J.S. JUNG AND T.H. KIM
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STRONG CONVERGENCE OF APPROXIMATING FIXED POINTS FOR NONEXPANSIVE NONSELF-MAPPINGS IN BANACH SPACES

JONG SOO JUNG AND TAE HWA KIM

Abstract

Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T\ C \to E$ a nonexpansive mapping satisfying the inwardness condition. Assume that every weakly compact convex subset of E has the fixed point property. For $u \in C$ and $t \in (0,1)$, let x_t be a unique fixed point of a contraction $G_t\ C \to E$, defined by $G_t x = tTx + (1-t)u$, $x \in C$. It is proved that if $\{x_t\}$ is bounded, then the strong $\lim_{t\to 1} x_t$ exists and belongs to the fixed point set of T Furthermore, the strong convergence of other two schemes involving the sunny nonexpansive retraction is also given in a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm.

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E, and let $T: C \to E$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$). Given a $u \in C$ and a $t \in (0,1)$, we can define a contraction $G_t : C \to E$ by

$$(1) \qquad G_t x = tTx + (1-t)u, \quad x \in C.$$

If T is a self-mapping (i.e., $T(C) \subset C$), then G_t maps C into itself, and hence, by Banach's contraction principle, G_t has a unique fixed point x_t in C, that is, we have

$$(2) \qquad x_t = tTx_t + (1-t)u.$$

(Such a sequence $\{x_t\}$ is said to be an approximating fixed point of T since it possesses the property that if $\{x_t\}$ is bounded, then $\lim_{t\to 1}\|Tx_t - x_t\| = 0$.) The strong convergence of $\{x_t\}$ as $t \to 1$ for a self-mapping T of a bounded C was proved in a Hilbert space independently by Browder [2] and Halpern [10] and in

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a uniformly smooth Banach space by Reich [18]. Thereafter, Singh and Watson [21] extended the result of Browder and Halpern to a nonexpansive nonself-mapping T satisfying Rothe's boundary condition: $T(\partial C) \subset C$ (here ∂C denotes the boundary of C).

Recently, Xu and Yin [27] proved that if C is a nonempty closed convex (not necessarily bounded) subset of Hilbert space H, if $T : C \to H$ is a nonexpansive nonself-mapping, and if $\{x_t\}$ is the sequence defined by (2) which is bounded, then $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T. They also studied other two schemes involving the nearest point projection P from H onto C, which were introduced by Marino and Trombetta [10]. Jung and Kim [11] extended Xu and Yin's results [27] to a uniformly convex Banach space with a uniformly Gâteaux differentiable norm with the additional condition upon C. Kim and Takahashi [12] also generalized Xu and Yin's results [27] to a smooth and reflexive Banach space with a weakly sequentially continuous duality mapping.

Very recently, Xu [26] showed that if E is a uniformly smooth Banach space, if C is a nonempty closed convex subset of E, and if $T : C \to E$ is a nonexpansive nonself-mapping with a fixed point, which satisfies the inwardness condition, then the sequence $\{x_t\}$ defined by (2) converges strongly as $t \to 1$ to a fixed point of T. He also gave the strong convergence theorem in a uniformly convex and uniformly smooth Banach space with the weak inwardness condition upon the mapping T.

In this paper, we establish the strong convergence of $\{x_t\}$ defined by (2) for a nonexpansive nonself-mapping T in a reflexive Banach space with a uniformly Gâteaux differentiable norm. We also prove the strong convergence of other two schemes studied in [12, 13, 27] in a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Our results extend and improve the results in [18, 26, 27].

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $x^* \in E^*$ at $x \in E$ will be denoted by (x, x^*).

A Banach space E is called *strictly convex* if its unit sphere $U = \{x \in E : \|x\| = 1\}$ does not contain any linear segment. For every ε with $0 \le \varepsilon \le 2$, the modulus $\delta(\varepsilon)$ of convexity of E is defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1,\ \|y\| \le 1,\ \|x-y\| \ge \varepsilon\right\}.$$

E is said to be *uniformly convex* if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then E is reflexive and strictly convex.

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{3}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be *uniformly Gâteaux differentiable* if, for each $y \in U$, this limit is attained uniformly for $x \in U$. Finally, the norm is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit in (3) is attained uniformly for $(x, y) \in U \times U$. Since the dual E^* of E is uniformly convex if and only if the norm of E is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm. The converse implication is false. A discussion of these and related concepts may be found in [5].

