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VANISHING THEOREMS FOR CONSTRUCTIBLE SHEAVES II

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Introduction

In [H-L2], we have proved theorems on the vanishing of the higher cohomology of direct images of complexes with constructible cohomology sheaves (constructible complexes). As a particular case of these theorems, we have obtained a theorem of Artin-Grothendieck which states that the direct image by an affine map of a constructible sheaf satisfying the support condition ([B-B-D] (4.0.1) or [G-M2] §4.1 Definition), also satisfies the support condition. P. Deligne has obtained the Lefschetz Theorem for hyperplane sections for the Z_ℓ-cohomology (cf [D2] (4.1.6)) by considering the theorem of Artin-Grothendieck in the case the affine map has a point as target and by using the duality. In fact, Lefschetz Theorem can be obtained directly by proving theorems on the vanishing of the lower cohomology of direct images of constructible complexes. Following Deligne's idea, one would use duality theorems and apply them to the results of [H-L2], as K. Fieseler and L. Kaup do in [F-K2] for the intersection homology complex of a complex analytic space by applying their results of [F-K1]. However, when the base ring is not a field or a principal ideal domain, such duality theorems are delicate. Furthermore, when the topology of the maps is complicated, we cannot apply duality arguments. For instance, to obtain Lefschetz Theorems of Zariski type, i.e. on open varieties, especially when one does not have good transversality conditions on the hyperplane section, we definitely have to get a direct approach. The advantage of this viewpoint is that we can state and prove general Theorems of Lefschetz type for constructible complexes and that we can also have such generalizations for relative situations.

In this paper we first "dualize" the proofs and concepts introduced in [H-L2]. Some of these proofs are similar to those we have developped to prove Lefschetz Theorems for homotopy groups ([H-L1] e.g. Theorem 3.4.1). We also obtain results with respect to maps. An important case is when the maps are inclusions, from which we obtain vanishing of global cohomologies from the vanishing of local ones. Then, we obtain a natural statement which can be understood as the "dual" statement of the Artin-Grothendieck Theorem. In fact, we prove a more general statement (Theorem 3.1.4) which applies to q-complete maps in the sense of Andreotti and Grauert. In the last section, we

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show how to obtain all kinds of Lefschetz type theorems for the cohomology of constructible sheaves.

Beware that the classical Lefschetz Theorem on hyperplane sections is true for the usual cohomology, because it concerns non-singular varieties. When the space has singularities, A. Grothendieck in [G] (Exposé XIII) has already noticed that such a theorem holds when we have good topological depth conditions on the singularities of the space, e.g. in the case of local complete intersections (see [H-L1] Corollary 3.2.2). In order to have a nice formulation for a Theorem of Lefschetz type for constructible complexes we have to impose local vanishing conditions for these complexes which are analogous to good depth conditions. Here the condition is the co-support condition (see §2 2.2.1). This is why there are Lefschetz type theorems for perverse sheaves. By definition, perverse sheaves (in particular the intersection cohomology complex) satisfy the co-support condition and the dual condition, which is the support condition.

1. Vanishing bounds for lower cohomology groups

We shall essentially consider the same topological setting as in [H-L2].

1.1. Let X be a topological space endowed with a constant sheaf of rings $\mathscr{R}$ defined by a ring R. Let $(X_k)_{k\in K}$ be a locally finite partition of X into non-empty locally closed connected subsets of X. The sets X_k, $k \in K$, are called the strata of the partition. We denote the inclusion map by i_k

$$i_k : X_k \to X.$$

We assume that the partition satisfies the frontier condition.

We replace the finiteness condition (*) of [H-L2] (1.1) by the following one:

(**) For any $k \in K$ there is an integer $\delta(X_k)$ which is the maximum of all numbers $r \geq 0$ such that there is a chain

$$\overline{X}_{k_0} \subset \cdots \subset \overline{X}_{k_r}$$

of closures of distinct strata of length r with $X_k = X_{k_r}$.

As in [H-L2], $D(X,R)$ is the derived category of the abelian category $M(X,R)$ of sheaves on X which are left $\mathscr{R}$-modules and $D^+(X,R)$ denotes the full subcategory of $D(X,R)$ whose objects are complexes of left $\mathscr{R}$-modules which are bounded from below. We always use derived functors without explicit indication, so we write f_* instead of Rf_*, etc. As usual, left $\mathscr{R}$-modules are considered as complexes in these categories where all terms are trivial except in degree zero where the left $\mathscr{R}$-module is placed.

In all this paper we shall only consider left $\mathscr{R}$-modules. Therefore we shall only say $\mathscr{R}$-module instead of left $\mathscr{R}$-module.

Let $\boldsymbol{F}$ be an object in $D^+(X,R)$.

Let $\overline{\boldsymbol{Z}} = \boldsymbol{Z} \cup \{+\infty\} \cup \{-\infty\}$. By convention the supremum of the void set is $-\infty$ and the infimum of the void set is $+\infty$.

We define:

DENITION 1.1.1. a) For any locally closed subset Y of X, let $i_Y : Y \to X$ be the inclusion map, and define

$$\bar{p}_{F,Y} = \inf\{s \mid h^s(i_Y^! F) \neq 0\}.$$

b) In the case $Y = \{x\}$ we write $\bar{p}_{F,x}$ instead of $\bar{p}_{F,\{x\}}$.

c) If, for $k \in K$, the cohomology sheaves of $i_k^! F$ are locally constant, we put $\bar{p}_F(k) = \bar{p}_{F,X_k}$.

LEMMA 1.1.2. *Suppose that for any $k \in K$ the space X_k is a topological manifold and that the cohomology sheaves of $i_k^! F$, $k \in K$, are locally constant. Then we have*

$$\bar{p}_{F,x} = \bar{p}_F(k) + \dim X_k$$

for any $k \in K$, $x \in X_k$, where $\dim X_k$ is the topological dimension of X_k.

Proof. Let $\ell_x : \{x\} \to X_k$ and $i_x : \{x\} \to X$ be the inclusion maps. Then we have

$$i_x = i_k \circ \ell_x,$$

so $i_x^! F = \ell_x^! i_k^! F$. Since $i_k^! F$ is locally constant on the topological manifold X_k we have $\ell_x^! i_k^! F = \ell_x^* i_k^! F[-\dim X_k]$ (see [G-M2] § 1.10), so $H^\nu(i_x^! F) = H^{\nu + \dim X_k}((i_k^! F)_x)$.

Let $f : X \to Z$ be a continuous map. We introduce:

DENITION 1.1.3. For any locally closed subset T of Z we define $\bar{d}_{f,T}(k) = \bar{d}_{f,T}(X_k)$ as the infimum in $\bar{\mathbf{Z}}$ of the set of integers s such that there is a locally constant $\mathscr{R}$-module $\mathscr{L}$ on X_k such that

$$h^s(j^! f_! (i_k)_* (\mathscr{L})) \neq 0$$

where j is the inclusion of T in Z. If $T = \{z\}$ we write $\bar{d}_{f,z}(k)$ instead of $\bar{d}_{f,\{z\}}(k)$.

Remark 1.1.4. There is the following relation between $\bar{d}_{f,T}(k)$ and $\bar{p}_{F,T}$:

$$\bar{d}_{f,T}(k) = \inf \bar{p}_{f_!(i_k)_* \mathscr{L}, T}$$

where the infimum is taken on the class of all locally constant R-modules $\mathscr{L}$ on X_k.

The following theorem gives the behaviour of $\bar{p}_F$ under direct image with proper supports:

THEOREM 1.1.5. *Assume that for any $k \in K$ the cohomology sheaves $i_k^! F$ are locally constant. Let T be a locally closed subset of Z. Then, we have*

$$\bar{p}_{f_! F, T} \geq \inf_k (\bar{p}_F(k) + \bar{d}_{f,T}(k)).$$

Proof. We formally dualize the proof of Theorem 1.1.3 in [H-L2].

We have to prove the vanishing of $h^q(j^! f_! \boldsymbol{F})$, for

$$q < \inf_k (\bar{p}_{\boldsymbol{F}}(k) + \bar{d}_{f,T}(k))$$

where j is the inclusion of T into Z.

For any $m \geq -1$ let X^m denote the union of the strata X_k such that $\delta(X_k) \leq m$:

$$X^m = \bigcup_{\delta(X_k) \leq m} X_k$$

According to [H-L2] Lemma 1.1.4, the space X^m is closed in X, and X_k is an open subset of $X^m - X^{m-1}$ if $\delta(X_k) = m$.

For $m = -1$, $X^{-1} = \emptyset$.

Call ι^m the inclusion map from X^m into X, u^m the inclusion of X^{m-1} into X^m and v_m the inclusion of $X^m - X^{m-1}$ into X^m. We define:

$$f^m := f \circ \iota^m$$

$$\boldsymbol{F}^m := (\iota^m)^! \boldsymbol{F}$$

We prove by induction on $m \geq -1$ that the cohomology sheaves $h^q(j^!(f^m)_! \boldsymbol{F}^m)$ vanish, when

$$q < \inf_k (\bar{p}_{\boldsymbol{F}}(k) + \bar{d}_{f,T}(k)).$$

The theorem will follow from the fact that

$$\varinjlim_m h^q(j^!(f^m)_! \boldsymbol{F}^m) = h^q(j^! f_! \boldsymbol{F}).$$

Notice that, the partition $(X_k)_{k \in K}$ being locally finite, the hypothesis (**) implies

$$\varinjlim_m (f^m)_! \boldsymbol{F}^m = f_! \boldsymbol{F}.$$

For $m = -1$, there is nothing to be proved. To make the induction, we consider the distinguished triangle, see [B-B-D] p. 43:

$$\to (u^m)_!(u^m)^! \boldsymbol{F}^m \to \boldsymbol{F}^m \to (v_m)_*(v_m)^! \boldsymbol{F}^m \xrightarrow{+1}$$

We use the identity $\iota^m \circ u^m = \iota^{m-1}$ to get the distinguished triangle

$$\to (u^m)_! \boldsymbol{F}^{m-1} \to \boldsymbol{F}^m \to (v_m)_*(v_m)^! \boldsymbol{F}^m \xrightarrow{+1}$$

to which we apply the (derived) functor $j^!(f^m)_!$. We obtain the long exact sequence of cohomology sheaves

$$\to h^q(j^!(f^{m-1})_! \boldsymbol{F}^{m-1}) \to h^q(j^!(f^m)_! \boldsymbol{F}^m) \to h^q(j^!(f^m)_!(v_m)_*(v_m)^! \boldsymbol{F}^m) \to$$

because $f^m \circ u^m = f^{m-1}$ implies

$$(f^m)_!(u^m)_! \boldsymbol{F}^{m-1} = (f^{m-1})_! \boldsymbol{F}^{m-1}.$$

We have to show that $h^q(j^!(f^m)_!(v_m)_*(v_m)^!F^m) = 0$ for

$$q < \inf_k(\bar{p}_F(k) + \bar{d}_{f,T}(k)).$$

We have the other identity

$$(v_m)^!F^m = (v_m)^!(\imath^m)^!F.$$

This implies

$$(f^m)_!(v_m)_*(v_m)^!F^m = f_!(\imath^m)_!(v_m)_*(v_m)^!(\imath^m)^!F.$$

Since $\imath^m \circ v_m$ is the inclusion w_m of $X^m - X^{m-1}$ into X and since X^m is closed in X (see above), we have $(\imath^m)_!(v_m)_* = (\imath^m)_*(v_m)_* = (w_m)_*$, so

$$j^!(f^m)_!(v_m)_*(v_m)^!F^m = j^!f_!(\imath^m)_*(v_m)_*(v_m)^!(\imath^m)^!F = j^!f_!(w_m)_*(w_m)^!F.$$

