bar{h})\varphi_X, (\bar{g} \vee k)\varphi_Y).$$

Proposition 2. *$\theta = j_* : \mathscr{E}(X \vee Y) \to U$ is an epimorphism, $s : \bar{U} \to \mathscr{E}(X \vee Y)$ is a homomorphism and the composite θs is the epimorphism $\pi : \bar{U} \to U$.*

Proof. Let $\sigma, \sigma_1 \in \mathscr{E}(X \vee Y)$, and $\theta(\sigma) = \begin{pmatrix} f & g \\ h & k \end{pmatrix}$, $\theta(\sigma_1) = \begin{pmatrix} f_1 & g_1 \\ h_1 & k_1 \end{pmatrix}$. Choose decompositions $h = \bar{h}p$, $h_1 = \bar{h}_1 p$, $g = \bar{g}p'$, $g_1 = \bar{g}_1 p'$.

The component $X \to Y$ of $\theta(\sigma\sigma_1)$ has the form

$$X \xrightarrow{\sigma_1 i_X} X \vee Y \xrightarrow{p \vee 1} S^{2n} \vee Y \xrightarrow{(\bar{h}, k)} Y.$$

The elements $(p \vee 1)\sigma_1 i_X$ and $(\tilde{f}_1 \vee h_1)\varphi_X$ are mapped to the same element by the induced function $[X, S^{2n} \vee Y] \to [X, S^{2n} \times Y]$: this latter function is a bijection since $S^{2n} \vee X \to S^{2n} \times X$ is $(3n-1)$-connected. Therefore we have

$$(h,k)\sigma_1 i_X = (\bar{h},k)(p \vee 1)\sigma_1 i_X = (\bar{h}\tilde{f}_1, k h_1)\varphi_X = (\bar{h}\tilde{f}_1, k\bar{h}_1)(1 \vee p)\varphi_X.$$

A similar argument for $(1 \vee p)\varphi_X$ and $X \xrightarrow{p} S^{2n} \xrightarrow{\nabla} S^{2n} \vee S^{2n}$ in $[X, S^{2n} \vee S^{2n}]$ proves that these elements coincide and hence

$$(h,k)\sigma_1 i_X = (\bar{h}\tilde{f}_1, k\bar{h}_1)(1 \vee p)\varphi_X = (\bar{h}\tilde{f}_1 + k\bar{h}_1)p.$$

Using the standard properties of the induced cofibre sequence, we have that this construction is independent of the choices of $\bar{h}$ and $\bar{h}_1$ satisfying $\bar{h}p = h$ and $\bar{h}_1 p = h_1$. Applying similar arguments for the other components, we obtain

$$\theta(\sigma\sigma_1) = \begin{pmatrix} ff_1 & (f\bar{g}_1 + \bar{g}\tilde{k}_1)p' \\ (\bar{h}\tilde{f}_1 + k\bar{h}_1)p & kk_1 \end{pmatrix},$$

and therefore θ is a homomorphism. Since θs is the epimorphism $\pi : \bar{U} \to U$, it follows that θ is surjective.

Given $u = \begin{pmatrix} f & \bar{g} \\ \bar{h} & k \end{pmatrix}$ and $u_1 = \begin{pmatrix} f_1 & \bar{g}_1 \\ \bar{h}_1 & k_1 \end{pmatrix}$ consider the composite

$$\begin{aligned} s(u)s(u_1)i_X &= ((f \vee \bar{h})\varphi_X, (\bar{g} \vee k)\varphi_Y)((f_1 \vee \bar{h}_1)\varphi_X) \\ &= ((f \vee \bar{h})\varphi_X f_1, (\bar{g} \vee k)\varphi_Y \bar{h}_1)\varphi_X. \end{aligned}$$

Now $\bar{h}_1 : S^{2n} \to Y$ factors through the n-skeleton of Y and therefore, by cellular considerations, $(\bar{g} \vee k)\varphi_Y \bar{h}_1 = i_Y k\bar{h}_1$. Also $\varphi_X f_1 \simeq (f_1 \vee \tilde{f}_1)\varphi_X$ since they have the same image under $[X, S^{2n} \vee X] \to [X, S^{2n} \times X]$, which is a bijection since $S^{2n} \vee X \to S^{2n} \times X$ is $(3n-1)$-connected. Thus $(f \vee \bar{h})\varphi_X f_1 = (\bar{h}\tilde{f}_1 \vee ff_1)\varphi_X$. Therefore

$$\begin{aligned} s(u)s(u_1)i_X &= ((ff_1 \vee \bar{h}\tilde{f}_1)\varphi_X, i_Y k\bar{h}_1)\varphi_X \\ &= (i_X ff_1, i_Y \bar{h}\tilde{f}_1, i_Y k\bar{h}_1)(\varphi_X, 1)\varphi_X \\ &= (i_X ff_1, i_Y(k\bar{h}_1 + \bar{h}\tilde{f}_1))\varphi_X \\ &= (ff_1 \vee (k\bar{h}_1 + \bar{h}\tilde{f}_1))\varphi_X = s(u \diamond u_1)i_X. \end{aligned}$$

Similarly $s(u)s(u_1)i_Y = s(u \diamond u_1)i_Y$. Hence, s is a homomorphism. □

We now investigate the kernel of θ. In the following diagram, induced by $\iota \vee \iota' : A = \bigvee S^n \vee \bigvee S^n \to X \vee Y$, the horizontal sequences of pointed sets are exact and the diagram is commutative by [10, (3.2.2) and §3.3]: the preferred elements for exactness are as indicated. Also the vertical sequence is exact, and, by obstruction theory, the left and right vertical maps are isomorphisms as indicated.

$$\begin{array}{ccccccc}
[SA, X \times Y] & \xrightarrow{\Gamma'} & [S^{2n} \vee S^{2n}, X \times Y] & \xrightarrow{\mu'} & [X \vee Y, X \times Y]_j & \longrightarrow & [A, X \times Y]_{j(\iota \vee \iota')} \\
{\scriptstyle j_*} \uparrow \cong & & {\scriptstyle j_*} \uparrow & & {\scriptstyle j_*} \uparrow & & {\scriptstyle j_*} \uparrow \cong \\
[SA, X \vee Y] & \xrightarrow{\Gamma} & [S^{2n} \vee S^{2n}, X \vee Y] & \xrightarrow{\mu} & [X \vee Y, X \vee Y]_1 & \longrightarrow & [A, X \vee Y]_{\iota \vee \iota'} \\
 & & {\scriptstyle i_*} \uparrow \text{mono} & & & & \\
 & & [S^{2n} \vee S^{2n}, \Omega X * \Omega Y] & & & &
\end{array}$$

The functions μ and μ' and given by $\mu(\alpha) = \alpha \perp 1$ and $\mu'(\beta) = \beta \perp j$. The image of μ consists precisely of those classes which extend the identity on the n-skeleton. Also $\Gamma = \Gamma(\iota \vee \iota', \alpha \vee \beta)$ and $\Gamma' = \Gamma(j(\iota \vee \iota'), \alpha \vee \beta)$. We recall the definition of $\Gamma(u, f)$ in §3.