The (normalized) *duality* mapping J from E into the family of nonempty (by Hahn-Banach theorem) weak-star compact subsets of its dual E^* is defined by

$$J(x) = \{f \in E^* : (x, f) = \|x\|^2 = \|f\|^2\}.$$

for each $x \in E$. It is single valued if and only if E is smooth. It is also well-known that if E has a uniformly Gâteaux differentiable norm, J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak-star topology of E^*. This fact is explicitly proved in Lemma 2.2 of [19] (see also [4, 6, 7]).

Let μ be a mean on positive integers $\mathbf{N}$, i.e., a continuous linear functional on ℓ^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we known that μ is a mean on $\mathbf{N}$ if and only if

$$\inf\{a_n : n \in \mathbf{N}\} \le \mu(a) \le \sup\{a_n : n \in \mathbf{N}\}$$

for every $a = (a_1, a_2, \ldots) \in \ell^\infty$. According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on $\mathbf{N}$ is called a *Banach limit* if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, \ldots) \in \ell^\infty$. Using the Hahn-Banach theorem, we can prove the existence of a Banach limit. We know that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \le \mu_n(a_n) \le \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, \ldots) \in \ell^\infty$. Let $\{x_n\}$ be a bounded sequence in E. Then we can define the real valued continuous convex function ϕ on E by

$$\phi(z) = \mu_n\|x_n - z\|^2$$

for each $z \in E$.

The following lemma which was given in [8, 9, 23] is, in fact, a variant of Lemma 1.3 in [17] (cf. [20, p. 171]).

LEMMA 1. *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm and let $\{x_n\}$ be a bounded sequence in E. Let μ be a Banach limit and $u \in C$. Then*

$$\mu_n\|x_n - u\|^2 = \min_{y\in C} \mu_n\|x_n - y\|^2$$

if and only if

$$\mu_n(x - u, J(x_n - u)) \leq 0$$

for all $x \in C$.

Let $I_C(x)$ be the inward set of a closed convex subset C of E at x given by

$$I_C(x) = \{z \in E : z = x + \lambda(y - x) \text{ for some } y \in C, \lambda \geq 0\}.$$

A nonself-mapping $T : C \to E$ is said to satisfy the *inwardness condition* if $Tx \in I_C(x)$ for all $x \in C$. T is also said to satisfy the *weak inwardness condition* if $Tx \in \operatorname{cl} I_C(x)$ for all $x \in C$, where $\operatorname{cl} I_C(x)$ is the closure of $I_C(x)$ in norm topology.

Recall that a closed convex subset C of E is said to have the fixed point property for nonexpansive self-mappings (FPP for short) if every nonexpansive mapping $T : C \to C$ has a fixed point, that is, there is a point $p \in C$ such that $Tp = p$. It is well-known that every bounded closed convex subset of a uniformly convex Banach space has the FPP (cf. [7, p. 22]).

Finally, let C be a nonempty closed convex subset of E. A mapping Q of C into C is said to be a *retraction* if $Q^2 = Q$. If a mapping Q of C into C is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q. Let Q be a retraction of E onto a closed subset C of E. Q is said to be *sunny* if each point on the ray $\{Qx + t(x - Qx) : t > 0\}$ is mapped by Q back onto Qx, in other words,

$$Q(Qx + t(x - Qx)) = Qx$$

for all $t \geq 0$ and $x \in E$. If there exists a retraction $Q : E \to C$ which is both sunny and nonexpansive, then C is said to be a *sunny nonexpansive retract*. Sunny nonexpansive retracts appear in [16, 17].

The following lemma is well-known (cf. [7, p. 48; 14, p. 65]).

LEMMA 2. *Let C be a closed convex subset of a smooth Banach space E and let $Q : E \to C$ be a retraction. Then the following the equivalent*:

(a) $(x - Qx, J(y - Qx)) \leq 0$ *for all* $x \in E$ *and* $y \in C$;

(b) $\|Qz - Qw\|^2 \leq (z - w, J(Qz - Qw))$ *for all* z *and* w *in* E;

(c) Q *is both sunny and nonexpansive.*

3. Main results

In this section, we study the strong convergence of $\{x_t\}$ defined by (2) in a reflexive Banach space with a uniformly Gâteaux differentiable norm.

Now, we state and prove the first main result.