Now we have to show that $h^q(j^!(f_!(w_m)_*(w_m)^!F)) = 0$, for

$$q < \inf_k(\bar{p}_F(k) + \bar{d}_{f,T}(k)).$$

To obtain this vanishing, we use a spectral sequence. In fact, by assumption, the cohomology sheaves of $(i_k)^!F$ are locally constant, so that

$$h^q((i_k)^!F) = 0$$

for any $q < \bar{p}_F(k)$ and, for any q, by definition of the function $\bar{d}_{f,T}$, the cohomology groups

$$h^p(j^!f_!(i_k)_*h^q((i_k)^!F)) = 0$$

when $p < \bar{d}_{f,T}(k)$. The spectral sequence

$$h^p(j^!f_!(i_k)_*h^q((i_k)^!F)) \Rightarrow h^{p+q}(j^!f_!(i_k)_*(i_k)^!F)$$

converges when F is in $D^+(X, R)$ (see [Go] Chap. II Théorème 4.17.1) and this gives the vanishing of the cohomology

$$h^q(j^!f_!(i_k)_*(i_k)^!F)$$

for $q < \bar{p}_F(k) + \bar{d}_{f,T}(k)$, for any k, and therefore for

$$q < \inf_\kappa(\bar{p}_F(\kappa) + \bar{d}_{f,T}(\kappa)).$$

Now the space $X^m - X^{m-1}$, endowed with the topology induced by X, is the disjoint union of the strata X_k for which $\delta(X_k) = m$, and these strata are open in $X^m - X^{m-1}$, as mentioned above. This implies that

$$h^q(j^!f_!(w_m)_*(w_m)^!F = \sum_{\delta(X_k)=m} h^q(j^!f_!(i_k)_*(i_k)^!F)$$

which is 0 if

$$q < \inf_\kappa(\bar{p}_F(\kappa) + \bar{d}_{f,T}(\kappa)).$$

1.2. As in 1.1, we consider a topological space X endowed with the constant sheaf of rings $\mathscr{R}$ defined by R and with a locally finite partition $\mathscr{S} = (X_k)_{k \in K}$ by non-empty locally closed connected subsets of X.

Now we fix a perversity of the partition $\mathscr{S}$, i.e. a map $p : \mathscr{S} \to \overline{\boldsymbol{Z}}$ (see [B-B-D] (2.1.1)), but notice that we admit $\overline{\boldsymbol{Z}}$ instead of $\boldsymbol{Z}$). As above, consider a complex of $\mathscr{R}$-modules $\boldsymbol{F}$ in $D^+(X, R)$, but we assume furthermore that the cohomology sheaves of $i_k^! \boldsymbol{F}$ are locally constant on the locally closed connected subsets X_k $(k \in K)$, where i_k is the inclusion of X_k into X.

Associated to the complex $\boldsymbol{F}$, we define a perversity $\bar{p}_{\boldsymbol{F}} : \mathscr{S} \to \overline{\boldsymbol{Z}}$ of the partition $\mathscr{S}$ by

$$\bar{p}_{\boldsymbol{F}}(X_k) = \bar{p}_{\boldsymbol{F}}(k).$$

for any $x \in X_k$.

Following the definition 2.1.2 of [B-B-D], for a given perversity p, we have:

DENITION 1.2.1. The subcategory ${}^pD^{\geq 0}(X, R)$ of $D^+(X, R)$ is the subcategory of complexes $\boldsymbol{K}$ such that, for any S in the partition $\mathscr{S}$, with i_S being the inclusion of S into X, we have $h^n(i_S{}^! \boldsymbol{K}) = 0$, for any $n < p(S)$.

Therefore by definition $\boldsymbol{F}$ is an object of the category ${}^{\bar{p}_{\boldsymbol{F}}}D^{\geq 0}(X, R)$. In fact we have:

LEMMA 1.2.2. *The complex $\boldsymbol{F}$ belongs to the category ${}^pD^{\geq 0}(X, R)$ if and only if*

$$\bar{p}_{\boldsymbol{F}} \geq p$$

and we have

$$\bar{p}_{\boldsymbol{F}} = \sup\{p \in \overline{\boldsymbol{Z}}^{\mathscr{S}} \mid \boldsymbol{F} \in {}^pD^{\geq 0}(X, R)\}$$

where $\overline{\boldsymbol{Z}}^{\mathscr{S}}$ is the set of maps of $\mathscr{S}$ into $\overline{\boldsymbol{Z}}$.

Recall that when the elements S in the partition $\mathscr{S}$ are topological manifolds of even dimension, there is a particular perversity called the middle perversity $p_{1/2}$ defined by:

$$p_{1/2}(S) = -(1/2)\dim(S)$$

1.3. Let us use the notations of § 1.1. In particular, let $f : X \to Z$ be a continuous map, but here we suppose that Z consists only of one point 0. We put $\bar{d}(k) = \bar{d}(X_k) = \bar{d}_{f,0}(k)$, and obviously we have that $\bar{d}(X_k) \in \overline{\boldsymbol{Z}}$ is the infimum of all integers q for which there is a locally constant sheaf $\mathscr{L}$ of R-modules on X such that

$$\boldsymbol{H}_c^q(X, i_{k*}\mathscr{L}) \neq 0.$$

Here, $\boldsymbol{H}_c^q(X, i_{k*}\mathscr{L})$ is the q-th hypercohomology group of $i_{k*}\mathscr{L}$ on X with compact supports.

Furthermore, for any complex F in $D^+(X,R)$ for which the cohomology sheaves of the complexes $i_k^! F$ are locally constant we have

$$\bar{p}_{f,F,0} = \inf\{q \in \boldsymbol{Z} \mid \boldsymbol{H}_c^q(X,F) \neq 0\}.$$

Let l_k be the inclusion of X_k into $\overline{X}_k$. Since $\overline{X}_k$ is closed in X we have

$$\boldsymbol{H}_c^q(X, i_{k*}\mathscr{L}) = \boldsymbol{H}_c^q(\overline{X}_k, l_{k*}\mathscr{L})$$

for any left $\mathscr{R}$-module L on X_k. Therefore we obtain:

Lemma 1.3.1. *The invariant $\bar{d}(X_k) \in \overline{\boldsymbol{Z}}$ is the infimum of all integers q for which there is a locally constant sheaf $\mathscr{L}$ of R-modules on X_k such that*

$$\boldsymbol{H}_c^q(\overline{X}_k, l_{k*}\mathscr{L}) \neq 0.$$

2. q-completeness of spaces and related properties

2.1. Let us assume that X is a complex analytic space. Let A be a closed complex analytic subset of X, $q \geq 0$.

Let us recall the definition of q-completeness.

(C_q) The space X is called q-complete if there is a proper q-convex function $\varphi : X \to \boldsymbol{R}^+$.

Recall that a function $\varphi : X \to \boldsymbol{R}^+$ is called q-convex if for any point $x \in X$ there is an open neighbourhood U, a complex manifold $\tilde{U}$ and a $\mathscr{C}^\infty$ function $\tilde{\varphi} : \tilde{U} \to \boldsymbol{R}$ such that U is a closed complex analytic subspace of $\tilde{U}$, $\varphi = \tilde{\varphi}|U$ and the Levi form of $\tilde{\varphi}$ at x has at most q eigenvalues which are not positive. For a discussion of this notion see [V].

Be aware of the fact that in [A-G] the actual definition of A. Andreotti and H. Grauert of a q-complete space gives a $(q-1)$-complete space in the sense of (C_q).

There is a weaker topological notion introduced in [H-L2], Definition 2.2.1:

(TC_q) The space X is topologically q-complete with respect to A if, for any locally closed connected subspace Y of X not contained in A, for which the closure $\overline{Y}$ and $\overline{Y} - Y$ are complex analytic subspaces of X, there is a complex analytic open dense subset Y_0 of $Y - A$ such that, for any locally constant $\mathscr{R}$-module $\mathscr{L}$ on Y_0, the cohomology group $H^k(X, i_!\mathscr{L})$ vanishes for any $k > \dim_C Y + q$, where i is the inclusion of Y_0 in X and $\dim_C Y$ is the complex dimension of Y.

In this paper, however, we need a slightly different condition:

$(\overline{TC}_q)$ The space X is dually topologically q-complete with respect to A if, for any locally closed connected subspace Y of X not contained in A, for which the closure $\overline{Y}$ and $\overline{Y} - Y$ are complex analytic subspaces of X, there is a complex analytic open dense subset Y_0 of $Y - A$ such that, for any locally constant

$\mathscr{R}$-module $\mathscr{L}$ on Y_0, the hypercohomology group $\boldsymbol{H}_c^k(X, i_*\mathscr{L})$ vanishes for any $k < \dim^- Y - q$, where i is the inclusion of Y_0 in X.

Here, $\dim^- Y := \inf\{\dim_{\boldsymbol{C}}(Y, y) \mid y \in Y\}$, whereas $\dim_{\boldsymbol{C}} Y = \sup\{\dim_{\boldsymbol{C}}(Y, y) \mid y \in Y\}$.

In fact we shall see later on that we could have taken $Y_0 = Y - A$ in the definition of the condition $(\overline{TC}_q)$, see Corollary 2.2.5.

As in the case of topological q-completeness, we have

Remark 2.1.1. If X is dually topologically q-complete with respect to A, any closed analytic subspace Y is dually topologically q-complete with respect to $A \cap Y$.

In [H-L2] it has been proved that a space X which is q-complete is also topologically q-complete with respect to any closed complex analytic subspace A. Here we show that this holds also for the condition $(\overline{TC}_q)$:

Theorem 2.1.2. *If the space X is q-complete, it is dually topologically q-complete with respect to any closed complex analytic subspace A.*

Proof. Let Y be a locally closed connected subspace of X not contained in A, for which the closure $\overline{Y}$ and $\overline{Y} - Y$ are complex analytic subspaces of X. Let Y_0 be the smooth part of $Y - A$ (with respect to the reduced structure). Call l the inclusion of Y_0 into $\overline{Y}$. Consider a locally constant $\mathscr{R}$-module $\mathscr{L}$ on Y_0. Since X is q-complete, the closed complex analytic subspace $\overline{Y}$ is also q-complete. We claim

$$H_c^k(\overline{Y}, l_*\mathscr{L}) = 0$$

for any $k < \dim^- Y - q$. Then, if i denotes the inclusion of Y_0 into X we have $H_c^k(\overline{Y}, l_*\mathscr{L}) = H_c^k(X, i_*\mathscr{L})$, which finishes the proof.