PROPOSITION 3. *The sequence*

$$i_*[S^{2n} \vee S^{2n}, \Omega X * \Omega Y] \xrightarrow{\mu} \mathscr{E}(X \vee Y) \xrightarrow{\theta} U \to 1$$

is an exact sequence of groups.

Proof. Since $p \vee p'$ is trivial on the image of i_*, it follows from [13, pages 276–277] that $\mu : i_*[S^{2n} \vee S^{2n}, \Omega X * \Omega Y] \to [X \vee Y, X \vee Y]$ is a homomorphism from the usual group structure to composition. As easy argument using homological considerations shows that the image of this homomorphism is contained in $\mathscr{E}(X \vee Y)$. Let $\theta(\sigma) = j_*(\sigma) = j$. Then, by the commutativity and exactness of the above diagram, we have $\sigma = \mu(d)$ say. Furthermore, we have $j_*(d) = \Gamma' j_*(c)$ say, and therefore $d = \Gamma(c) + i_*(b)$ say. But $\mu(\Gamma(c)) = 1$, so that $\sigma = \mu(d) = \mu(i_*(b))$. This proves the inclusion $\operatorname{Ker} \theta \subset \operatorname{Im} \mu$. The proposition now follows since $j_* \mu i_* = \mu' j_* i_*$ is constant. □

From this proposition and the diagram above we obtain an exact sequence

$$0 \to G \xrightarrow{\mu} \mathscr{E}(X \vee Y) \xrightarrow{\theta} U \to 1$$

where

$$G = \frac{i_*[S^{2n} \vee S^{2n}, \Omega X * \Omega Y]}{\operatorname{im} \Gamma(\iota \vee \iota', \alpha \vee \beta) \cap i_*[S^{2n} \vee S^{2n}, \Omega X * \Omega Y]}.$$

Note that the kernel of $[S^{2n} \vee S^{2n}, X \vee Y] \xrightarrow{\mu} \mathscr{E}(X \vee Y)$ is $\operatorname{im} i_* + \operatorname{im} \Gamma$, and that $\mu(\operatorname{im} i_*) = \mu(\operatorname{im} i_* + \operatorname{im} \Gamma)$. The construction $d \mapsto i_*(b)$ induces the isomorphism

$$\frac{\operatorname{im} i_* + \operatorname{im} \Gamma}{\operatorname{im} \Gamma} \cong \frac{\operatorname{im} i_*}{\operatorname{im} i_* \cap \operatorname{im} \Gamma}.$$

In the homotopy fibre sequence $\Omega X * \Omega Y \xrightarrow{i} X \vee Y \xrightarrow{j} X \times Y$, the map i may be regarded as the generalized Whitehead product $[-\varepsilon_X, \varepsilon_Y]$ of evaluation maps (see [11, §3]). Moreover, by obstruction theory, we have that the canonical map

$$\chi : \bigvee^{m_1 m_2} S^{2n-1} \simeq (\bigvee^{m_1} S^{n-1}) * (\bigvee^{m_2} S^{n-1}) \\ \to \Omega S(\bigvee^{m_1} S^{n-1}) * \Omega S(\bigvee^{m_2} S^{n-1}) \to \Omega X * \Omega Y$$

is $(3n-2)$-connected, and, without loss of generality, it may be regarded as the inclusion into $\Omega X * \Omega Y$ of its $(3n-2)$-skeleton.

PROPOSITION 4. *The map χ induces an isomorphism*

$$\chi_* : \bigoplus^{2m_1 m_2} \mathbf{Z}/2 \cong [S^{2n} \vee S^{2n}, \bigvee^{m_1 m_2} S^{2n-1}] \to [S^{2n} \vee S^{2n}, \Omega X * \Omega Y].$$

Proof. Since χ is $(3n-2)$-connected, χ_* is an isomorphism for $n \geq 4$ and an epimorphism for $n = 3$. We consider the case $n = 3$. For a homotopy coloop Z, the evaluation map $S\Omega Z \to Z$ has a homotopy section (see for example [12]). Thus $\Omega S(\bigvee^{m_1} S^{n-1}) * \Omega S(\bigvee^{m_2} S^{n-1}) \cong S\Omega S(\bigvee^{m_1} S^{n-1}) \wedge \Omega S(\bigvee^{m_2} S^{n-1}) \to S(\bigvee^{m_1} S^{n-1}) \wedge \Omega S(\bigvee^{m_2} S^{n-1}) \cong (\bigvee^{m_1} S^{n-1}) \wedge S\Omega S(\bigvee^{m_2} S^{n-1}) \to (\bigvee^{m_1} S^{n-1}) \wedge S(\bigvee^{m_2} S^{n-1}) \cong (\bigvee^{m_1} S^{n-1}) * (\bigvee^{m_2} S^{n-1})$ has a homotopy section. Up to a homotopy self-equivalence of $(\bigvee^{m_1} S^{n-1}) * (\bigvee^{m_2} S^{n-1})$, this composite is a homotopy co-section of χ. Hence χ_* is an isomorphism. □

By Proposition 4, $\iota_*[S^{2n} \vee S^{2n}, \Omega X * \Omega Y] \cong \bigoplus^{2m_1 m_2} \mathbf{Z}/2$, and $G \cong \bigoplus^{N} \mathbf{Z}/2$, with $N \leq 2m_1 m_2$ (see Proposition 10). The composite $w = i\chi$ factors as

$$w = i\chi : \bigvee^{m_1 m_2} S^{2n-1} \to \bigvee^{m_1} S^n \vee \bigvee^{m_2} S^n \subset X \vee Y,$$

and, after a suitable choice of orientation of the $(2n-1)$-spheres, the $m_1 m_2$ components of w are easily shown to be the Whitehead products $w_{rs} = [\iota_r, \iota_s]$ of $S^n \xrightarrow{\iota_r} \bigvee^{m_1} S^n \subset X$ and $S^n \xrightarrow{\iota_s} \bigvee^{m_2} S^n \subset Y$, $r = 1 \cdots m_1$, $s = 1 \cdots m_2$.

Finally, in the following pull-back square of groups

$$\begin{array}{ccccc}
 & & G & \xrightarrow{\cong} & G \\
 & & \downarrow & & \downarrow \\
R_{\beta,\alpha} \times R_{\alpha,\beta} & \longrightarrow & E & \longrightarrow & \mathscr{E}(X \vee Y) \\
\cong\downarrow & & \bar{\theta}\downarrow & & \theta\downarrow \\
R_{\beta,\alpha} \times R_{\alpha,\beta} & \longrightarrow & \bar{U} & \xrightarrow{\pi} & U
\end{array}$$

the map $s : \bar{U} \to \mathscr{E}(X \vee Y)$ induces a cross-section of $\bar{\theta}$, so that

$$E \cong G \rtimes \bar{U}, \quad \text{and} \quad \mathscr{E}(X \vee Y) \cong \frac{E}{R_{\beta,\alpha} \times R_{\alpha,\beta}}.$$

This completes the proof of Theorem A.