THEOREM 1. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to E$ a nonexpansive nonself-mapping satisfying the inwardness condition. Assume that* *every weakly compact convex subset of E has the FPP. Suppose that for each $u \in C$ and $t \in (0,1)$, the contraction G_t defined by (1) has a (unique) fixed point $x_t \in C$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$ and in this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point to T.*

Proof. If the fixed point set $F(T)$ of T is nonempty, then $\{x_t\}$ is bounded. In fact, we have $\|x_t - v\| \le \|u - v\|$ for all $t \in (0,1)$ and $v \in F(T)$.

Suppose conversely that $\{x_t\}$ remains bounded as $t \to 1$. We now show that $F(T)$ is nonempty and that $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T. To this end, we follow ideas of [22] and [23]. Let $t_n \to 1$ and $x_n = xt_n$. Define $\phi : E \to [0, \infty)$ by $\phi(z) = \mu_n\|x_n - z\|^2$. Since ϕ is continuous and convex, $\phi(z) \to \infty$ as $\|z\| \to \infty$, and E is reflexive, ϕ attains its infimum over C (cf. [1, p. 79]). Let $z \in C$ be such that

$$\mu_n\|x_n - z\|^2 = \min_{y \in C} \mu_n\|x_n - y\|^2$$

and let

$$M = \left\{x \in C : \mu_n\|x_n - x\|^2 = \min_{y \in C} \mu_n\|x_n - y\|^2\right\}.$$

Then M is a nonempty bounded closed convex subset of C. Since

$$\|x_t - Tx_t\| = (1-t)\|Tx_t - x\| \to 0 \text{ as } t \to 1, \tag{4}$$

we have for $x \in C$

$$\begin{aligned} \phi(Tx) = \mu_n\|x_n - Tx\|^2 &= \mu_n\|Tx_n - Tx\|^2 \\ &\le \mu_n\|x_n - x\|^2 = \phi(x). \end{aligned} \tag{5}$$

Now we prove that the inwardness condition of T on C implies the inwardness condition of T on M; that is,

$$Tx \in I_M(x) \quad \text{for } x \in M. \tag{6}$$

In fact, let $x \in M$. The inwardness condition of T on C implies that $Tx = x + \lambda(y - x)$ for some $y \in C$ and $\lambda \ge 0$. If $\lambda \le 1$, then $Tx \in C$ by convexity of C. From (5), it follows that $Tx \in M \subset I_M(x)$ and (6) is verified. Assume $\lambda > 1$, we can write y in the form $y = rTx + (1-r)x$, where $r = \lambda^{-1} \in (0,1)$. By convexity of f and (5), we obtain

$$\phi(y) \le r\phi(Tx) + (1-r)\phi(x) \le \phi(x) \quad \text{for } x \in M.$$

This implies that $y \in M$ and therefore $Tx = x + \lambda(y - x)$ belongs to $I_M(x)$ for $x \in M$ and (6) is proved. Thus it follows from Theorem 16.1 of Goebel and Reich [7] that T has a fixed point $z \in M$, that is, $F(T)$ is nonempty. On the other hand, for $v \in F(T)$, we have

$$\begin{aligned}(x_n - Tx_n, J(x_n - v) &= (x_n - Tv + Tv - Tx_n, J(x_n - v)) \\ &= \|x_n - Tv\|^2 - (Tx_n - Tv, J(x_n - v)) \\ &\geq \|x_n - Tv\|^2 - \|Tx_n - Tv\| \, \|x_n - v\| \\ &\geq \|x_n - Tv\|^2 - \|x_n - Tv\|^2 = 0\end{aligned}$$

for all n. Since $x_n - Tx_n = (1 - t_n)(u - Tx_n)$, we get from the above inequality

$$\begin{aligned}0 &\leq (x_n - Tx_n, J(x_n - v)) \\ &= (1 - t_n)(u - Tx_n, J(x_n - v))\end{aligned} \tag{7}$$

for all $v \in F(T)$ and all n. Thus from (4) and (7), we obtain

$$\mu_n(x_n - u, J(x_n - v)) \leq 0 \tag{8}$$

for $v \in F(T)$. From Lemma 1, it follows that

$$\mu_n(x - z, J(x_n - z)) \leq 0$$

for all $x \in C$. In particular, we have

$$\mu_n(u - z, J(x_n - z)) \leq 0. \tag{9}$$

Combining (8) and (9), we get

$$\mu_n(x_n - z, J(x_n - z)) = \mu_n\|x_n - z\|^2 \leq 0.$$

Therefore, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges strongly to z. To complete the proof, suppose that there is another subsequence $\{x_{n_l}\}$ of $\{x_n\}$ which converges strongly to (say) y. Then y is a fixed point of T by (4). It follows from (8) that