Our claim is a consequence of the following proposition:

Proposition 2.1.3. *Let X be a q-complete space, A a closed complex analytic subspace,* $\dim^-(X - A) = n$, *$X - A$ smooth. Let $\mathscr{L}$ be a locally constant $\mathscr{R}$-module on $X - A$, and let $i : X - A \to X$ be the inclusion map. Then*

$$\boldsymbol{H}_c^k(X, i_*\mathscr{L}) = 0$$

for $k < n - q$.

This proposition follows from a statement in homotopy:

Lemma 2.1.4. *Let X and A be as in Proposition* 2.1.3. *Let us fix a complex analytic Whitney stratification of (X, A) such that $X - A$ is a stratum, and let $\varphi : X \to \boldsymbol{R}^+$ be proper and q-convex. Let $r > 0$ be a regular value of φ, i.e. of its restrictions to the strata, and let $X_{\geq r} := \{x \in X \mid \varphi(x) \geq r\}$. Then, the pair $(X - A, X_{\geq r} - A)$ is $(n - q - 1)$-connected.*

Proof. We are using stratified Morse theory. We may assume that φ is a Morse function in the sense of Goresky-MacPherson ([G-M1] I §2.1). In fact, we use $-\varphi$ instead of φ. Let x be a critical point of $-\varphi$ with $\varphi(x) < r$, and let S be the stratum containing x. If $S = X - A$ the index of the Hessian of $-\varphi$ at x is $\geq n - q$, by the usual argument of Andreotti-Frankel type (see [A-F]). So let us assume $S \subset A$, $s = \dim S$. Again, the index of the Hessian of the restriction $-\varphi|S$ at x is $\geq s - q$, so the tangential Morse data is $(s - q - 1)$-connected. Let N be a normal slice of S in X at x and L the corresponding complex link. Then $N - A$ is smooth, $\dim^{-}(N - A) \geq n - s$. By the Strong local Lefschetz theorem [H-L1] (Theorem 2.12) we have that the pair $(N - A, L - A)$ is $(n - s - 1)$-connected, so the normal Morse data is also $(n - s - 1)$-connected (see [G-M1] I §2.4). In total, by using Lemma 1.8 of [H-L1], we obtain that the local Morse data at x is $(n - q - 1)$-connected, which implies our lemma.

Proof of Proposition 2.1.3. Since X is q-complete, there is a proper q-convex map $\varphi : X \to \boldsymbol{R}^{+}$ and we can apply Lemma 2.1.4.

First we prove that, under the assumptions of Lemma 2.1.4, the relative cohomology

$$H^k(X - A, X_{\geq r} - A, \mathscr{L})$$

is 0, for $k < n - q$ and any regular value r of φ.
According to [Sw] §6.13, for a pair (Y, T) of CW-complexes which is s-connected, there is a map

$$g : (B, C) \to (Y, T)$$

from a pair (B, C) of CW-complexes, where B is obtained from C by adding cells of dimension $> s$, to (Y, T), such that g is a homotopy equivalence. We apply this result to the case when $Y = X - A$, $T = X_{\geq r} - A$ and $s = n - q - 1$. Now Theorem 2.6 of [H2] shows that this homotopy equivalence implies an isomorphism

$$H^k(X - A, X_{\geq r} - A, \mathscr{L}) \to H^k(B, C, g^*\mathscr{L})$$

of the corresponding relative cohomology groups. As the inverse image of a locally constant sheaf by g is obviously locally constant, the cohomology group $H^k(B, C, g^*\mathscr{L})$ vanishes for $k \leq n - q - 1$ (apply 3.11 of [H2]).

Again, let r be a regular value of φ, and let $X_{<r} := \{x \in X \mid \varphi(x) < r\}$. Consider the diagram of inclusion mappings

$$\begin{array}{ccc} X_{<r} - A & \xrightarrow{\tilde{j}_r} & X - A \\ \downarrow{\scriptstyle \iota_r} & & \downarrow{\scriptstyle \iota} \\ X_{<r} & \xrightarrow{j_r} & X. \end{array}$$

Observe that

$$H^k(X - A, X_{\geq r} - A, \mathscr{L}) = H^k(X - A, \tilde{j}_{r!}\tilde{j}_r^* \mathscr{L})$$

and, by Leray spectral sequence,

$$H^k(X - A, \tilde{j}_{r!}\tilde{j}_r^* \mathscr{L}) = \boldsymbol{H}^k(X, i_* \tilde{j}_{r!}\tilde{j}_r^* \mathscr{L}).$$

Now we have the following base change isomorphism (see Appendix, Lemma A.3) which is true here, since r is a regular value of φ:

$$\boldsymbol{H}^k(X, i_* \tilde{j}_{r!}\tilde{j}_r^* \mathscr{L}) \simeq \boldsymbol{H}^k(X, j_{r!} i_{r*} \tilde{j}_r^* \mathscr{L}).$$

On the other hand j_r is an open map, so $j_r^* = j_r^!$. Another base change gives

$$i_{r*}\tilde{j}_r^! \mathscr{L} = j_r^! i_* \mathscr{L}.$$

So, we obtain

$$\boldsymbol{H}^k(X, j_{r!} i_{r*} \tilde{j}_r^* \mathscr{L}) = \boldsymbol{H}^k(X, j_{r!} j_r^! i_* \mathscr{L}).$$

Altogether this implies

$$\boldsymbol{H}^k(X, j_{r!} j_r^! i_* \mathscr{L}) = 0 \quad \text{for any } k < n - q.$$

Since $X_{<r}$ is relatively compact in X, we can replace here the usual hypercohomology by hypercohomology with compact supports:

$$\boldsymbol{H}^k(X, j_{r!} j_r^* i_* \mathscr{L}) = \boldsymbol{H}_c^k(X, j_{r!} j_r^* i_* \mathscr{L}).$$

Now

$$\varinjlim_r j_{r!} j_r^* i_* \mathscr{L} = i_* \mathscr{L},$$

therefore

$$\varinjlim_r \boldsymbol{H}_c^k(X, j_{r!} j_r^* i_* \mathscr{L}) = \boldsymbol{H}_c^k(X, i_* \mathscr{L}),$$

which gives our proposition.

2.2. Now we want to apply the result of § 1.1 to the case where X is dually topologically q-complete with respect to A and where $Z = \{0\}$.

We shall prove a theorem analogous to a dual statement of Theorem 2.3.2 in [H-L2] which implies the classical Weak Lefschetz Theorem.

We shall deal with weakly constructible complexes (see 2.3.1 of [H-L2]), i.e. complexes in $D^+(X, R)$ whose cohomology sheaves are locally constant on a complex analytic partition of X. In this paper for the sake of simplicity, we shall assume that all the cohomology sheaves of a weakly constructible complex are locally constant on the same complex analytic partition. All the proofs on the vanishing of the cohomology in what follows can be extended to general weakly constructible complexes by using truncation (see e.g. [B-B-D] 1.3.2). Notice that we have

Remark 2.2.1. Let $(X_k)_{k\in K}$ be a complex analytic Whitney stratification of X such that the restrictions of the cohomology sheaves of the complex $\boldsymbol{F} \in D^+(X, R)$ are locally constant along each X_k. Then the cohomology sheaves of $i_k^!\boldsymbol{F}$ are locally constant, where $i_k : X_k \to X$ is the inclusion map.

Let Z be a complex analytic space. We define the category ${}^{1/2}D^{\geq m}(Z, R)$ (compare to [B-B-D] 2.1.2) to be the full subcategory of weakly constructible complexes $\boldsymbol{F}$ in the category $D^+(Z, R)$ such that, for any stratum Z_k of some complex analytic Whitney stratification of Z on which the cohomology sheaves of $\boldsymbol{F}$ are locally constant, we have

$$\bar{p}_{\boldsymbol{F},Z_k} \geq m - \dim_{\boldsymbol{C}} Z_k.$$

Such sheaves are said to satisfy the co-support condition, when the base ring is a field (see [B-B-D] (4.0.2) or [G-M2] §4.1 Definition).

The following lemma shows that the definition of ${}^{1/2}D^{\geq m}$ does not depend on the Whitney stratification (see [B-B-D] (2.1.14)).

LEMMA 2.2.2. *Let X be a complex analytic space, $m \in \boldsymbol{Z}$, and let $\boldsymbol{F}$ be a weakly constructible complex in $D^+(X, R)$. Let us fix a complex analytic Whitney stratification $(X_k)_{k\in K}$ of (X, A) such that the restrictions of the cohomology sheaves of $\boldsymbol{F}$ to each stratum are locally constant. Let K' be the set of all $k \in K$ such that X_k is not contained in A. For $x \in X$ call i_x the inclusion of $\{x\}$ into X. The following conditions are equivalent*:

a) $\bar{p}_{\boldsymbol{F}}(k) := \bar{p}_{\boldsymbol{F},X_k} \geq m - \dim_{\boldsymbol{C}} X_k$ *for any* $k \in K'$;

b) $\dim\{x \in X - A \mid H_c^q(i_x^!\boldsymbol{F}) \neq 0\} \leq q - m$, *for any integer* q.

Proof. For any integer q, let Z_q be the set

$$Z_q := \{x \in X - A \mid H_c^q(i_x^!\boldsymbol{F}) \neq 0\}.$$

Then we observe that Z_q is stratified by strata of the given Whitney stratification $(X_k)_{k\in K}$.

Let us suppose a). By Lemma 1.1.2 we have that, for any $k \in K'$ and $x \in X_k$,

$$\bar{p}_{\boldsymbol{F},x} = \bar{p}_{\boldsymbol{F}}(k) + 2\dim_{\boldsymbol{C}} X_k \geq m + \dim_{\boldsymbol{C}} X_k$$

because the topological dimension $\dim X_k$ of X_k is twice the complex dimension of X_k. Let X_ℓ be a stratum in Z_q. Of course, ℓ belongs to K'. Pick a point $x \in X_\ell$. By definition $H_c^q(i_x^!\boldsymbol{F}) \neq 0\}$, therefore, as $\ell \in K'$, we have

$$q \geq \bar{p}_{\boldsymbol{F},x} \geq m + \dim_{\boldsymbol{C}} X_\ell$$

which yields

$$q - m \geq \dim_C X_\ell.$$

This implies b).

Now suppose b). Let $k \in K'$ and $x \in X_k$. Observe that $\bar{p}_{F,x} \neq -\infty$ because F belongs to $D^+(X, R)$. If $\bar{p}_{F,x} = +\infty$, for this $k \in K'$, the inequality a) is trivially obtained. Otherwise, we have by definition that

$$x \in Z_{\bar{p}_{F,x}}.$$

Therefore $X_k \subset Z_{\bar{p}_{F,x}}$. By b) this yields that

$$\bar{p}_{F,x} - m \geq \dim_C Z_{\bar{p}_{F,x}} \geq \dim_C X_k.$$

Now $\bar{p}_{F,x} = \bar{p}_F(k) + 2\dim_C X_k$, so that

$$\bar{p}_{F,x} - m = \bar{p}_F(k) + 2\dim_C X_k - m \geq \dim_C X_k$$

which means a).

Now we can state

THEOREM 2.2.3. *Assume that X is dually topologically q-complete with respect to A. Let $i : X - A \to X$ be the inclusion map. Let F be a complex in ${}^{1/2}D^{\geq m}(X - A, R)$ such that $i_!F$ is weakly constructible. Then*

$$H_c^k(X, i_*F) = 0$$

for any $k < m - q$.