In order to describe the inclusion of $R_{\beta,\alpha} \times R_{\alpha,\beta}$ into E let us choose the isomorphism $G \rtimes \bar{U} \cong E = \mathscr{E}(X \vee Y) \times_U \bar{U}$ to be given by $(\gamma, u) = \gamma + u \mapsto (\mu(\gamma), 1) + (s(u), u) = (\mu(\gamma)s(u), u)$. The inverse isomorphism is given by $(\sigma, u) \mapsto (\gamma, u)$, where $\mu(\gamma) = \sigma s(u)^{-1}$. Thus the element $(1, u) \in \ker(E \to \mathscr{E}(X \vee Y))$, is identified under this isomorphism with the element $(\gamma, u) \in G \rtimes \bar{U}$ where $\mu(\gamma) = s(u)^{-1} = s(u^{-1})$ and $u \in \ker \pi$.

We decompose γ in G as $\gamma = (\gamma^1, \gamma^2)$ corresponding to the isomorphism $[S^{2n} \vee S^{2n}, X \vee Y] \to \pi_{2n}(X \vee Y) \times \pi_{2n}(X \vee Y)$.

Proposition 5. *The monomorphism $R_{\beta,\alpha} \times R_{\alpha,\beta} \to G \rtimes \bar{U}$ is given by*

$$(\bar{g}, \bar{h}) \mapsto \left(\gamma, \begin{pmatrix} 1_X & \bar{g} \\ \bar{h} & 1_Y \end{pmatrix}\right),$$

where $\mu(\gamma) = s\left(\begin{pmatrix} 1_X & -\iota_* \bar{g} \\ -\iota'_* \bar{h} & 1_Y \end{pmatrix}\right) \in \mathscr{E}(X \vee Y)$. *Furthermore, if* $\bar{g} = (S\beta)^*(\zeta_X)$ *and* $\bar{h} = (S\alpha)^*(\zeta_Y)$, *then*

$$\gamma = (\Gamma(i_Y \iota', \beta)(\zeta_X) - (S\beta)^*(\zeta_X), \quad \Gamma(i_X \iota, \alpha)(\zeta_Y) - (S\alpha)^*(\zeta_Y)).$$

Proof. The first part is already proved. For the second observe that γ is a sum of Whitehead products which lie in the kernel of j_*. The result follows on applying [9, 3.4.3] or using the computation of $\Gamma(i_X \iota, \alpha)(\zeta_Y) - (S\alpha)^*(\zeta_Y)$ given in §3 below. □

In the next proposition we describe the action of $\bar{U}$ on G in the extension $G \rightarrowtail E \twoheadrightarrow \bar{U}$.

Proposition 6. *Let* $u = \begin{pmatrix} f & \bar{g} \\ \bar{h} & k \end{pmatrix} \in \bar{U}$ *and* $\gamma = (\gamma_1, \gamma_2) \in G$. *Then* $\gamma' = (\gamma'_1, \gamma'_2) = u \cdot \gamma$ *is given by*

$$\gamma'_1 = (f \vee k)\gamma_1 \tilde{f}^{-1}$$
$$\gamma'_2 = (f \vee k)\gamma_2 \tilde{k}^{-1}.$$

In particular the subgroup $R_{\beta,\alpha} \times R_{\alpha,\beta}$ of $\bar{U}$ acts trivially on G.

Proof. By definition, the action of $\bar{U}$ on G is given by $u \cdot (\gamma_1, \gamma_2) = (\gamma'_1, \gamma'_2)$ say, where $(\gamma'_1, \gamma'_2) \perp 1 = s(u)((\gamma_1, \gamma_2) \perp 1)s(u^{-1})$. Since $[X, S^{2n} \vee X] \to [X, S^{2n} \times X]$ is bijective by obstruction theory, we have $\varphi_X f = (\tilde{f} \vee f)\varphi_X$ for example. Similarly, for example, $\varphi_X \bar{g} = p\bar{g} + \bar{g} = \bar{g}$. Let $u^{-1} = \begin{pmatrix} f^{-1} & -f^{-1}\bar{g}\tilde{k}^{-1} \\ -k^{-1}\bar{h}\tilde{f}^{-1} & k^{-1} \end{pmatrix}$ $= \begin{pmatrix} f_1 & g_1 \\ h_1 & k_1 \end{pmatrix}$ say. Since γ_1 and γ_2 are sums of (proper) Whitehead products, we have

$$\begin{aligned} s(u)((\gamma_1,\gamma_2)\perp 1)s(u^{-1}) &= s(u)((\gamma_1,\gamma_2),1)\varphi_{X\vee Y}((f_1 \vee \bar{h}_1)\varphi_X,(\bar{g}_1 \vee k_1)\varphi_Y) \\ &= s(u)((\gamma_1\tilde{f}_1+\bar{h}_1,f_1)\varphi_X,(\gamma_2\tilde{k}_1+\bar{g}_1,k_1)\varphi_Y) \\ &= s(u)((\gamma_1\tilde{f}_1,\gamma_2\tilde{k}_1),s(u^{-1}))\varphi_{X\vee Y} \\ &= ((f\vee k)\gamma_1\tilde{f}_1,(f\vee k)\gamma_2\tilde{k}_1)\perp 1, \end{aligned}$$

where $\tilde{f}_1=\tilde{f}^{-1}$ and $\tilde{k}_1=\tilde{k}^{-1}$. □

We now give a necessary and sufficient condition that the semi-direct product structure on $G\rtimes\bar{U}$ on E carries over to a semi-direct product structure $G\rtimes U$ on $\mathscr{E}(X\vee Y)$. By Proposition 6, the action of $\bar{U}$ on G induces the action of U on G. So we can consider the 5-term exact sequence [5, Theorem VI 8.1]

$$0\to \mathrm{Der}(U,G)\to \mathrm{Der}(\bar{U},G)\to \mathrm{Hom}_U(R_{\beta,\alpha}\times R_{\alpha,\beta},G)\to H^2(U,G)\to H^2(\bar{U},G)$$

associated to the group extension $R_{\beta,\alpha}\times R_{\alpha,\beta}\rightarrowtail \bar{U}\twoheadrightarrow U$ and the U-module G. Here $\mathrm{Der}(\bar{U},G)$ is the group of derivations (crossed homomorphisms) from $\bar{U}$ to G, that is the group of functions $d:\bar{U}\to G$ such that $d(u_1\cdot u_2)=d(u_1)+u_1\cdot d(u_2)$ for all $u_1,u_2\in\bar{U}$. The group $H^2(U,G)$ classifies the extensions of the group U by the U-module G. We denote the restriction of the section s to $R_{\beta,\alpha}\times R_{\alpha,\beta}$ by

$$s':R_{\beta,\alpha}\times R_{\alpha,\beta}\to\mu(G)\cong G.$$

PROPOSITION 7. *The group of homotopy self-equivalences $\mathscr{E}(X\vee Y)$ is a semidirect-product, or more precisely, θ has a right-inverse, if and only if s' extends to some derivation from $\bar{U}$ into G.*

Proof. It follows from the diagram after Proposition 4 that the cohomology class in $H^2(U,G)$ of the extension $G\rightarrowtail\mathscr{E}(X\vee Y)\twoheadrightarrow U$ maps to the cohomology class corresponding to the semidirect-product E, that is to the zero element of $H^2(\bar{U},G)$. We now show that the cohomology class which classifies the extension $G\rightarrowtail\mathscr{E}(X\vee Y)\twoheadrightarrow U$ is given by s'. The section s' is a U-module homomorphism since, for $u=\pi(\bar{u})\in U$ and $r\in R_{\beta,\alpha}\times R_{\alpha,\beta}$, we have

$$s'(u\cdot r)=s'(\bar{u}r\bar{u}^{-1})=s(\bar{u})s(r)s(\bar{u}^{-1})=u\cdot s(r).$$