$$(z - u, J(z - y)) \leq 0$$

and

$$(y - u, J(y - z)) \leq 0.$$

Adding these two inequalities yields

$$(z - y, J(z - y)) = \|z - y\|^2 \leq 0$$

and thus $z = y$. This prove the strong convergence of $\{x_t\}$ to z.

COROLLARY 1 [26]. *Let E be a uniformly smooth Banach space, C a nonempty closed convex subset of E, and $T : C \to E$ a nonexpansive nonself-mapping satisfying the inwardness condition. Suppose that for each $u \in C$ and $t \in (0, 1)$, the contraction G_t defined by (1) has a (unique) fixed point $x_t \in C$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$ and in this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.*

For the second main result, we need the following result which was essentially proved by Takahashi and Jeong [24] and here present the brief proof for the sake of completeness.

LEMMA 3. *Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E, and $\{x_n\}$ a bounded sequence of E. Then the set*

$$M = \left\{ u \in C : \mu_n \|x_n - u\|^2 = \min_{z \in C} \mu_n \|x_n - z\|^2 \right\}$$

consists of one point.

Proof. Let $\phi(z) = \mu_n \|x_n - z\|^2$ for each $z \in E$ and $r = \inf\{\phi(z) : z \in C\}$. Then, since the function ϕ on C is convex and continuous, $\phi(z) \to \infty$ as $\|z\| \to \infty$, and E is reflexive, it follows from [1, p. 79] that there exists $u \in C$ with $\phi(u) = r$. Therefore M is nonempty. By Theorem 2 of [25], $\|\cdot\|^2$ is uniformly convex on any bounded subset of E; especially, we have a continuous increasing function $g = g_r : [0, \infty) \to [0, \infty)$, with $g(0) = 0$, such that

$$\|\lambda x + (1-\lambda) y\|^2 \le \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda) g(\|x - y\|), \quad 0 \le \lambda \le 1, \quad x, y \in B_r,$$

where B_r is the closed ball centered at 0 and with radius r that is big enough so that B_r contains $\{x_n\}$. It follows that

$$\phi(\lambda x + (1-\lambda) y) \le \lambda \phi(x) + (1-\lambda)\phi(y) - \lambda(1-\lambda) g(\|x - y\|), \quad 0 \le \lambda \le 1, \quad x, y \in B_r.$$

This implies that ϕ is a strictly convex function on E. Thus the minimum point u of ϕ is unique, that is, M consists of one point.

THEOREM 2. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to E$ a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that for each $u \in C$ and $t \in (0,1)$, The contraction G_t defined by (1) has a (unique) fixed point $x_t \in C$. If the fixed point set $F(T)$ of T is nonempty, then $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.*

Proof. Let $w \in F(T)$. As in proof of Theorem 1, we have $\|x_t - w\| \le \|u - w\|$ for all $t \in (0,1)$ and hence $\{x_t\}$ is bounded. We now show that $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T. To this end, let $t_n \to 1$ and $x_n = x_{t_n}$. As in the proof of Theorem 1, we define the same function $\phi : E \to [0, \infty)$ by $\phi(z) = \mu_n \|x_n - z\|^2$ and let

$$M = \left\{ x \in C : \mu_n \|x_n - x\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2 \right\}.$$

Then, by Lemma 3, we know that M consists of one point, say z. We must show that this z is a fixed point of T. Since T satisfies the weak inwardness condition, there are some $v_n \in C$ and $\lambda_n \geq 0$ such that

$$w_n := z + \lambda_n(v_n - z) \to Tz \text{ strongly.}$$

If $\lambda_n \leq 1$ for infinitely many n and these n, then we have $w_n \in C$ and hence $Tz \in C$. We have $Tz = z$ by (5). So, we may assume $\lambda_n > 1$ for all sufficiently large n. We then write