Proof. It is sufficient to prove this theorem only for the case where X is finite dimensional, because we can go over to the direct limit in general.

We first assert that we can find a complex analytic Whitney stratification $(X_k)_{k \in K}$ of (X, A) adapted to the constructible complex F such that

$$\bar{d}(k) := \bar{d}(X_k) \geq \dim_C X_k - q$$

for any $k \in K'$, where K' denotes the set of all k such that X_k is not contained in A and where $\bar{d}(X_k)$ is defined as in 1.3. Since F is a complex in ${}^{1/2}D^{\geq m}(X - A, R)$, we have

$$\bar{p}_F(k) \geq m - \dim_C X_k$$

for any $k \in K'$. Furthermore, $\bar{p}_{i_*F}(k) = \infty$ for any $k \in K - K'$. By Theorem 1.1.5, applied to the case where Z is a point, we obtain the desired result.

It remains to prove the above assertion. This is a consequence of

LEMMA 2.2.4. *Let us assume that X is dually topologically q-complete with respect to A and that X is finite dimensional. Let $(X_l')_{l \in L}$ be a complex analytic partition by smooth connected strata of X. Then there is a finer complex analytic Whitney stratification $(X_k)_{k \in K}$ of X as in* 1.1 *such that A is a union of strata which has the following property*:

Let K' be the set of all $k \in K$ such that X_k is not contained in A, then, for any $k \in K'$,

$$\bar{d}(k) \geq \dim_{C} X_k - q.$$

Proof. By induction over $n = \dim_C(X - A)$. For $X = A$ there has nothing to be proved. Otherwise, let L_n be the set of all l such that $\dim_C X'_l = n$ and X'_l is not contained in A. Since X is dually topologically q-complete with respect to A, for any $l \in L_n$ there is a complex analytic open dense subset X_l of $X'_l - A$ such that, for any locally constant $\mathcal{R}$-module $\mathcal{L}$ on X_l, the cohomology group $H^k_c(X, i_*\mathcal{L})$ vanishes for any $k < n - q$, where i is the inclusion of X_l in X. Let $X^* = X - \bigcup_{l \in L_n} X_l$. The partition $(X'_l)_{l \in L}$ induces a partition $(X'_l \cap X^*)_{l \in L}$ of X^*, after refining this latter we may assume that this partition, together with $(X_l)_{l \in L_n}$, is a Whitney stratification because, for $l \in L_n$, $\dim_C X_l$ is maximal. By induction hypothesis we may refine further the partition of X^* such that we obtain in total the desired partition of X.

Theorem 2.2.3 has the following corollary

COROLLARY 2.2.5. *Assume that X is dually topologically q-complete with respect to A. Let Y be a locally closed connected subspace Y of X not contained in A, for which the closure $\bar{Y}$ and $\bar{Y} - Y$ are complex analytic subspaces of X. Let $\mathcal{L}$ be a locally constant $\mathcal{R}$-module on $Y - A$. Then the hypercohomology groups $\boldsymbol{H}^k_c(X, i_*\mathcal{L})$ vanish for any $k < \dim_C Y - q$, where i is the inclusion of $Y - A$ in X.*

In fact this means that we could have taken $Y_0 = Y - A$ in the definition of the condition $(\overline{TC}_q)$.

3. Dually topological q-complete maps

3.1. By analogy with the notion of q-completeness for spaces introduced by A. Andreotti and H. Grauert, we can define q-complete complex analytic maps (see [K-S], [H-L2]). In [H-L2], we have also extended the notion of topological completeness to maps. Of course, in a dual way we can define

DENITION 3.1.1. $(\overline{TC}_q(f))$ Let $f : X \to Z$ be a complex analytic morphism and A be a closed complex analytic subspace of X.

a) Let z be a point of Z. We say that f is dually topologically q-complete at z with respect to A if there is a fundamental system of open neighbourhoods $(U_\alpha(z))$ of z in Z such that, for any α, the space $(f^{-1}(U_\alpha(z))$ is dually topologically q-complete with respect to $A \cap f^{-1}(U_\alpha(z)))$.

b) We say that the map f is dually topologically q-complete with respect to A if it is so at every point $z \in Z$.

Remark 3.1.2. Remark 2.1.1 yields that, if f is dually topologically q-complete with respect to A, for any closed subspace Y of Z, f induces a map f_Y from $f^{-1}(Y)$ to Y which is dually topologically q-complete with respect to $A \cap f^{-1}(Y)$.

From Theorem 2.1.2 we obtain immediately

LEMMA 3.1.3. *Let $f : X \to Z$ be a complex analytic morphism. Let z be a point of Z. If f is q-complete at z, then it is dually topologically q-complete at z with respect to any closed complex analytic subspace A of X.*

The main result of this section is the following dual to the generalization of the theorem of Artin-Grothendieck stated in [H-L2] (Theorem 3.1.4).

THEOREM 3.1.4. *Let $f : X \to Z$ be dually topologically q-complete at z with respect to a closed complex analytic subspace A of X. Let $i : X - A \to X$ be the inclusion. Let $\boldsymbol{F}$ be a complex in ${}^{1/2}D^{\geq 0}(X - A, R)$. Assume that $i_! \boldsymbol{F}$ and $f_! i_* \boldsymbol{F}$ are weakly constructible. Then $f_! i_* \boldsymbol{F}$ is in ${}^{1/2}D^{\geq -q}(Z, R)$.*

Proof. Because of Lemma 1.2.2, Theorem 3.1.4 follows directly from

PROPOSITION 3.1.5. *Let $f : X \to Z$ be a complex analytic map and let A be a closed complex analytic subspace of X. Let $i : X - A \to X$ be the inclusion. Let $\boldsymbol{F}$ be a complex in ${}^{1/2}D^{\geq 0}(X - A, R)$. Assume that $i_! \boldsymbol{F}$ and $f_! i_* \boldsymbol{F}$ are weakly constructible. Let us fix a Whitney stratification $(X_k)_{k \in K}$ of X adapted to A and to $i_! \boldsymbol{F}$. Furthermore, let us fix a Whitney stratification $(Z_h)_{h \in H}$ of Z adapted to $f_! i_* \boldsymbol{F}$. Let Z_h be a stratum of Z. Let us assume that there is a point $z \in Z_h$ such that f is dually topologically q-complete at z with respect to A and such that for each stratum X_k of X there is no critical point x of $f|X_k$ with $f(x) = z$. Then*

$$\bar{p}_{f_! i_* \boldsymbol{F}}(h) \geq -\dim_{\boldsymbol{C}} Z_h - q.$$

In order to prove Proposition 3.1.5, we need to prove the following two Lemmas 3.1.6 and 3.1.7.

LEMMA 3.1.6. *Let $f : X \to Z$ be a complex analytic morphism and A be a closed complex analytic subspace of X. Let z be a point of Z. Assume that f is dually topologically q-complete at z with respect to A. Let $(X_k)_{k \in K}$ be a complex analytic Whitney stratification as in* (1.1). *Then we have*

$$\bar{d}_{f,z}(k) \geq \dim_{\boldsymbol{C}} X_k - q$$

for any $k \in K$ with $X_k \subset X - A$.

Proof of Lemma 3.1.6. Since f is dually topologically q-complete at z with respect to A, Corollary 2.2.5 shows that we can choose a fundamental system

$(U_\alpha(z))$ of open and relatively compact neighbourhoods of z in Z, such that, for any locally constant $\mathcal{R}$-module $\mathcal{L}$ on a stratum X_k contained in $X - A$ and for $s < \dim_{\mathbf{C}} X_k - q$, we have

$$\boldsymbol{H}_c^s(f^{-1}(U_\alpha(z)), i_{k*}\mathcal{L}) = 0,$$

where $i_k : X_k \to X$ is the inclusion. Now, we notice that

$$\boldsymbol{H}_c^s(f^{-1}(U_\alpha(z)), i_{k*}\mathcal{L}) \simeq \boldsymbol{H}_c^s(U_\alpha(z), f_! i_{k*}\mathcal{L})$$

and we use the exact sequence

$$\to \boldsymbol{H}_c^s(U_\alpha(z), f_! i_{k*}\mathcal{L}) \to \boldsymbol{H}^s(Z, f_! i_{k*}\mathcal{L}) \to \boldsymbol{H}^s(Z - U_\alpha(z), f_! i_{k*}\mathcal{L})$$
$$\to \boldsymbol{H}_c^{s+1}(U_\alpha(z), f_! i_{k*}\mathcal{L}) \to .$$

Therefore, for $s < \dim_{\mathbf{C}} X_k - q - 1$, the natural map

$$\boldsymbol{H}^s(Z, f_! i_{k*}\mathcal{L}) \to \boldsymbol{H}^s(Z - U_\alpha(z), f_! i_{k*}\mathcal{L})$$

is an isomorphism. So, $\boldsymbol{H}^s(Z, f_! i_{k*}\mathcal{L})$ is isomorphic to the projective limit

$$\varprojlim_\alpha \boldsymbol{H}^s(Z - U_\alpha(z), f_! i_{k*}\mathcal{L}).$$

In particular the projective system $(\boldsymbol{H}^s(Z - U_\alpha(z), f_! i_{k*}\mathcal{L}))_\alpha$ satisfies the Mittag-Leffler condition for $s < \dim_{\mathbf{C}} X_k - q - 1$, which yields by Theorem 3.7 of [H2] applied to complexes instead of sheaves

$$\boldsymbol{H}^s(Z - \{z\}, f_! i_{k*}\mathcal{L}) \simeq \varprojlim_\alpha \boldsymbol{H}^s(Z - U_\alpha(z), f_! i_{k*}\mathcal{L}) \simeq \boldsymbol{H}^s(Z, f_! i_{k*}\mathcal{L}),$$

for $s < \dim_{\mathbf{C}} X_k - q - 1$. Let $j : \{z\} \to Z$ be the inclusion. From the exact sequence

$$\cdots \to h^s(j^! f_! i_{k*}\mathcal{L}) \to \boldsymbol{H}^s(Z, f_! i_{k*}\mathcal{L}) \to \boldsymbol{H}^s(Z - \{z\}, f_! i_{k*}\mathcal{L})$$
$$\to h^{s+1}(j^! f_! i_{k*}\mathcal{L}) \to \cdots,$$

we obtain

$$h^s(j^! f_! i_{k*}\mathcal{L})_z = 0,$$

for $s < \dim_{\mathbf{C}} X_k - q$. Note that, for $s = \dim_{\mathbf{C}} X_k - q - 1$, the map

$$h^s(j^! f_! i_{k*}\mathcal{L}) \to \boldsymbol{H}^s(Z, f_! i_{k*}\mathcal{L})$$

is zero, since it factorizes through $\boldsymbol{H}_c^s(U_\alpha(z), f_! i_{k*}\mathcal{L})$ which is zero.

By definition 1.1.3, this means that $\bar{d}_{f,z}(k) \geq \dim_{\mathbf{C}} X_k - q$ and proves 3.1.6.

If the map $f : X \to Z$ can be compactified, we may obtain a better bound for $\bar{d}_{f,z}(k)$, see Proposition 3.2.1 below.