To see that s' maps to the extension $G\rightarrowtail\mathscr{E}(X\vee Y)\twoheadrightarrow U$, observe that the commutative diagram of group extensions

$$\begin{array}{ccccc} R_{\beta,\alpha}\times R_{\alpha,\beta} & \longrightarrow & \bar{U} & \xrightarrow{\pi} & U \\ \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle 1} \\ G & \longrightarrow & \mathscr{E}(X\vee Y) & \xrightarrow{\theta} & U \end{array}$$

induces the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_U(G,G) & \longrightarrow & H^2(U;G) & \longrightarrow & H^2(\mathscr{E}(X\vee Y);G) \\ {\scriptstyle (s')^*}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle s^*}\downarrow \\ \mathrm{Hom}_U(R_{\beta,\alpha}\times R_{\alpha,\beta},G) & \longrightarrow & H^2(U;G) & \longrightarrow & H^2(\bar{U};G). \end{array}$$

The proposition now follows. □

As an aid to calculation, we note by Proposition 5 that

$$\begin{aligned} &s'((S\beta)^*(\zeta_X),(S\alpha)^*(\zeta_Y)) \\ &\quad = (\Gamma(i_Y\iota',\beta)(\zeta_X)-(S\beta)^*(\zeta_X),\Gamma(i_X\iota,\alpha)(\zeta_Y)-(S\alpha)^*(\zeta_Y)). \end{aligned}$$

In §4 Example 5 we give examples of spaces for which θ has no right-inverse.

We now consider the case where $Y=X$. We define $\rho:\mathscr{E}(X\vee X)\to \boldsymbol{Z}_2$ as follows: let $\theta(\sigma)=\begin{pmatrix} f & g \\ h & k\end{pmatrix}$, then $\rho(\sigma)=+1$ if f and $k\in\mathscr{E}(X)$ and $\rho(\sigma)=-1$ if g and $h\in\mathscr{E}(X)$. That ρ is a homomorphism follows easily using the techniques of the proof of Proposition 2. This homomorphism has a section given by $-1\mapsto T$, where $T(x,y)=(y,x)$. The action in the split extension $\mathscr{S}(X\vee X)\rightarrowtail\mathscr{E}(X\vee X)\twoheadrightarrow \boldsymbol{Z}/2$ is given by $(-1)\cdot\sigma=T\sigma T$. We have, as above, the isomorphism $G\rtimes\bar{U}\mapsto\mathscr{S}(X\vee Y)\times_U\bar{U}$ given by $(\gamma,u)\mapsto(\mu(\gamma)s(u),u)$: the inverse of this isomorphism is given by $(\sigma,u)\mapsto(\gamma,u)$, where $\mu(\gamma)=\sigma s(u)^{-1}$. The proof of the following proposition is straightforward.

Proposition 8. *The action in the split extension* $G\rtimes\bar{U}/R_{\beta\alpha}\times R_{\alpha\beta}\rightarrowtail\mathscr{E}(X\vee X)\twoheadrightarrow\boldsymbol{Z}/2$ *is given by*

$$(-1)\cdot\left((\gamma_1,\gamma_2),\begin{pmatrix} f & \bar{g} \\ \bar{h} & k\end{pmatrix}\right)=\left((\gamma_2,\gamma_1),\begin{pmatrix} k & \bar{h} \\ \bar{g} & f\end{pmatrix}\right).$$

§3. The group G

Let Z and W be (pointed) spaces. For any map $u:Z\to W$, the *u-based track group* $\pi_1^Z(W;u)$ is the set of homotopy classes in the space of functions $\zeta:Z\wedge I^+=Z\times I/z_0\times I\to W$, satisfying $\zeta(z,0)=\zeta(z,1)=u(z)$, for all $z\in Z$. The set $\pi_1^Z(W;u)$ is a group with the obvious operation. If Z is a co-H-space, W^Z is an H-space and there exists an isomorphism

$$u_b:\pi_1^Z(W;u)\to\pi_1^Z(W;u.u^{-1})\cong\pi_1^Z(W;*),$$

defined in the following way. Let F be a homotopy $u\cdot u^{-1}\sim *$, then

$$u_b(\xi)(z,t) = \begin{cases} F(z, 1-4t), & \text{for } 0 \le t \le \frac{1}{4} \\ (\xi, u^{-1}) \circ \Phi\left(z, 2t - \frac{1}{2}\right), & \text{for } \frac{1}{4} \le t \le \frac{3}{4} \\ F(z, 4t-3), & \text{for } \frac{3}{4} \le t \le 1, \end{cases}$$

where $\Phi : Z \wedge I^+ \to (Z \vee Z) \wedge I^+ \cong (Z \wedge I^+) \vee (Z \wedge I^+) \to (Z \wedge I^+) \vee Z$ is the map induced by the comultiplication of Z followed by the projection.

Given co-H-spaces A and B and pointed maps $f : B \to A$, $u : A \to X$, we define

$$\Gamma(u,f) : [SA, X] \xrightarrow{u_b^{-1}} \pi_1^A(X;u) \xrightarrow{f^*} \pi_1^B(X;uf) \xrightarrow{(uf)_b} [SB, X].$$

For a detailed account of the properties of $\Gamma(u,f)$ see [10, §3].

In our case $B = S^{2n-1} \vee S^{2n-1}$ and $A = \bigvee^{m_1} S^n \vee \bigvee^{m_2} S^n$. By [10], for any $(\zeta^1, \zeta^2) \in [\bigvee^{m_1} S^{n+1}, X \vee Y] \times [\bigvee^{m_2} S^{n+1}, X \vee Y] \cong [\bigvee^{m_1} S^{n+1} \vee \bigvee^{m_2} S^{n+1}, X \vee Y]$ we have

$$\Gamma(\iota \vee \iota', \alpha \vee \beta)(\zeta^1, \zeta^2) = (\Gamma(i_X\iota, \alpha)(\zeta^1), \Gamma(i_Y\iota', \beta)(\zeta^2)).$$

Hence, as a subgroup of $[S^{2n}, X \vee Y] \oplus [S^{2n}, X \vee Y]$,

$$\operatorname{im}\Gamma(\iota \vee \iota', \alpha \vee \beta) \cap i^*[S^{2n} \vee S^{2n}, \Omega X * \Omega Y]$$
$$\cong (\operatorname{im}\Gamma(i_X\iota, \alpha) \cap \ker j_*, \operatorname{im}\Gamma(i_Y\iota', \beta) \cap \ker j_*).$$

Here $\ker\{j_* : [S^{2n}, X \vee Y] \to [S^{2n}, X \times Y]\} = i_*[S^{2n}, \Omega X * \Omega Y]$.