$$v_n = r_n w_n + (1 - r_n)z,$$

where $r_n = \lambda_n^{-1}$. Suppose $r_n \to 1$. Then $v_n \to Tz$ and hence $Tz \in C$. By (5), we have $Tz = z$. So, without loss of generality, we may assume $r_n \leq a < 1$. By Theorem 2 of [25], $\|\cdot\|^2$ is uniformly convex on any bounded subset of E; especially, we have a continuous increasing function $g = g_r : [0, \infty) \to [0, \infty)$, with $g(0) = 0$, such that

$$\|\lambda x+(1-\lambda)y\|^2 \leq \lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)g(\|x-y\|), \quad 0 \leq \lambda \leq 1, \quad x, y \in B_r,$$

where B_r is the closed ball centered at 0 and with radius r such that B_r contains z and $\{w_n\}$. It follows that

$$\phi(\lambda x+(1-\lambda)y) \leq \lambda\phi(x)+(1-\lambda)\phi(y)-\lambda(1-\lambda)g(\|x-y\|) \quad 0 \leq \lambda \leq 1, \quad x, y \in B_r.$$

Noting $v_n \in C$, we derive that

$$\begin{aligned}\phi(z) &\leq \phi(v_n)\\ &\leq r_n\phi(w_n) + (1 - r_n)\phi(z) - r_n(1 - r_n)g(\|w_n - z\|)\end{aligned}$$

and hence

$$(1 - a)g(\|w_n - z\|) \leq (1 - r_n)g(\|w_n - z\|).$$

Taking limit as $n \to \infty$, we obtain

$$g(\|Tz - z\|) \leq \phi(Tz) - \phi(z) \leq 0$$

by (5). Therefore, $Tz = z$, that is, z is a fixed point of T. The proof of the strong convergence of $\{x_t\}$ to z is the same as given in the proof of Theorem 1.

Remark 1. (1) Theorem 1 generalizes Xu and Yin's result [27, Theorem 1] to a Banach space setting.

(2) Corollary 1 extends Reich's result [18] to nonself-mappings.

(3) Theorem 2 also improves slightly Theorem 2 in [26].

(4) To guarantee the existence of a fixed point of the contraction G_t defined by (1), the weak inwardness condition upon the mapping T is used. In fact, it is well-known (cf. [7, 15]) that if C, a bounded closed convex subset of a Banach space E, has the FPP and a nonexpansive $T : C \to E$ is weakly inward, then the contraction G_t has a fixed point for every $t \in (0, 1)$. Hence we have the following corollary.

COROLLARY 2. *Let E, C, T be as in Theorem 2. Suppose in addition that C is bounded. Then for each $u \in C$, the sequence $\{x_t\}$ defined by* (2) *converges strongly as $t \to 1$ to a fixed point of T.*

Remark 2. (1) Corollary 2 generalizes Corollary 1 in [27] to a Banach space setting.

(2) Since Rothe's boundary condition: $T(\partial C) \subset C$ implies the weak inwardness condition, Corollary 2 also improves upon Theorem in [21].

Next, we denote by Q the sunny and nonexpansive retraction of E onto C. Now let $T : C \to E$ be nonexpansive and let $u \in C$ be fixed. Following Marino and Trombetta [10], we define the contraction U_t from C into itself by

$$U_t x = tQT(x) + (1-t)u, \quad x \in C$$

and

$$R_t x = Q(tTx + (1-t)u), \quad x \in C.$$

Then Banach's contraction principle yields a unique point x_t (resp. y_t) $\in C$ that is fixed by U_t (resp. R_t), that is, we have

$$x_t = tQT(x_t) + (1-t)u \tag{10}$$

and

$$y_t = Q(tTy_t + (1-t)u). \tag{11}$$

The following lemma is well-known (cf. [1, p. 79; 7, p. 12]).

LEMMA 4. *Let C be a closed convex of a reflexive and strictly convex Banach space E. Then $C^o = \{x \in C : \|x\| = \inf\{\|y\| : y \in C\}\}$ is a singleton.*

THEOREM 3. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to E$ a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E, and that for some $u \in C$ and each $t \in (0,1)$, x_t is a (unique) fixed point of the contraction U_t defined by* (10), *where Q is a sunny nonexpansive retraction of E onto C. If the fixed point set of T ıs nonempty, then $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.*