LEMMA 3.1.7. *Let $p : T \to S$ be a subanalytic map onto a homology manifold S. Let $\boldsymbol{F}$ be a weakly constructible complex of $\mathcal{R}$-modules on T and let $s \in S$.*

We suppose that, for every point $t \in T_s := p^{-1}(s)$, *there is a fundamental system of good neighbourhoods* $(\overline{U}_\alpha(t))_{\alpha \in A}$ *relatively to* $\boldsymbol{F}$ *and* T_s, *such that the restrictions of* p *to* $\overline{U}_\alpha(t)$ *induce proper maps* p_α *from* $\overline{U}_\alpha(t)$ *onto* $p(\overline{U}_\alpha(t))$, *the images* $p(\overline{U}_\alpha(t))$ *are a fundamental system of neighbourhoods of* $s = p(t)$ *in* S *and the direct images* $(p_\alpha)_*(\boldsymbol{F}|\overline{U}_\alpha(t))$ *have locally constant cohomology sheaves on* $p(\overline{U}_\alpha(t))$. *Let* i *denote the inclusion of the fiber* T_s *into* T. *Then,*

$$i^*\boldsymbol{F}[-\dim S] = i^!\boldsymbol{F}$$

where $\dim S$ *is the topological dimension of* S.

Proof of Lemma 3.1.7. Let $z \in T_s$. We have to prove that, at z, we have

$$i^*\boldsymbol{F}[-\dim S]_z = i^!\boldsymbol{F}_z.$$

As $\boldsymbol{F}$ is weakly constructible, we have, for adequate α,

$$h^k(i^*\boldsymbol{F}_z) = \boldsymbol{H}^k(\overline{U}_\alpha(z) \cap T_s, \boldsymbol{F})$$

and, p_s denoting the constant map of $T_s \cap \overline{U}_\alpha(z)$ onto the point $\{s\}$, we have

$$\boldsymbol{H}^k(\overline{U}_\alpha(z) \cap T_s, \boldsymbol{F}) = h^k(p_{s*} i_\alpha^*(\boldsymbol{F}|\overline{U}_\alpha(z)))$$

where i_α denotes the inclusion of $\overline{U}_\alpha(z) \cap T_s$ in $\overline{U}_\alpha(z)$. As p_α is proper, by base change we have

$$h^k(p_{s*} i_\alpha^*(\boldsymbol{F}|\overline{U}_\alpha(z))) = h^k(i_s^*(p_\alpha)_*(\boldsymbol{F}|\overline{U}_\alpha(z)))$$

where i_s is the inclusion of $\{s\}$ into $p(\overline{U}_\alpha(z))$. Now by hypothesis we have that the complex $(p_\alpha)_*(\boldsymbol{F}|\overline{U}_\alpha(z))$ has locally constant cohomology sheaves on $p(\overline{U}_\alpha(z))$ which is supposed to be a homology manifold. The standard identity (15) of § 1.13 in [G-M2] gives that

$$i_s^*(p_\alpha)_*(\boldsymbol{F}|\overline{U}_\alpha(z))[-\dim S] = i_s^!(p_\alpha)_*(\boldsymbol{F}|\overline{U}_\alpha(z)).$$

Therefore

$$h^k(i_s^*(p_\alpha)_*(\boldsymbol{F}|\overline{U}_\alpha(z))) = h^{k-\dim S}(i_s^!(p_\alpha)_*(\boldsymbol{F}|\overline{U}_\alpha(z))).$$

Now by base change we have

$$k^{k-\dim S}(i_s^!(p_\alpha)_*(\boldsymbol{F}|\overline{U}_\alpha(z))) = h^{k-\dim S}(p_{s*} i_\alpha^!(\boldsymbol{F}|\overline{U}_\alpha(z)))$$

and by definition

$$\begin{aligned} h^{k-\dim S}(p_{s*} i_\alpha^!(\boldsymbol{F}|\overline{U}_\alpha(z))) &= \boldsymbol{H}^{k-\dim S}(\overline{U}_\alpha(z) \cap T_s, i_\alpha^!(\boldsymbol{F}|\overline{U}_\alpha(z))) \\ &= \boldsymbol{H}^{k-\dim S}(\overline{U}_\alpha(z), (i_\alpha)_*(i_\alpha)^!(\boldsymbol{F}|\overline{U}_\alpha(z)) \\ &= \boldsymbol{H}^{k-\dim S}(\overline{U}_\alpha(z), \overline{U}_\alpha(z) - T_s, \boldsymbol{F}). \end{aligned}$$

The weak constructibility of $\boldsymbol{F}$ implies, for the interior $U_\alpha(z)$ of $\overline{U}_\alpha(z)$

$$\boldsymbol{H}^{k-\dim S}(\overline{U}_\alpha(z), \overline{U}_\alpha(z) - T_s, \boldsymbol{F}) = \boldsymbol{H}^{k-\dim S}(U_\alpha(z), U_\alpha(z) - T_s, \boldsymbol{F})$$

and the weak constructibility of $i^! \boldsymbol{F}$ yields, for adequate α

$$\boldsymbol{H}^{k-\dim S}(U_\alpha(z), U_\alpha(z) - T_s, \boldsymbol{F}) = \boldsymbol{H}^{k-\dim S}(U_\alpha(z) \cap T_s, i^! \boldsymbol{F}) = h^{k-\dim S}(i^! \boldsymbol{F}_z)$$

which shows that $h^k(i^* \boldsymbol{F}_z) = h^{k-\dim S}(i^! \boldsymbol{F}_z)$ and so $i^! \boldsymbol{F}[\dim S] = i^* \boldsymbol{F}$.

Now, we can proceed to prove Proposition 3.1.5.

Proof of Proposition 3.1.5. Let $z \in Z_h \subset Z$ be chosen as in the statement of 3.1.5. Let $j_z : \{z\} \to Z$ be the inclusion. Because of Lemma 1.1.2, we have to show

$$\bar{p}_{f_! i_* \boldsymbol{F}, z} := \inf\{s \in \boldsymbol{Z} \mid h^s(j_z^! f_! i_* \boldsymbol{F}_z) \neq 0\} \geq \dim_{\boldsymbol{C}} Z_h - q.$$

From the definition and general results about Whitney stratifications, there is an open neighbourhood U of z in Z and a normal slice $\mathcal{N}$ of Z_h at z such that $\mathcal{N}$ is closed in U, the space $f^{-1}(U)$ is dually topologically q-complete with respect to $A \cap f^{-1}(U)$, and the map $f_{\mathcal{N}}$ induced by f from $f^{-1}(\mathcal{N})$ to $\mathcal{N}$ is dually topologically q-complete with respect to $A \cap f^{-1}(\mathcal{N})$. Moreover, we have a stratified product structure $\mathcal{N} \times V$ of U, where V is a suitable neighbourhood of z in Z_h.

Now, let l be the inclusion of $\{z\}$ into $\mathcal{N}$ and k the inclusion of $\mathcal{N}$ into Z:

$$\{z\} \xrightarrow{l} \mathcal{N} \xrightarrow{k} Z.$$

Then we have $j_z = k \circ l$. For any weakly constructible complex $\boldsymbol{G}$ on Z for which the Whitney stratification on Z is adapted to $\boldsymbol{G}$, we have the quasi-isomorphism

$$(*) \qquad k^! \boldsymbol{G} \simeq k^* \boldsymbol{G}[-2 \dim_{\boldsymbol{C}} Z_h]$$

because of the stratified product structure $\mathcal{N} \times V$ of U, as one can see e.g. from [H2] (3) of Proposition 5.3 or Lemma 3.1.7. Namely, the normal slice $\mathcal{N}$ can be defined as the fiber $r^{-1}(z)$ of a local retraction $r : U \to V$ of the open neighbourhood U of z in Z onto an open neighbourhood V of z in Z_h. In fact, because of the properties of Whitney stratifications, Thom-Mather first isotopy lemma (see [M]) shows that r can be assumed to be a locally trivial stratified continuous fibration on V and therefore one can show that, for any weakly constructible complex $\boldsymbol{G}$ on Z, for which the Whitney stratification (Z_h) is adapted, the complex $\boldsymbol{G}|U$ and r satisfy the hypothesis of the Lemma 3.1.7. Hence, we obtain the result quoted above from [H2] (3) of Proposition 5.3

$$(k_1)^*(\boldsymbol{G}|U)[-2 \dim_{\boldsymbol{C}} Z_h] = (k_1)^!(\boldsymbol{G}|U)$$

if k_1 denotes the inclusion of the normal slice $\mathcal{N}$ into U. Since U is open in Z, this yields the isomorphism $(*)$ above.

Now, applying this result to the weakly constructible complex $\boldsymbol{G} := f_! i_* \boldsymbol{F}$, we get:

$$h^s(j_z^! f_! i_* \boldsymbol{F}_z) = h^s(l^! k^! f_! i_* \boldsymbol{F}_z) = h^s((l^! k^* f_! i_* \boldsymbol{F}[-2 \dim_{\boldsymbol{C}} Z_h])_z).$$

Let $f_{\mathcal{N}}$ denote the map induced by f from $f^{-1}(\mathcal{N})$ to $\mathcal{N}$. We call j the inclusion of $f^{-1}(\mathcal{N})$ into X, $i_{\mathcal{N}}$ the inclusion of $f^{-1}(\mathcal{N}) - A$ into $f^{-1}(\mathcal{N})$, and j_A the inclusion of $f^{-1}(\mathcal{N}) - A$ into $X - A$.

Now, by base change (see [B] (4) of 10.7, p. 159), we have

$$k^* f_! i_* \boldsymbol{F} = (f_{\mathcal{N}})_! j^* i_* \boldsymbol{F}.$$

Because of the assumption about z, after shrinking $\mathcal{N}$ (or the neighbourhood U) if necessary, we have an induced stratification of $f^{-1}(\mathcal{N})$ and we have some relation between $j^!$ and j^*

$$(**) \qquad j^* i_* \boldsymbol{F} = j^! i_* \boldsymbol{F}[2 \dim_{\boldsymbol{C}} Z_h].$$

since the restrictions of f to the strata X_ι are non-critical at every point of $f^{-1}(\mathcal{N})$. In fact, let j_U be the inclusion of $f^{-1}(\mathcal{N})$ into $f^{-1}(U)$ and q be the map from $f^{-1}(U)$ into V defined by $q(x) = r(f(x))$, for any $x \in f^{-1}(U)$. The hypothesis made on f above the point $z \in Z_h$ implies that for sufficiently small U, we have that q is locally in $f^{-1}(U)$ a locally trivial stratified continuous fibration on V by Thom-Mather first isotopy lemma again. This implies that, for any weakly constructible complex $\boldsymbol{K}$ on X for which the Whitney stratification X_k is adapted, q and $\boldsymbol{K}|f^{-1}(U)$ satisfy the hypothesis of the Lemma 3.1.7 and therefore

$$(j_U)^* \boldsymbol{K}|f^{-1}(U)[-2 \dim_{\boldsymbol{C}} Z_h] = j_U^! \boldsymbol{K}|f^{-1}(U).$$

However $f^{-1}(U)$ is open in X, so we have

$$j^* \boldsymbol{K}[-2 \dim_{\boldsymbol{C}} Z_h] = j^! \boldsymbol{K}.$$

We apply this result to the complex $i_* \boldsymbol{F}$. This gives $(**)$.