Observe that $[\bigvee S^{n+1}, X \vee Y] \cong [\bigvee S^{n+1}, X] \oplus [\bigvee S^{n+1}, Y]$. So we only need to study the image of $\Gamma(i_X\iota, \alpha)$ and $\Gamma(i_Y\iota', \beta)$ on these two direct summands. Let $\zeta^1 = \zeta^1_X + \zeta^1_Y \in [\bigvee^{m_1} S^{n+1}, X] \times [\bigvee^{m_1} S^{n+1}, Y] \cong [\bigvee^{m_1} S^{n+1}, X \vee Y]$ and similarly $\zeta^2 = \zeta^2_X + \zeta^2_Y$. We have

$$\Gamma(i_X\iota, \alpha)(i_X)_* = (i_X)_*\Gamma(\iota, \alpha).$$

Hence $\ker j_* \cap \Gamma(\iota_X\iota, \alpha)\{\zeta^1_X\} = 0$, and, similarly, $\ker j_* \cap \Gamma(i_Y\iota', \beta)\{\zeta^2_Y\} = 0$. Now $j_*\Gamma(i_X\iota, \alpha)(\zeta^1_Y) = (S\alpha)^*(\zeta^1_Y)$ by [10, (3.4.3)] and, therefore,

$$\operatorname{im}\Gamma(i_X\iota, \alpha) \cap \ker j_* = \Gamma(i_X\iota, \alpha)\{\zeta^1_Y : (S\alpha)^*(\zeta^1_Y) = 0\}.$$

Similarly

$$\operatorname{im}\Gamma(i_Y\iota', \beta) \cap \ker j_* = \Gamma(i_Y\iota', \beta)\{\zeta^2_X : (S\beta)^*(\zeta^2_X) = 0\}.$$

We have proved the following Proposition.

PROPOSITION 9.

$$\operatorname{im}\Gamma(\iota \vee \iota', \alpha \vee \beta) \cap i_*[S^{2n} \vee S^{2n}, \Omega X * \Omega Y]$$
$$= \Gamma(i_X\iota, \alpha)\{\zeta^1_Y : (S\alpha)^*(\zeta^1_Y) = 0\} + \Gamma(i_Y\iota', \beta)\{\zeta^2_X : (S\beta)^*(\zeta^2_X) = 0\}.$$

Let us compute these groups. Write α in the form $\alpha = \sum \iota_i\alpha' +$

$\sum_{\iota<j}[\iota_i, \iota_j]\alpha^{\iota}j$. For n odd $H(\alpha^{\iota}) = 0$, $\alpha^{\iota} = t^{\iota}$ is a suspension element and $a_{ii} = 0$. Here we use the notation introduced in §1 to define the matrix $Q(\alpha)$. For n even $(n \neq 2, 4, 8)$, since $H[\iota_i, \iota_i] = \pm 2$, we have $\alpha^{\iota} = t^{\iota} + (1/2)a_{ii}[\iota_i, \iota_i]$ where t^{ι} is a suspension element. For $n = 2, 4, 8$, we have $\alpha^{\iota} = t^{\iota} + a_{ii}\vartheta$ where t^{ι} is a suspension element and ϑ is the Hopf map: observe that in this case $S(\alpha^{\iota}) \neq S(t^{\iota})$ in general. With this notation we have by [10, (3.3.3) and (3.3.6)]

$$\Gamma(i_X\iota, \alpha) = \Gamma(i_X\iota, \sum \iota_i\alpha^{\iota}) + \sum_{\iota<j} \alpha^{ij}\Gamma(i_X\iota, [\iota_i, \iota_j])$$

$$= \begin{cases} (S\alpha)^* + \sum_{\iota} \frac{1}{2} a_{ii}\Gamma(i_X\iota, [\iota_i, \iota_i]) + \sum_{\iota<j} a_{ij}\Gamma(i_X\iota, [\iota_i, \iota_j]), & \text{for } n \text{ even}, \neq 2, 4, 8, \\ (St)^* + \sum_{\iota} a_{ii}\Gamma(i_X\iota, \iota_i\vartheta) + \sum_{\iota<j} a_{ij}\Gamma(i_X\iota, [\iota_i, \iota_j]), & \text{for } n = 2, 4, 8, \\ (S\alpha)^* + \sum_{\iota<j} a_{ij}\Gamma(i_X\iota, [\iota_i, \iota_j]), & \text{for } n \text{ odd}. \end{cases}$$

Consider the map $[\iota_i, \iota_j]$ as the composition of $w : S^{2n-1} \to S^n \vee S^n$ and $(\iota_i, \iota_j) : S^n \vee S^n \to \bigvee^{m_1} S^n$. By [10, (3.4.2)], for $\zeta_Y^1 = (\zeta_1, \ldots, \zeta_{m_1}) \in \bigoplus^{m_1}[S^{n+1}, Y]$, we have

$$\begin{aligned} \Gamma(i_X\iota, [\iota_i, \iota_j])(\zeta_Y^1) &= \Gamma(i_X\iota(\iota_i, \iota_j), w)\Gamma(i_X\iota, (\iota_i, \iota_j))(\zeta_Y^1) \\ &= \Gamma(i_X\iota(\iota_i, \iota_j), w)(\zeta_\iota, \zeta_j) \\ &= [\zeta_\iota, i_X u_j] + [i_X u_i, \zeta_j], \end{aligned}$$

since all elements are of order 2. Also, for the Hopf maps $\vartheta : S^{2n-1} \to S^n$ $(n = 2, 4, 8)$, we have $(\iota_1 + \iota_2)\vartheta = \iota_1\vartheta + \iota_2\vartheta \pm [\iota_1, \iota_2]$. Therefore, by [10, 3.4.3],

$$\Gamma(i_X\iota, \iota_i\vartheta)(\zeta_Y^1) = (S\vartheta)^*(\zeta_\iota) + [\zeta_\iota, i_X u_i].$$

Since $\pi_{n+1} Y \cong \bigoplus^{m_2}\pi_{n+1}S^n$, we can write ζ_ι in the form $\zeta_\iota = \sum_\lambda e_\iota^\lambda c\iota_{m_1+\lambda}\eta$, where η is the generator of $\pi_{n+1}S^n$, $e_\iota^\lambda = 0, 1$, $\iota_i : S^n \to \bigvee^{m_1} S^n \vee \bigvee^{m_2} S^n$ is the inclusion onto the i-th sphere of the union and $c = (\iota \vee \iota') : \bigvee^{m_1} S^n \vee \bigvee^{m_2} S^n \to X \vee Y$ is the inclusion onto the n-skeleton. With this notation $i_X u_i = c\iota_i$ and we have

$$\Gamma(i_X\iota, [\iota_i, \iota_j])(\zeta_Y^1) = c_*\left(\sum_{\lambda=1}^{m_2}(e_\iota^\lambda[\iota_j, \iota_{m_1+\lambda}\eta] + e_j^\lambda[\iota_i, \iota_{m_1+\lambda}\eta])\right), \quad \text{and}$$

$$\Gamma(i_X\iota, \iota_i\vartheta)(\zeta_Y^1) = (S\vartheta)^*(\zeta_\iota) + c_*\left(\sum_{\lambda=1}^{m_2} e_\iota^\lambda[\iota_i, \iota_{m_1+\lambda}\eta]\right).$$