Proof. We follow an idea of [22]. Let $w \in F(T)$. Then it is easily seen that $\|x_t - w\| \le \|u - w\|$ for all $t \in (0,1)$ and hence $\{x_t\}$ is bounded. As in the proof of Theorem 1, we define the same function $\phi \cdot C \to [0, \infty)$ by $\phi(z) = \mu_n \|x_n - z\|^2$ and let

$$M = \left\{ x \in C : \mu_n \|x_n - x\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2 \right\}.$$

Then M is invariant under QT. In fact, since

$$\|x_t - QTx_t\| = (1-t)\|QTx_t - x\| \to 0 \quad \text{as } t \to 1,$$

we have for $x \in M$

$$\begin{aligned}\phi(QTx) = \mu_n\|x_n - QTx\|^2 &= \mu_n\|QTx_n - QTx\|^2 \\ &\leq \mu_n\|x_n - x\|^2 = \phi(x),\end{aligned}$$

and hence $QTx \in M$ because $QTx \in C$. Furthermore, M contains a fixed point of QT. To this end, define

$$M^o = \left\{ v \in M : \|v - w\| = \min_{y \in M} \|w - y\| \right\}.$$

Then, by Lemma 4, M^o is a singleton. Denote such a singleton by z. Then we have

$$\|QTz - w\| = \|QTz - QTw\| \leq \|z - w\|$$

and hence $QTz = z$. Applying the method of the proof of Theorem 1 to the nonexpansive mapping QT, we obtain that $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point z of QT. It remains to show that z is a fixed point of T. Indeed, since Q is sunny and nonexpansive retraction, from Lemma 2, we get

(12) $$(Tz - z, J(z - y)) \geq 0 \quad \text{for all } y \in C.$$

On the other hand, Tz belongs to $\operatorname{cl} I_C(z)$ by the weak inwardness condition. Hence for each integer $n \geq 1$, there exist $z_n \in C$ and $a_n \geq 0$ such that

(13) $$y_n := z + a_n(z_n - z) \to Tz \text{ strongly}.$$

Since E has a uniformly Gâteaux differentiable norm, J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak-star topology of E^*. Thus it follows from (12) and (13) that

$$\begin{aligned}0 &\leq (Tz - z, a_nJ(z - z_n)) \\ &= (Tz - z, J(a_n(z - z_n))) \\ &= (Tz - z, J(z - y_n)) \to (Tz - z, J(z, Tz)) = -\|Tz - z\|^2.\end{aligned}$$

Hence we have $Tz = z$.

THEOREM 4. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to E$ a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E, and that for some $u \in C$ and each $t \in (0,1)$, y_t is a (unique) fixed point of the contraction R_t defined by* (11), *where Q is a sunny nonexpansive retraction of E onto C. If the fixed point set of T is nonempty, then $\{y_t\}$ converges strongly as $t \to 1$ to a fixed point of T.*

Proof. The proof follows an idea of [22]. Let x be a fixed point of T. Then we have

$$\begin{aligned}\|x - y_t\| &= \|Qx - Q(tTy_t + (1-t)u)\| \\ &\le t\|x - Ty_t\| + (1-t)\|x - u\| \\ &\le t\|x - y_t\| + (1-t)\|x - u\|\end{aligned}$$

and hence $\|x - y_t\| \le \|x - u\|$ for all $t \in (0,1)$. So $\{y_t\}$ is bounded. We now show that $\{y_t\}$ converges strongly as $t \to 1$ to a fixed point of T. To this end, let $t_n \to 1$ and $y_n = y_{t_n}$. As in proof of Theorem 1, define $\phi : C \to [0,\infty)$ by $\phi(z) = \mu_n\|y_n - z\|^2$ for each $z \in C$ and let

$$M = \{u \in C : \mu_n\|y_n - u\|^2 = \min_{y\in C} \mu_n\|y_n - y\|^2\}.$$

Then M is invariant under QT. In fact, since $\{Ty_t\}$ is bounded and

$$\begin{aligned}\|y_t - QTy_t\| &\le \|tTy_t + (1-t)u - Ty_t\| \\ &= (1-t)\|u - Ty_t\|,\end{aligned} \tag{14}$$

we have $y_t - QTy_t \to 0$. So, we have for $z \in M$,

$$\begin{aligned}\|y_n - QTz\| &\le \|y_n - QTy_n\| + \|QTy_n - QTz\| \\ &\le \|y_n - z\| + \|y_n - QTy_n\|\end{aligned}$$