By base change, we have

$$j^! i_* \boldsymbol{F}[2 \dim_{\boldsymbol{C}} Z_h] = (i_{\mathcal{N}})_* (j_A)^! \boldsymbol{F}[2 \dim_{\boldsymbol{C}} Z_h].$$

A similar reasoning as above gives

$$(j_A)^! \boldsymbol{F}[2 \dim_{\boldsymbol{C}} Z_h] = (j_A)^* \boldsymbol{F}$$

and, with $(**)$, it yields

$$j^* i_* \boldsymbol{F} = j^! i_* \boldsymbol{F}[2 \dim_{\boldsymbol{C}} Z_h] = (i_{\mathcal{N}})_* (j_A)^* \boldsymbol{F} = (i_{\mathcal{N}})_* (\boldsymbol{F}|f^{-1}(\mathcal{N}) - A).$$

Now, $f_{\mathcal{N}}$ is dually topologically q-complete at z with respect to $f^{-1}(\mathcal{N}) \cap A$ and Lemma 3.1.6 implies that

$$\bar{d}_{f_{\mathcal{N}},z}(X_k \cap f^{-1}(\mathcal{N})) \geq \dim_{\boldsymbol{C}}(X_k \cap f^{-1}(\mathcal{N})) - q = \dim_{\boldsymbol{C}} X_k - \dim_{\boldsymbol{C}} Z_h - q$$

for any stratum X_k of X which is contained in $X - A$ and intersects $f^{-1}(\mathcal{N})$. Since $\boldsymbol{F}$ belongs ${}^{1/2}D^{\geq 0}(X - A, R)$, we assert that

$$(***) \qquad (i_{\mathcal{N}})_* (\boldsymbol{F}|f^{-1}(\mathcal{N}) - A) \in {}^{1/2}D^{\geq -\dim_{\boldsymbol{C}} Z_h}(f^{-1}(\mathcal{N}), R).$$

This fact is rather long to prove.

The assertion means by definition that, for all the strata S of a Whitney stratification of $f^{-1}(\mathcal{N})$, we have

$$h^s(\tilde{\imath}_S^!(i_{\mathcal{N}})_*(\boldsymbol{F}|f^{-1}(\mathcal{N})-A))=0$$

for all $s\leq -\dim_{\mathbf{C}} S-\dim_{\mathbf{C}} Z_h$ and where $\tilde{\imath}_S$ denotes the inclusion of S into $f^{-1}(\mathcal{N})$. Because of the transversality assumption made on $f^{-1}(\mathcal{N})$ in the Proposition 3.1.5, we have a Whitney stratification induced by the one we have on X. As the latter is adapted to A, the stratification induced on $f^{-1}(\mathcal{N})$ is adapted to $A\cap f^{-1}(\mathcal{N})$. For a stratum S of this stratification of $f^{-1}(\mathcal{N})$, there exists a stratum X_k of the Whitney stratification of X such that

$$S=X_k\cap f^{-1}(\mathcal{N}).$$

First, we notice that for any stratum S of the induced stratification of $f^{-1}(\mathcal{N})-A$, we have, for any integer s,

$$h^s(\tilde{\imath}_S^!(i_{\mathcal{N}})_*(\boldsymbol{F}|f^{-1}(\mathcal{N})-A))=h^s(i_S'^!(j_A)^*\boldsymbol{F}),$$

where i_S' is the inclusion of S into $f^{-1}(\mathcal{N})-A$. This comes from the fact that, $i_{\mathcal{N}}$ being open, we have $i_{\mathcal{N}}^!=i_{\mathcal{N}}^*$ (see [B-B-D] 1.4.1). So

$$\tilde{\imath}_S^!=i_S'^!i_{\mathcal{N}}^!=i_S'^!i_{\mathcal{N}}^*.$$

As the stratum S is in $f^{-1}(\mathcal{N})-A$, the inclusion of $S\cap(f^{-1}(\mathcal{N})-A)$ into S is the identity. By base change we have

$$\tilde{\imath}_S^!(i_{\mathcal{N}})_*=i_S'^!.$$

In fact the same proof shows that, if S is contained in A,

$$h^s(\tilde{\imath}_S^!(i_{\mathcal{N}})_*(\boldsymbol{F}|f^{-1}(\mathcal{N})-A))=0,$$

for any s.

Again, let $S=X_k\cap f^{-1}(\mathcal{N})$ be contained in $f^{-1}(\mathcal{N})-A$. Call j_k the inclusion of S into $X_k\cap f^{-1}(U)$ and $\hat{\imath}_k$ the inclusion of $X_k\cap f^{-1}(U)$ into $X-A$. Of course, we have

$$\hat{\imath}_k\circ j_k=j_A\circ i_S'.$$

Lemma 3.1.7 gives

$$(j_A)^*\boldsymbol{F}=(j_A)^!\boldsymbol{F}[2\dim_{\mathbf{C}} Z_h],$$

as we have seen above. Therefore

$$i_S'^!(j_A)^*\boldsymbol{F}=i_S'^!(j_A)^!\boldsymbol{F}[2\dim_{\mathbf{C}} Z_h]=j_k^!\hat{\imath}_k^!\boldsymbol{F}[2\dim_{\mathbf{C}} Z_h].$$

Applying again Lemma 3.1.7 to $q|X_k\cap f^{-1}(U)$, we have

$$j_k^!\hat{\imath}_k^!\boldsymbol{F}[2\dim_{\mathbf{C}} Z_h]=j_k^*\hat{\imath}_k^!\boldsymbol{F}.$$

So, finally

$$i'^{!}_{S}(j_A)^*\boldsymbol{F} = j_k^*\hat{i}_k^!\boldsymbol{F}$$

which gives that

$$h^s(i'^{!}_{S}(j_A)^*\boldsymbol{F}) = h^s(j_k^*\hat{i}_k^!\boldsymbol{F}) = j_k^* h^s(\hat{i}_k^!\boldsymbol{F})$$

because, j_k being closed, j_k^* is exact (see [B-B-D] 1.4.1).

We have assumed that the complex $\boldsymbol{F}$ belongs to ${}^{1/2}D^{\geq 0}(X - A, R)$. This implies that $h^s(\hat{i}_k^!\boldsymbol{F}) = 0$, for $s < -\dim_{\boldsymbol{C}} X_k$, which shows that

$$h^s(\tilde{i}_S^!(i_{\mathcal{N}})_*(\boldsymbol{F}|f^{-1}(\mathcal{N}) - A)) = h^s(i'^{!}_{S}(j_A)^*\boldsymbol{F}) = 0$$

for $s < -\dim_{\boldsymbol{C}} X_k = -\dim_{\boldsymbol{C}} S - \dim_{\boldsymbol{C}} Z_h$, as desired for any S in $f^{-1}(\mathcal{N}) - A$. When S is contained in A, we saw above that the vanishing of

$$h^s(\tilde{i}_S^!(i_{\mathcal{N}})_*(\boldsymbol{F}|f^{-1}(\mathcal{N}) - A))$$

is true of any s. This ends the proof of the assertion $(***)$ above.

Moreover, by definition $\bar{p}_{(i_{\mathcal{N}})_*(\boldsymbol{F}|f^{-1}(\mathcal{N})-A)}(X_\ell \cap f^{-1}(\mathcal{N})) = +\infty$ for any stratum X_ℓ contained in A. Then, by 1.2.2,

$$\bar{p}_{(i_{\mathcal{N}})_*(\boldsymbol{F}|f^{-1}(\mathcal{N})-A)}(X_k \cap f^{-1}(\mathcal{N})) \geq -\dim_{\boldsymbol{C}} Z_h - \dim_{\boldsymbol{C}}(X_k \cap f^{-1}(\mathcal{N})) = -\dim_{\boldsymbol{C}} X_k.$$

Theorem 1.1.5 and Lemma 3.1.6 imply

$$\bar{p}_{f_{\mathcal{N}}!(i_{\mathcal{N}})_*(\boldsymbol{F}|f^{-1}(\mathcal{N})-A),z} \geq -\dim_{\boldsymbol{C}} Z_h - q$$

and, since $(**)$ yields

$$\begin{aligned} f_{\mathcal{N}!}(i_{\mathcal{N}})_*(\boldsymbol{F}|f^{-1}(\mathcal{N}) - A) &= f_{\mathcal{N}!}(i_{\mathcal{N}})_*(j_A)^*(\boldsymbol{F}) = f_{\mathcal{N}!}(i_{\mathcal{N}})_*(j_A)^!(\boldsymbol{F})[2\dim_{\boldsymbol{C}} Z_h] \\ &= f_{\mathcal{N}!}j^!i_*(\boldsymbol{F})[2\dim_{\boldsymbol{C}} Z_h] = f_{\mathcal{N}!}j^*i_*(\boldsymbol{F}) = k^*f_!i_*\boldsymbol{F}, \end{aligned}$$

this implies

$$\inf\{s \in \boldsymbol{Z} \mid h^s(l^!k^*f_!i_*\boldsymbol{F}_z) \neq 0\} \geq -\dim_{\boldsymbol{C}} Z_h - q$$

or equivalently, by $(*)$,

$$\begin{aligned} \inf\{s \in \boldsymbol{Z} \mid h^s(j_z^!f_!i_*\boldsymbol{F}_z) \neq 0\} &= \inf\{s \in \boldsymbol{Z} \mid h^s(l^!k^*f_!i_*\boldsymbol{F}[-2\dim_{\boldsymbol{C}} Z_h]_z) \neq 0\} \\ &\geq \dim_{\boldsymbol{C}} Z_h - q, \end{aligned}$$

because $h^s(j_z^!f_!i_*\boldsymbol{F}_z) = h^s(l^!k^*f_!i_*\boldsymbol{F}[-2\dim_{\boldsymbol{C}} Z_h]_z)$ implies

$$\inf\{s \in \boldsymbol{Z} \mid h^s(j_z^!f_!i_*\boldsymbol{F}_z) \neq 0\} = \inf\{s \in \boldsymbol{Z} \mid h^s(l^!k^*f_!i_*\boldsymbol{F}[-2\dim_{\boldsymbol{C}} Z_h]_z) \neq 0\}.$$

This proves our proposition.

3.2. In particular, we may consider the case where the map $f : X \to Z$ can be compactified.

PROPOSITION 3.2.1. *Let $f : X \to Z$ be a complex analytic morphism which has an analytic compactification $\bar{f} : \bar{X} \to Z$. Let A be a closed complex analytic subspace of X whose closure $\bar{A}$ in $\bar{X}$ is a closed complex analytic subspace of $\bar{X}$. We assume that f is dually topologically q-complete with respect to A. Let us fix Whitney stratifications $\bar{\mathcal{S}} = (X_k)_{k\in\bar{K}}$ of $\bar{X}$ and $\mathcal{U} = (Z_l)_{l\in L}$ of Z adapted to $\bar{A}$, to X and to the map $\bar{f}$. Let z be a point of some stratum Z_l of Z, and let us assume that the map f is dually topologically q-complete with respect to A at z. Let $\bar{d}_{f,Z_l}(k)$ be defined as in* (1.1.3). *Then we have*

$$\bar{d}_{f,Z_l}(k) \geq \dim_C X_k - \dim_C Z_l - q$$

for every $k \in \bar{K}$ such that $X_k \subset X - A$.