Therefore, for all n,

$$\Gamma(i_X\iota, \alpha)(\zeta_Y^1) = (S\alpha)^*(\zeta_Y^1) + c_*\left(\sum_{\lambda, \iota, j} a_{ij}e_\iota^\lambda[\iota_j, \iota_{m_1+\lambda}\eta]\right).$$

where the sum runs over all $1 \leq i,\ j \leq m_1,\ 1 \leq \lambda \leq m_2$. Observe that the Whitehead products $[\iota_i, \iota_{m_1+\lambda}\eta]$ generate some of the last summands in

$$\pi_{2n}(\bigvee^{m_1} S^n \vee \bigvee^{m_2} S^n) \cong \pi_{2n}(\bigvee^{m_1} S^n) \oplus \pi_{2n}(\bigvee^{m_2} S^n) \oplus \sum_{\lambda<\mu} \pi_{2n}(S^{2n-1}).$$

Now the natural isomorphism

$$\pi_{2n+1}(X \vee Y, \bigvee S^n \vee \bigvee S^n) \cong \pi_{2n+1}(X, \bigvee S^n) \oplus \pi_{2n+1}(Y, \bigvee S^n)$$

commutes with the connecting homomorphism δ:

$$\begin{array}{ccccc} \pi_{2n+1}(X \vee Y, \bigvee S^n \vee \bigvee S^n) & \xrightarrow{\delta} & \pi_{2n}(\bigvee S^n \vee \bigvee S^n) & \xrightarrow{c_*} & \pi_{2n}(X \vee Y) \\ \cong \uparrow & & \uparrow & & \\ \pi_{2n+1}(X, \bigvee S^n) \oplus \pi_{2n+1}(Y, \bigvee S^n) & \longrightarrow & \pi_{2n}(\bigvee S^n) \oplus \pi_{2n}(\bigvee S^n) & & \end{array}$$

Since $\ker c_* = \operatorname{im} \delta$, the map c_* is injective on the subgroup $\sum_{\lambda<\mu} \pi_{2n}(S^{2n-1})$ of $\pi_{2n}(\bigvee S^n \vee \bigvee S^n)$.

Also $\iota_*[S^{2n} \vee S^{2n}, \Omega X * \Omega Y]$ is the subgroup of $c_*[S^{2n} \vee S^{2n}, \bigvee S^n \vee \bigvee S^n]$ given by

$$\iota_*[S^{2n} \vee S^{2n}, \Omega X * \Omega Y] \cong \bigoplus_{j,\lambda} \langle [\iota_j, \iota_{m_1+\lambda}\eta] \rangle \oplus \bigoplus_{j,\lambda} \langle [\iota_{m_1+\lambda}, \iota_j \eta] \rangle$$

where $1 \leq j \leq m_1$ and $1 \leq \lambda \leq m_2$. So, we can describe its elements as couples (D_1, D_2) of non-square matrices over $\boldsymbol{Z}/2$. With this notation the element

$$(\Gamma(i_X \iota, \alpha) - (S\alpha)^*)(\zeta_Y^1) = \sum_{j,\lambda} \left(\sum_{t=1}^{m_1} a_{tj} e_t^\lambda \right) [\iota_j, \iota_{m_1+\lambda}\eta],$$

is represented by the matrix $D_1 = E\bar{Q}(\alpha)$ where E is the $m_2 \times m_1$-matrix with entries $e_t^\lambda \in \boldsymbol{Z}/2$ and $\bar{Q}(\alpha)$ denotes the reduction of the matrix $Q(\alpha)$ modulo 2. In particular

$$\dim(\operatorname{im}\{\Gamma(i_X \iota, \alpha) - (S\alpha)^*\}) = m_2 \operatorname{rank} \bar{Q}(\alpha).$$

On the other hand, $\zeta_Y^1 \in \ker(S\alpha)^*$ if and only if $\sum_t \zeta_t S\alpha^t = 0$, that is, if and only if $ET_\alpha = 0$, where T_α the one-column matrix with entries $S\alpha^t \in \pi_{2n}(S^{n+1})$. We define

$$r_\alpha = \dim\left\{ e \in \bigoplus^{m_1} \boldsymbol{Z}/2 : eT_\alpha = 0 \right\} - \dim\left\{ e \in \bigoplus^{m_1} \boldsymbol{Z}/2 : eT_\alpha = 0, e\bar{Q}(\alpha) = 0 \right\}$$

and

$$r_\beta = \dim\left\{ e \in \bigoplus^{m_2} \boldsymbol{Z}/2 : eT_\beta = 0 \right\} - \dim\left\{ e \in \bigoplus^{m_2} \boldsymbol{Z}/2 : eT_\beta = 0, e\bar{Q}(\beta) = 0 \right\}.$$

We have

$$\dim\{\zeta_Y^1 : (S\alpha)^*\zeta_Y^1 = 0\} = \dim\{E \in \mathcal{M}(m_2 \times m_1, \boldsymbol{Z}/2) : ET_\alpha = 0\}$$
$$= m_2 \dim\left\{e \in \bigoplus^{m_1} \boldsymbol{Z}/2 : eT_\alpha = 0\right\}$$

and

$$\dim \Gamma(\iota_X \iota, \alpha)\{\zeta_Y^1 : (S\alpha)^*\zeta_Y^1 = 0\} = m_2 r_\alpha.$$

This together with Proposition 9 proves the following proposition.

PROPOSITION 10. *The group*

$$G = \frac{i_*[S^{2n} \vee S^{2n}, \Omega X * \Omega Y]}{\operatorname{im} \Gamma(\iota \vee \iota', \alpha \vee \beta) \cap i_*[S^{2n} \vee S^{2n}, \Omega X * \Omega Y]},$$

is a free $\boldsymbol{Z}/2$*-module of dimension* $N = 2m_1m_2 - m_2r_\alpha - m_1r_\beta$.

COROLLARY. *Let* $(S\alpha)^* = 0$ *and* $(S\beta)^* = 0$, *then the dimension of the free* $\boldsymbol{Z}/2$*-module* G *is* $N = 2m_1m_2 - m_2 \operatorname{rank} \bar{Q}(\alpha) - m_1 \operatorname{rank} \bar{Q}(\beta))$. *Given further that* $\det Q(\alpha)$ *and* $\det Q(\beta)$ *are both odd, then* $G = 0$ *and*

$$\mathscr{E}(X \vee Y) \cong U, \qquad \text{for } X \not\simeq Y$$
$$\mathscr{E}(X \vee Y) \cong U \rtimes \boldsymbol{Z}/2, \quad \text{for } X \simeq Y.$$

Remark. Where $X = \bigvee S^n \cup_\alpha e^{2n}$ is a manifold, we have $\det Q(\alpha) = \pm 1$ and hence $\operatorname{rank} \bar{Q}(\alpha) = m$. More generally $\operatorname{rank} \bar{Q}(\alpha) = m$ in case $\det Q(\alpha)$ is odd. In the case where n is odd, we only can have $\det Q(\alpha) \neq 0$ if m is even.