and hence

$$\mu_n\|y_n - QTz\|^2 \le \mu_n\|y_n - z\|^2.$$

Then $QTz \in M$ because $QTz \in C$. Furthermore, by the proof of Theorem 3, we know that M contains a fixed point of QT, that is, there is a point z such that $QTz = z$. Since Q is sunny and nonexpansive retraction, from Lemma 2, we have

$$(Tz - z, J(z - w)) \ge 0 \quad \text{for all } w \in C.$$

On the other hand, Tz belongs to $\mathrm{cl}\, I_C(z)$ by the weak inwardness condition. Hence for each integer $n \ge 1$, there exist $z_n \in C$ and $a_n \ge 0$ such that

$$x_n := z + a_n(z_n - z) \to Tz \text{ strongly}.$$

As in the proof of Theorem 3, we have $Tz = z$. For any $v \in F(T)$, we have

$$t(v - u) + u = tv + (1-t)u = Q(tv + (1-t)u)$$

and hence

$$\|(y_t - u) + t(v - u)\|^2 = \|Q(tTy_t + (1-t)u) - u - t(v - u)\|^2$$

$$
\begin{aligned}
&= \|Q(tTy_y - u) + u) - Q(t(v - u) + u)\|^2 \\
&\le \|t(Ty_t - u) - t(v - u)\|^2 \\
&\le t^2\|y_t - v\|^2 \\
&= t^2\|(y_t - u) - (v - u)\|^2.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
0 &\ge \|(y_t - u) - t(v - u)\|^2 - \|t(y_t - u) - t(v - u)\|^2 \\
&\ge 2((1 - t)(y_t - u), J(t(y_t - v))) \\
&= 2(1 - t)t(y_t - u, J(y_t - v))
\end{aligned}
$$

and hence

$$(y_t - u, J(y_t - v)) \le 0 \tag{15}$$

for $v \in F(T)$. From Lemma 1, it follows that

$$\mu_n(x - z, J(y_n - z)) \le 0$$

for all $x \in C$. In particular, we have

$$\mu_n(u - z, J(y_n - z)) \le 0. \tag{16}$$

Combining (15) with $v = z$ and (16), we get

$$\mu_n(y_n - z, J(y_n - z)) = \mu_n\|y_n - z\|^2 \le 0.$$

Therefore, there is a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ which converges strongly to z. Suppose that there is another subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which converges strongly to (say) y. Then y is a fixed point of QT by (14) and also of T. Thus, as in the proof of Theorem 1, we have $z = y$ and hence $y_t \to z$.

COROLLARY 3. *Let E, C, T, Q be as in Theorem 3 (resp., Theorem 4). Suppose in addition that C is bounded and that every weakly compact convex subset of E has the FPP. Then for each $u \in C$, the sequence $\{x_t\}$ (resp., $\{y_t\}$)* defined by (10) *(resp., (11)) converges strongly as $t \to 1$ a fixed point of T.*

COROLLARY 4 [27]. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $T : C \to H$ a nonexpansive nonself-mapping satisfying the weak inwardness condition, $P : H \to C$ the nearest point projection, and $\{x_t\}$ the sequence (resp., $\{y_t\}$) defined by (10) (resp., (11)) with P instead of Q. If T has a fixed point, then $\{x_t\}$ (resp., $\{y_t\}$) converges strongly as $t \to 1$ to a fixed point of T.*

Proof. Note that the nearest point projection P of a Hilbert space H onto a closed convex subset C is a sunny and nonexpansive retraction. Thus the result follows from Theorem 3 (resp., Theorem 4).

Remark 3. Theorem 2, Theorem 3 and Theorem 4 apply to all uniformly convex and uniformly smooth Banach spaces and in particular, to all L^p spaces, $1 < p < \infty$.

Note added in proof. 1. Since E is uniformly convex, the existence of the minimum in proofs of Lemma 3 and Theorem 2 also follows from [7, Proposition 2.2].

2. Since $\{x_t\}$ is a bounded approximating sequence and E is uniformly convex, the existence of a fixed point of T in proof of Theorem 2 also follows from Browder's demiclosedness principle [3].

3. The authors noticed, in the process of referring, the fact that Theorem 3 and 4 were proved in [22] with no assumption of strict convexity of E, using the stronger version of Theorem 1 for the self-mapping $T : C \to C$, where C is a nonempty closed convex subset of E which has normal structure.