Proof. We shall apply Theorem 3.1.4 to the complex $\boldsymbol{F}$ on $X - A$ defined in the following way. Let X_k be a stratum contained in $X - A$. Let $\mathcal{L}$ be a locally constant sheaf on X_k. Let $\tilde{\imath}_k$ be the inclusion of X_k into $X - A$. Define

$$\boldsymbol{F} = (\tilde{\imath}_k)_*\mathcal{L}.$$

We notice that

$$\boldsymbol{F} \in {}^{1/2}D^{\geq \dim_C X_k}(X - A, R)$$

since $(\tilde{\imath}_l)^!(\tilde{\imath}_k)_*\mathcal{L}$ is 0, if $l \neq k$, and $\mathcal{L}$, when $l = k$. Since f is compactifiable, $f_! i_*\boldsymbol{F}$ is weakly constructible and $i_!\boldsymbol{F}$ is obviously weakly constructible. So Theorem 3.1.4 gives that

$$f_! i_*\boldsymbol{F} \in {}^{1/2}D^{\geq \dim_C X_k - q}(Z, R).$$

This implies that (see 2.2)

$$\bar{p}_{f_! i_*\boldsymbol{F},Z_l} \geq \dim_C X_k - q - \dim_C Z_l.$$

Since we have $i_*\boldsymbol{F} = (i_k)_*\mathcal{L}$, this leads to

$$\bar{d}_{f,Z_l}(k) \geq \dim_C X_k - q - \dim_C Z_l$$

by Remark 1.1.4. This ends the proof of 3.2.1.

For any $z \in Z_l$, we can get a lower bound for $\bar{d}_{f,z}(k)$. By Lemma 1.1.2 we have

$$\bar{p}_{f_! i_*\boldsymbol{F},Z_l} = \bar{p}_{f_! i_*\boldsymbol{F},z} - 2\dim_C Z_l$$

so that

$$\bar{d}_{f,z}(k) \geq \dim_C X_k - q + \dim_C Z_l$$

which improves the inequality of 3.1.6 in the case f is compactifiable.

Now we obtain

THEOREM 3.2.2. *Let $f : X \to Z$ be a complex analytic morphism which has a compactification $\bar{f} : \bar{X} \to Z$. Let A be a closed complex analytic subspace of*

X whose closure $\bar{A}$ in $\bar{X}$ is a closed complex analytic subspace of $\bar{X}$. Let $F \in {}^{1/2}D^{\geq 0}(X - A, R)$. We assume that f is dually topologically q-complete with respect to A. Call $\bar{\imath}$ the inclusion of $X - A$ into *$\bar{X}$ and i the inclusion of $X - A$ into X. Assume that $\bar{\imath}_! F$ is weakly constructible. Then $f_! i_* F$ is in ${}^{1/2}D^{\geq -q}(Z, R)$.*

Proof. This theorem can be obtained in two ways: either as a consequence of Theorem 3.1.4, since $i_! F = \bar{\imath}_! F|X$ and $f_! i_* F = \bar{f}_! i_{1!} i_* F$ are weakly constructible, where i_1 is the inclusion of X in $\bar{X}$, or as a consequence of Proposition 3.2.1 and Theorem 1.1.5.

An important example is the case of an algebraic morphism f, algebraic varieties A and X and a complex F which is weakly constructible in the algebraic sense, because this situation is naturally compactifiable.

Another important example is the inclusion of $X - A$ into X:

Corollary 3.2.3. *Let $i : X - A \to X$ be the inclusion and $\mathcal{S} = (X_k)_{k \in K}$ a Whitney stratification of X adapted to A. Let x be a point of some stratum X_l of A and suppose that i is dually topologically q-complete with respect to the empty set at x. Let $\bar{d}_{\iota, X_l}(k)$ be defined as in* (1.1). *Then we have*

$$\bar{d}_{\iota, X_l}(k) \geq \dim_C X_k - \dim_C X_l - q$$

for any stratum X_k in $X - A$.

This obvious corollary is useful in the case where $X - A$ is locally q-complete along A, as defined in the following (see [H-L2] Definition 3.2.3):

Denition 3.2.4. We say that $X - A$ is locally q-complete along A if the inclusion $i : X - A \to X$ is q-complete, i.e. if for any point $x \in A$ there is an open neighbourhood U of x in X such that $U - A$ is q-complete. If the inclusion i is q-complete at a point $x \in A$ ([H-L2] Definition 3.1.2), i.e. if there is an open neighbourhood U of x in X such that $U - A$ is q-complete, we shall say that $X - A$ is locally q-complete along A at x.

We have as a consequence Lemma 3.1.3:

Lemma 3.2.5. *If $X - A$ is locally q-complete along A at x, then the inclusion $i : X - A \to X$ is dually topologically q-complete at x with respect to any closed complex analytic subspace of $X - A$.*

Example 3.2.6. If A is locally defined by at most $q + 1$ holomorphic equations, $X - A$ is locally q-complete along A [S-V].

As an application of this case, we have a vanishing theorem for the cohomology on $X - A$:

COROLLARY 3.2.7. *Let X be dually topologically q_1-complete with respect to the empty set, let $\boldsymbol{F}$ be a weakly constructible complex in ${}^{1/2}D^{\geq m}(X-A,R)$. Assume that the inclusion $i : X-A \to X$ is dually topologically q_2-complete with respect to the empty set at x and that $i_!\boldsymbol{F}$ is weakly constructible on X. Then, for any extension $\boldsymbol{F}_1$ of $\boldsymbol{F}$ to X:*

$$\boldsymbol{H}_c^s(X,A,\boldsymbol{F}_1) := \boldsymbol{H}_c^s(X,i_!\boldsymbol{F}) = 0, \quad \textit{for any } s < m - q_1 - q_2.$$

Proof. We have that $i_!\boldsymbol{F} \in {}^{1/2}D^{\geq m-q_2}(X,R)$. Then, we apply Theorem 2.2.3, when the set A considered in 2.2.3 is the empty set so that the inclusion i of 2.2.3 is the identity on X.

4. Theorems of Lefschetz type

In this paragraph we shall show that the main consequences of the Vanishing Theorems of §2 and §3 are the classical Theorem of Lefschetz on hyperplane sections for cohomology and several of its consequences. Our formulation enables us to state all these consequences in great generality.

4.1. For the sake of simplicity, we first consider the cohomological version of the classical Weak Lefschetz Theorem on hyperplane sections (see [D2] (4.1.6)):

THEOREM 4.1.1. *Let V be a non-singular complex projective variety and W a hyperplane section of V. Then we have that*

$$H^k(V,W;R) = 0$$

for any $k < \dim_{\mathbf{C}}(V-W)$.

Proof. The complement $X := V - W$ is an affine variety and can be endowed with a 0-convex exhaustion function as it has been shown by Andreotti and Frankel (see [A-F]), and therefore, it is 0-complete. Theorem 2.1.2 tells that X is dually topologically 0-complete with respect to the empty set $\emptyset$.

We choose $m := \dim_{\mathbf{C}} X$. On X we consider the complex $\boldsymbol{R}$ given by the constant sheaf generated by the ring R in degree 0. We may consider X as the unique stratum of the trivial Whitney stratification of X. Obviously we have

$$\bar{p}_{\boldsymbol{R},X} \geq 0$$

so that $\boldsymbol{R}$ belongs to ${}^{1/2}D^{\geq \dim_{\mathbf{C}} X}(X,R)$. Theorem 2.2.3 implies that, for the hypercohomology,

$$\boldsymbol{H}_c^k(X,\boldsymbol{R}) = 0$$

for any $k < \dim_{\mathbf{C}} X$. Since the sheaf complex $\boldsymbol{R}$ is trivial, we have, in fact

$$H_c^k(X;R) = 0$$

for any $k < \dim_{\mathbf{C}} X$. In this case, it is well known that, for any k,

$$H_c^k(X; R) = H^k(V, W; R)$$

which proves our theorem.

If A is a closed analytic subspace of a complex analytic space X, for a complex $\boldsymbol{F}$ in $D^+(X, R)$, we use the following notation

$$\boldsymbol{H}^k(X, A, \boldsymbol{F}) := \boldsymbol{H}^k(X, j_! j^* \boldsymbol{F})$$

where j is the inclusion of $X - A$ into X, as we already did in [H-L2] §2.2. When A is not closed in X, $\boldsymbol{H}^k(X, A, \boldsymbol{F})$ is given by the hypercohomology of the mapping cone of the natural morphism $\boldsymbol{F} \to i_* i^* \boldsymbol{F}$.

Then, we have a general lemma to compare the hypercohomology with compact supports with the relative hypercohomology:

LEMMA 4.1.2. *Let us assume that X is compact and A is a closed analytic subspace of X. Let $\boldsymbol{F}$ be the restriction of $\boldsymbol{F}_1$ to $X - A$. Then*

$$\boldsymbol{H}^k(X, A, \boldsymbol{F}_1) = \boldsymbol{H}_c^k(X - A, \boldsymbol{F}).$$

Proof. Let p be the mapping of X to a point 0. Then p is proper, therefore

$$\begin{aligned} \boldsymbol{H}^k(X, A, \boldsymbol{F}_1) &:= \boldsymbol{H}^k(X, j_! \boldsymbol{F}) = h^k((p_* j_! \boldsymbol{F})_0) \\ &= h^k((p_! j_! \boldsymbol{F})_0) = h^k(((p \circ j)_! \boldsymbol{F})_0) = \boldsymbol{H}_c^k(X - A, \boldsymbol{F}). \end{aligned}$$

Now a generalization of Theorem 4.1.1 to singular varieties and more general complexes (compare to [L] 4.10) is

THEOREM 4.1.3. *Let V be a complex projective variety and W a hyperplane section of V. Let j be the inclusion of $V - W$ into V. Let $\boldsymbol{F}$ be a complex in ${}^{1/2}D^{\geq \dim_{\mathbf{C}}(V - W)}(V - W, R)$, and let $\boldsymbol{F}_1$ be any extension to V. Then we have that*

$$\boldsymbol{H}^k(V, W, \boldsymbol{F}_1) = 0$$

for any $k < \dim_{\mathbf{C}}(V - W)$.

Proof. The proof is again a straightforward consequence of Theorem 2.2.3 by observing that $j^* \boldsymbol{F}_1 = \boldsymbol{F}$ and, by Lemma 4.1.2,

$$\boldsymbol{H}^k(V, W, \boldsymbol{F}_1) = \boldsymbol{H}_c^k(V - W, \boldsymbol{F}).$$

Remark 4.1.4. We shall give below other statements which generalize the Weak Lefschetz theorem on hyperplane sections. To avoid expressing bounds involving the complex dimension of the ambient space, we shall shift the complex by this dimension, so that the complex dimension does not appear in the statements. Doing so, we follow the conventions introduced in [B-B-D].