§4. Examples

Example 1. $X = Y = \boldsymbol{H}P^2 = S^4 \cup_{\nu_4} e^8$, the quaternionic projective plane. The group $\pi_8(S^4) \cong \boldsymbol{Z}/2 \times \boldsymbol{Z}/2$ is generated by elements $\nu_4\eta_7$ and $S\nu'\eta_7$, where η_k is the generator of $\pi_{k+1}(S^k)$. Since $\eta_3\nu_4 = \nu'\eta_6$ [18, page 44], we have

$$\iota_*\pi_8(S^4) \cong \frac{\pi_8(S^4)}{(\nu_4)_*\pi_8(S^7)} = \boldsymbol{Z}/2 = \{\eta_4 S\nu_4\},$$

and

$$p^*\iota_*\pi_8(S^4) \cong \frac{\pi_8(S^4)}{(\nu_4)_*\pi_8(S^7) + (S\nu_4)^*[S^5, S^4]} = 0.$$

Clearly $\bar{U} \xrightarrow{\pi} U$ has a right inverse and therefore $S(\boldsymbol{H}P^2 \vee \boldsymbol{H}P^2) \cong G \rtimes U$. By [6],

$$\boldsymbol{Z}/2 = \pi_8(\boldsymbol{H}P^2) \overset{\mu}{\cong} \mathscr{E}(\boldsymbol{H}P^2),$$

where the isomorphism is given by $\mu(\xi) = \xi \perp 1$, $\xi \in \pi_8(\boldsymbol{H}P^2)$. Thus

$$U = \mathscr{E}(\boldsymbol{H}P^2) \times \mathscr{E}(\boldsymbol{H}P^2) = \boldsymbol{Z}/2 \times \boldsymbol{Z}/2.$$

The generator of $\pi_5(\boldsymbol{H}P^2)$ is $\imath\eta_4$, and $(S\nu_4)^*\imath\eta_4 = \imath\eta_4(S\nu_4) = \imath(S\nu')\eta_7 \neq 0$ is the generator of $\pi_8(\boldsymbol{H}P^2)$. Therefore, by Proposition 9,

$$G = i_*[S^8 \vee S^8, \Omega \boldsymbol{H}P^2 * \Omega \boldsymbol{H}P^2] = \{[\imath, \imath']\eta_7\} \times \{[\imath, \imath']\eta_7\} = \boldsymbol{Z}/2 \times \boldsymbol{Z}/2.$$

The isomorphism μ shows that, for each self-equivalence f, $\hat{f} = 1$. Moreover, from $\hat{f}\nu_4 = \nu_4\tilde{f}$, we deduce that $\deg \tilde{f} = 1$. Therefore

$$(f \vee k)[\imath, \imath']\eta_7 = (\imath \vee \imath')(\hat{f} \vee \hat{k})[\imath_1, \imath_2]\eta_7 = [\imath, \imath']\eta_7,$$

and, by Proposition 6, the action of U on G is trivial. Finally, by Theorem B,

$$\mathscr{E}(\boldsymbol{H}P^2 \vee \boldsymbol{H}P^2) = (G \rtimes (\mathscr{E}(\boldsymbol{H}P^2) \times \mathscr{E}(\boldsymbol{H}P^2))) \rtimes \boldsymbol{Z}/2 = (\boldsymbol{Z}/2)^4 \rtimes \boldsymbol{Z}/2,$$

where the action is given by

$$(-1).(\gamma_1, \gamma_2; f, k) = (\gamma_2, \gamma_1; k, f).$$

Thus $\mathscr{E}(\boldsymbol{H}P^2 \vee \boldsymbol{H}P^2) \cong D(\boldsymbol{Z}/4 \times \boldsymbol{Z}/4)$, the dihedral extension, where the copies of $\boldsymbol{Z}/4$ are generated by $([\imath, \imath']\eta_7, -1)$ and $(\imath\eta_4(S\nu_4), -1)$ in $(\boldsymbol{Z}/2)^4 \rtimes \boldsymbol{Z}/2$.

Example 2. $X = Y = CP^2 = S^8 \cup_{\sigma_8} e^{16}$, the Cayley projective plane.

The group $\pi_{16}(S^8) = \boldsymbol{Z}/2 \times \boldsymbol{Z}/2 \times \boldsymbol{Z}/2 \times \boldsymbol{Z}/2$ is generated by elements $\sigma_8\eta_{15}$, $(S\sigma')\eta_{15}$, $\bar{\nu}_8$ and ε_8. Since $S(\eta_7\sigma_8) = (S\sigma')\eta_{15} + \bar{\nu}_8 + \varepsilon_8$, [18, page 64], we have

$$\imath_*\pi_{16}(S^8) \cong \frac{\pi_{16}(S^8)}{(\sigma_8)_*\pi_{16}(S^{15})} \cong \boldsymbol{Z}/2 \times \boldsymbol{Z}/2 \times \boldsymbol{Z}/2 \quad \text{and}$$

$$p^*\imath_*\pi_{16}(S^8) \cong \frac{\pi_{16}(S^8)}{(\sigma_8)_*\pi_{16}(S^{15}) + (S\sigma_8)^*[S^9, S^8]} \cong \boldsymbol{Z}/2 \times \boldsymbol{Z}/2$$

generated by $\{\eta_8(S\sigma_8), \bar{\nu}_8, \varepsilon_8\}$ and $\{\bar{\nu}_8, \varepsilon_8\}$ respectively. Therefore $\pi : \bar{U} \to U$ has a right inverse and $S(CP^2 \vee CP^2) = G \rtimes U$. By [9, Example 4.1], $\mathscr{E}(CP^2) \cong \boldsymbol{Z}/2 \times \boldsymbol{Z}/2$, generated by $\mu((S\sigma')\eta_{15})$ and $\mu(\nu_8) = \mu(\eta_8)$, where $\mu(\xi) = \xi \perp 1$. Hence

$$U = \left\{ \begin{pmatrix} f & g \\ h & k \end{pmatrix} \in \begin{pmatrix} \boldsymbol{Z}/2 \times \boldsymbol{Z}/2 & \boldsymbol{Z}/2 \times \boldsymbol{Z}/2 \\ \boldsymbol{Z}/2 \times \boldsymbol{Z}/2 & \boldsymbol{Z}/2 \times \boldsymbol{Z}/2 \end{pmatrix} \right\}.$$

As in Example 1, we have $\tilde{f} = 1$ and $\hat{f} = 1$, for each self-equivalence f. Therefore, U acts trivially on G (Proposition 6), and $U \cong (\boldsymbol{Z}/2)^8$. This is a consequence of Proposition 1, since $f\bar{g}\tilde{k} = f\imath g' = \imath\hat{f}g' = \bar{g}$, and similarly $k\bar{h}\tilde{f} = \bar{h}$. The generator of $\pi_9(CP^2)$ is $\imath\eta_8$ and $(S\sigma_8)^*\imath\eta_8 = \imath\eta_8(S\sigma_8) = \imath((S\sigma')\eta_{15} + \bar{\nu}_8 + \varepsilon_8) \neq 0$. Therefore, by Proposition 9,

$$G = \imath_*[S^{16} \vee S^{16}, \Omega CP^2 * \Omega CP^2] = \{[\imath, \imath']\eta_{15}\} \times \{[\imath, \imath']\eta_{15}\} = \boldsymbol{Z}/2 \times \boldsymbol{Z}/2.$$

Finally, by Theorem B,

$$\mathscr{E}(CP^2 \vee CP^2) = (G \times U) \rtimes \boldsymbol{Z}/2 = (\boldsymbol{Z}/2)^{10} \rtimes \boldsymbol{Z}/2,$$

where the action of $\boldsymbol{Z}/2$ is given by

$$(-1).\left((\gamma_1, \gamma_2), \begin{pmatrix} f & \bar{g} \\ \bar{h} & k \end{pmatrix}\right) = \left((\gamma_2, \gamma_1), \begin{pmatrix} k & \bar{h} \\ \bar{g} & f \end{pmatrix}\right).$$

Example 3. $X = Y = S^n \times S^n = (S^n \vee S^n) \cup_{[\iota_1, \iota_2]} e^{2n}$ $(n \geq 3)$.