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REFERENCES

[1] V Barbu and Th. Precupanu, Convexity and Optimization in Banach Spaces, Editura Academiei R. S. R., Bucharest, 1978.

[2] F.E. Browder, Convergence of approximations to fixed points of nonexpansive mappings in Banach spaces, Arch. Rational Mech. Anal., **24** (1967), 82–90.

[3] F.E. Browder, Semicontractive and semiaccretive nonlinear mappings in Banach spaces, Bull. Amer. Math. Soc., **74** (1968), 660–665.

[4] D.F Cudia, The geometry of Banach spaces, smoothness, Trans. Amer. Math. Soc., **110** (1964), 284–314.

[5] M.M. Day, Normed Linear Spaces, 3rd ed., Springer-Verlag, Berlin-New York, 1973.

[6] J. Diestel, Geometry of Banach Spaces, Lectures Notes in Math., **485**, Springer Verlag, Berlin-Heidelberg, 1975.

[7] K. Goebel and S. Reich, Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings, Marcel Dekker, New York-Basel, 1984.

[8] K.S. Ha and J.S. Jung, On generators and nonlinear semigroups in Banach spaces, J. Korean Math. Soc., **25** (1988), 245–257

[9] K.S. Ha and J.S. Jung, Strong convergence theorems for accretive operators in Banach spaces, J. Math. Anal. Appl., **147** (1990), 330–339.

[10] B. Halpern, Fixed points of nonexpansive maps, Bull. Amer. Math. Soc., **73** (1967), 957–961.

[11] J.S. Jung and S.S. Kim, Strong convergence theorems for nonexpansive nonself-mappings in Banach spaces, to appear in Nonlinear Anal.

[12] G.E. Kim and W Takahashi, Strong convergence theorems for nonexpansive nonself-mappings in Banach spaces, Nihonkai Math. J., **7** (1996), 63–72.

[13] G. Marino and G. Trombetta, On approximating fixed points for nonexpansive maps, Indian J. Math., **34** (1992), 91–98.

[14] S. Reich, Asymptotic behavior of contractions in Banach spaces, J. Math. Anal. Appl., **44** (1973), 227–290.

[15] S. Reich, Fixed points of condensing functions, J. Math. Anal. Appl., **41** (1973), 460–467

[16] S. Reich, An iterative procedure for constructing zeros of accretive sets in Banach spaces, Nonlinear Anal., **2** (1978), 85–92.

[17] S. Reich, Product formulas, nonlinear semigroups and accretive operators, J. Funct. Anal., **36** (1980), 147–168.

[18] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl., **75** (1980), 287–292.

[19] S. Reich, On the asymptotic behavior of nonlinear semigroups and the range of accretive operators, J. Math. Anal. Appl., **79** (1981), 113–126.

[20] S. Reich, Convergence, resolvent consistency, and the fixed point property for nonexpansive mappings, Contemp. Math., **18** (1983), 167–174.

[21] S.P Singh and B. Watson, On approximating fixed points, Nonlinear Functional Analysis and its Applications, Proc. Sympos. Pure Math., **45**, part 2, Amer. Math. Soc., Providence, 1988, 393–395.

[22] W Takahashi and G.E. Kim, Strong convergence of approximants to fixed points of non-expansive nonself-mappings in Banach space, Nonlinear Anal., **32** (1998), 447–454.

[23] W Takahashi and Y Ueda, On Reich's strong convergence theorems for resolvents of accretive operators, J. Math. Anal. Appl., **104** (1984), 546–553.

[24] W Takahashi and D.H. Jeong, Fixed point theorem for nonexpansive semigroups on Banach space, Proc. Amer. Math. Soc., **122** (1994), 1175–1179.

[25] H.K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal., **16** (1991), 1127–1138.

[26] H.K. Xu, Approximating curves of nonexpansive nonself mappings in Banach spaces, C. R. Acad. Sci. Paris Ser. I Math., **325** (1997), 179–184.

[27] H.K. Xu and X.M. Yin, Strong convergence theorems for nonexpansive nonself-mappings, Nonlinear Anal., **24** (1995), 223–228.

Department of Mathematics
Dong-A University
Pusan 604-714
Korea
E-mail: jungjs@seunghak.donga.ac.kr

Department of Applied Mathematics
Pukyong National University
Pusan 608-737
Korea
E-mail: taehwa@dolphin.pknu.ac.kr