In this case $S[\iota_1, \iota_2] = 0$ and X and Y are both manifolds. Therefore $G = 0$, $\mathscr{S}(S^n \times S^n \vee S^n \times S^n) \cong U$, and

$$\mathscr{E}(S^n \times S^n \vee S^n \times S^n) \cong U \rtimes \boldsymbol{Z}/2.$$

Using the isomorphism

$$\begin{aligned} p^* i'_* \pi_{2n}(S^n \vee S^n) &\cong \frac{\pi_{2n}(S^n \vee S^n)}{\beta_* \pi_{2n}(S^{2n-1}) + (S\alpha)^*[S^{n+1} \vee S^{n+1}, S^n \vee S^n]} \\ &\cong \pi_{2n}(S^n) \times \pi_{2n}(S^n), \end{aligned}$$

we have

$$U \cong \begin{pmatrix} \mathscr{E}(X) & \pi_{2n}(S^n) \times \pi_{2n}(S^n) \\ \pi_{2n}(S^n) \times \pi_{2n}(S^n) & \mathscr{E}(Y) \end{pmatrix}$$

with the semi-direct product structure given in Proposition 1. The action of $\boldsymbol{Z}/2$ on U is again given by

$$(-1).\begin{pmatrix} f & g \\ h & k \end{pmatrix} = \begin{pmatrix} k & h \\ g & f \end{pmatrix}.$$

The groups $\mathscr{E}(S^n \times S^n)$ have been computed (see [6] and [16]). For $n = 5$, we have

$$\mathscr{E}(S^5 \times S^5) \cong \mathrm{Sym} = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \subset GL(2, \boldsymbol{Z}).$$

Example 4. $X = Y = S^n \cup_{[\iota, \iota]} e^{2n}$ $(n \geq 3)$, where ι is the generator of $\pi_n(S^n)$. Again $S[\iota, \iota] = 0$ and $\bar{Q}([\iota, \iota]) = 0$. Thus, in Proposition 10, $r_{[\iota, \iota]} = 0$ and

$$G = i_*[S^{2n} \vee S^{2n}, \Omega X * \Omega X] = \{[\iota, \iota']\eta_{2n-1}\} \times \{[\iota, \iota']\eta_{2n-1}\} = \boldsymbol{Z}/2 \times \boldsymbol{Z}/2.$$

We also have $\bar{U} \cong U$ and $\mathscr{S}(X \vee X) \cong G \rtimes U$. By Proposition 6, the action is given by

$$(f \vee k)[\iota, \iota']\eta_{2n-1} = (\iota \vee \iota')(\hat{f} \vee \hat{k})[\iota_1, \iota_2]\eta_{2n-1} = (\deg \hat{f} \deg \hat{k})[\iota, \iota']\eta_{2n-1}.$$

Let K_n be the group $p'^* \iota_* \pi_{2n} S^n \cong \dfrac{\pi_{2n} S^n}{\{[\iota, \iota]\eta_{2n-1}\}}$, so that $U = \begin{pmatrix} \mathscr{E}(X) & K_n \\ K_n & \mathscr{E}(Y) \end{pmatrix}$ is a semi-direct product, as in Proposition 1. Here

$$\{[\iota,\iota]\eta_{2n-1}\} = \begin{cases} 0, & \text{for } n \equiv -1(4) \text{ or } n = 2,6, \\ \mathbf{Z}/2, & \text{otherwise.} \end{cases}$$

In the case where n is even, we have $\tilde{f} = 1$ for each self-equivalence f. In the case where $\iota_*\pi_{2n}S^n = \mathbf{Z}/2$ (e.g. $n = 2, 6, 12, \ldots$), $\mathscr{E}(X)$ acts trivially on this group and $U = \mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathscr{E}(X) \times \mathscr{E}(X)$. Also (see [9, Example 4.4])

$$\mathscr{E}(S^n \cup_{[\iota,\iota]} e^{2n}) = \begin{cases} D(K_n) \times \mathbf{Z}/2, & \text{for } n \text{ odd,} \\ D(K_n), & \text{for } n \text{ even.} \end{cases}$$

By Theorem B, we have

$$\mathscr{E}(X \vee X) \cong (G \rtimes U) \rtimes \mathbf{Z}/2.$$

Example 5. $X = S^3 \times S^3$, $Y = (S^3 \vee S^3) \cup_\beta e^6$, with $\beta = \iota_1\eta^2 + \iota_2\eta^2 + [\iota_1, \iota_2]$. We have $Q(\alpha) = Q(\beta) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $T_\alpha = 0$ and $T_\beta = \begin{pmatrix} \eta^2 \\ \eta^2 \end{pmatrix}$. Hence, with the notation of Proposition 10, $r_\alpha = 2$ and $r_\beta = 1$, so that $G = (\mathbf{Z}/2)^2$. On the other hand, $R_{\alpha,\beta} = 0$ and

$$R_{\beta,\alpha} = (S\beta)^*[S^4 \vee S^4, S^3 \vee S^3] = \langle\eta^3\rangle \oplus \langle\eta^3\rangle \subset \pi_6(S^3) \oplus \pi_6(S^3) \subset \pi_6(S^3 \vee S^3).$$

By obstruction theory we have that the composite $\pi_6(S^3 \vee S^3) \to \mathscr{E}(X \vee Y) \to [X \vee Y, X \times Y]$ has trivial kernel and therefore the map $s' : R_{\beta,\alpha} \times R_{\alpha,\beta} \to G$ is an isomorphism. By Proposition 7, $\mathscr{E}(X \vee Y)$ is a semidirect-product if and only if s' has a extension to a derivation from $\bar{U}$ to G. By Proposition 6, the subgroup $\iota_*\pi_{2n}(\bigvee^m S^n) \times \iota'_*\pi_{2n}(\bigvee^m S^n)$ of $\bar{U}$ always acts trivially on G. Hence, each derivation from $\bar{U}$ to G is a homomorphism on this subgroup. In our example s' has no extension to a homomorphism on $\iota_*\pi_6(S^3 \vee S^3) \times \iota'_*\pi_6(S^3 \vee S^3)$ since

$$R_{\beta,\alpha} = \mathbf{Z}/2 \times \mathbf{Z}/2 \subset \iota_*\pi_6(S^3 \vee S^3) \cong \mathbf{Z}/(12) \times \mathbf{Z}/(12).$$

Therefore $\mathscr{E}(X \vee Y)$ is not a semi-direct product of G by U.

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