In fact, the preceding Theorems of Lefschetz type are consequences of a general vanishing theorem of the relative hypercohomology:

THEOREM 4.1.5. *Let V be a compact complex analytic space and W be a complex analytic closed subspace of V. Suppose that $V - W$ is dually topologically q-complete with respect to the empty set and that $\boldsymbol{F}$ is a weakly constructible complex in ${}^{1/2}D^{\geq 0}(V - W, R)$. Let $\boldsymbol{F}_1$ be any extension of $\boldsymbol{F}$ to V. Then we have*

$$\boldsymbol{H}^s(V, W, \boldsymbol{F}_1) = 0, \quad \textit{for any } s < -q.$$

Proof. By Lemma 4.1.2, we have

$$\boldsymbol{H}^k(V, W, \boldsymbol{F}_1) = \boldsymbol{H}^k_c(V - W, \boldsymbol{F}).$$

Therefore we may apply Theorem 2.2.3.

4.2. More generally, there is a general theorem of Zariski-Lefschetz type for quasi-projective varieties. To get it, we shall apply the results of §2 and consider the case where X is of the form $V - A$, where A is closed in V.

We first need a topological statement

LEMMA 4.2.1. *Let V be a topological space, A and W closed subsets of V. Let i and j be the inclusions of $V - A \cup W$ into $V - W$ and into $V - A$, let i_1 and j_1 be the inclusions of $V - A$ and of $V - W$ into V. Let $\boldsymbol{F}$ be a complex on $V - A \cup W$ and $\boldsymbol{F}_1$ an extension to $V - A$. Let us assume that we have the following base change property*

$$(j_1)_! i_* \boldsymbol{F} = (i_1)_* j_! \boldsymbol{F}.$$

Then we have, for any $k : \boldsymbol{H}^k(V - A, W - A, \boldsymbol{F}_1) = \boldsymbol{H}^k(V, (j_1)_! i_* \boldsymbol{F})$.

Proof. $\boldsymbol{H}^k(V - A, W - A, \boldsymbol{F}_1) = \boldsymbol{H}^k(V - A, j_! \boldsymbol{F}) = \boldsymbol{H}^k(V, (i_1)_* j_! \boldsymbol{F})$ and by assumption $\boldsymbol{H}^k(V, (i_1)_* j_! \boldsymbol{F}) = \boldsymbol{H}^k(V, (j_1)_! i_* \boldsymbol{F})$.

In fact to apply this lemma we need a criterion to get the base change property which will be stated and proved in the appendix (Lemma A.2).

Now we can prove a vanishing theorem for the relative hypercohomology analogous to Theorem 4.1.5 which implies a generalized theorem of Zariski-Lefschetz type:

THEOREM 4.2.2. *Let V be a compact complex analytic space and A and W be complex analytic closed subspaces of V. Suppose that $V - W$ is dually topologically q-complete with respect to $A - W$. Let i and j be the inclusions of $V - A \cup W$ into $V - W$ and into $V - A$, let i_1 and j_1 be the inclusions of $V - A$ and of $V - W$ into V. Let $\boldsymbol{F}$ be a complex in ${}^{1/2}D^{\geq 0}(V - A \cup W, R)$ and $\boldsymbol{F}_1$ an* *extension to* $V - A$. *Assume that* $i_! \boldsymbol{F}$ *is weakly constructible. Let us assume that we have the following base change property*

$$(j_1)_! i_* \boldsymbol{F} = (i_1)_* j_! \boldsymbol{F}.$$

Then we have

$$\boldsymbol{H}^s(V - A, W - A, \boldsymbol{F}_1) = 0, \quad \textit{for any } s < -q.$$

Proof. This follows from Lemma 4.2.1, Lemma 4.1.2 and Theorem 2.2.3.

Since an affine complex variety is 0-complete, an obvious corollary of 4.2.2 is another theorem of Zariski-Lefschetz type:

COROLLARY 4.2.3. *Let* V *be a complex projective subvariety of* $\boldsymbol{P}^N$ *and* A *be a subvariety of* V. *Let* $\boldsymbol{F}$ *be a complex in* ${}^{1/2}D^{\geq 0}(V - A, R)$. *Let* i_1 *be the inclusion of* $V - A$ *into* V *and assume* $i_{1!} \boldsymbol{F}$ *is weakly constructible. There is a Zariski open dense set* Ω *of projective hyperplanes in* $\boldsymbol{P}^N$ *such that, for any* $H \in \Omega$, *we have*

$$\boldsymbol{H}^s(V - A, V \cap H - A, \boldsymbol{F}) = 0, \quad \textit{for any } s < 0.$$

The Zariski open dense set Ω is the set of projective hyperplanes transverse to a Whitney stratification adapted to A and to the complex $i_{1!} \boldsymbol{F}$ (compare with [C] Théorème 1.1). According to the appendix, this transversality condition implies the base change property of 4.2.2.

In fact, the base change condition in Theorem 4.2.2 can be weakened.

THEOREM 4.2.4. *Let* V *be a compact complex analytic space and* A *and* W *be complex analytic closed subspaces of* V. *Suppose that* $V - W$ *is dually topologically q-complete with respect to* $A - W$. *Let* i *and* j *be the inclusions of* $V - A \cup W$ *into* $V - W$ *and into* $V - A$, *let* i_1 *and* j_1 *be the inclusions of* $V - A$ *and of* $V - W$ *into* V. *Let* $\boldsymbol{F}$ *be a complex in* ${}^{1/2}D^{\geq 0}(V - A \cup W, R)$ *and* $\boldsymbol{F}_1$ *an extension to* $V - A$. *Assume that* $i_! \boldsymbol{F}$ *is weakly constructible. Let us assume that we have*

$$\boldsymbol{H}^s(W, (i_1)_* j_! \boldsymbol{F}) = 0, \quad \textit{for any } s < -q.$$

Then we have

$$\boldsymbol{H}^s(V - A, W - A, \boldsymbol{F}_1) = 0, \quad \textit{for any } s < -q.$$

Proof. We have $\boldsymbol{H}^s(V - A, W - A, \boldsymbol{F}_1) = \boldsymbol{H}^s(V - A, j_! \boldsymbol{F}) = \boldsymbol{H}^s(V, (i_1)_* j_! \boldsymbol{F})$. Because of our assumption it is therefore sufficient to prove that

$$\boldsymbol{H}^s(V, W, (i_1)_* j_! \boldsymbol{F}) = 0, \quad \text{for any } s < -q.$$

But $\boldsymbol{H}^s(V, W, (i_1)_* j_! \boldsymbol{F}) = \boldsymbol{H}^s(V, (j_1)_! j_1^* (i_1)_* j_! \boldsymbol{F}) = \boldsymbol{H}^s(V, (j_1)_! i_* j^* j_! \boldsymbol{F}) = \boldsymbol{H}^s(V, (j_1)_! i_* \boldsymbol{F}) = \boldsymbol{H}^s_c(V - W, i_* \boldsymbol{F}) = 0$ for $s < -q$ because of Theorem 2.2.3 and Lemma 4.1.2.

The assumption of Theorem 4.2.4 is fulfilled if $h^s((i_1)_* j_! F)|W = 0$ for any s. In particular, it is fulfilled if the base change property of Theorem 4.2.2 holds.

In fact, it is obviously sufficient to suppose that $h^s((i_1)_* j_! F) = 0$ for any $s < -q$. Such a hypothesis can be obtained using a local Lefschetz-Zariski theorem which will be developed in section 4.4. Using the results of that section we will be able to prove:

THEOREM 4.2.5. *Let V be a compact complex analytic space and A and W be complex analytic closed subspaces of V. Suppose that $V - W$ is dually topologically q-complete with respect to $A - W$. Let i and j be the inclusions of $V - A \cup W$ into $V - W$ and into $V - A$, let i_1 and j_1 be the inclusions of $V - A$ and of $V - W$ into V. Let F be a complex in ${}^{1/2}D^{\geq 0}(V - A \cup W, R)$ which admits a weakly constructible extension F' to V, and let F_1 be an extension to $V - A$. Let us assume that there is a complex analytic subset S of $A \cap W$ such that $W - S$ is transverse to the subset $A - S$ of $V - S$ and $F'|V - S$. Then we have*

$$H^s(V - A, W - A, F_1) = 0, \quad \textit{for any } s < -q - \dim_C S - 1.$$

Here, by definition, $\dim \emptyset = -1$, so, in the case $S = \emptyset$, Theorem 4.2.5 follows from Theorem 4.2.2 and Lemma A.2 of the appendix. Now, this theorem implies the following corollary which is more general than Corollary 4.2.3.

COROLLARY 4.2.6. *Let V be a complex projective subvariety of P^N, S a linear subspace of P^N and A a subvariety of V. Let F be a complex in ${}^{1/2}D^{\geq 0}(V - A, R)$. Let i_1 be the inclusion of $V - A$ into V and assume $i_{1!}F$ is weakly constructible. There is a Zariski open dense set Ω of projective hyperplanes in P^N containing S such that, for any $H \in \Omega$, we have*

$$H^s(V - A, V \cap H - A, F) = 0, \quad \textit{for any } s < -\dim_C S - 1.$$

The Zariski open dense set Ω is the set of projective hyperplanes containing S which are outside S transverse to a Whitney stratification adapted to A and S and to the complex $i_{1!}F$.

4.3. Now if the base change property (or the weaker assumption of Theorem 4.2.5) does not hold, it has been first observed by P. Deligne in [D1] that there is a Lefschetz type theorem on a comparison with a neighbourhood of the hyperplane section.

We need another topological lemma:

LEMMA 4.3.1. *Let V be a topological space, A and W closed subsets of V. Let $(U_l)_{l \in L}$ be a fundamental system of neighbourhoods of W in V. Let i be the inclusion of $V - A \cup W$ into $V - W$ and j_1 the inclusion of $V - W$ into V. Let F be a complex on $V - A \cup W$ and F_1 an extension to $V - A$. Then we* *have, for any k*

$$\varinjlim_{l} \boldsymbol{H}^k(V - A, U_l - A, \boldsymbol{F}_1) = \boldsymbol{H}^k(V, (j_1)_! i_* \boldsymbol{F}).$$

Proof. We may assume that the U_l are open. Let $l \in L$, let k and i_2 be the inclusions of $U_l - A$ into $V - A$ and into U_l, and let i_1 and k_1 be the inclusions of $V - A$ and of U_l into V. Since k_1 is open, we have $k_1^*(i_1)_* \boldsymbol{F}_1 = (i_2)_* k^* \boldsymbol{F}_1$. Therefore

$$\begin{aligned}\boldsymbol{H}^k(U_l - A, \boldsymbol{F}_1 | (U_l - A)) &= \boldsymbol{H}^k(U_l, (i_2)_* k^* \boldsymbol{F}_1) = \boldsymbol{H}^k(U_l, k_1^*(i_1)_* \boldsymbol{F}_1) \\ &= \boldsymbol{H}^k(U_l, ((i_1)_* \boldsymbol{F}_1) | U_l)\end{aligned}$$

which implies

$$\varinjlim_{l} \boldsymbol{H}^k(U_l - A, \boldsymbol{F}_1) = \boldsymbol{H}^k(W, (i_1)_* \boldsymbol{F}_1 | W).$$

Furthermore, $\boldsymbol{H}^k(V - A, \boldsymbol{F}_1 | (V - A)) = \boldsymbol{H}^k(V, (i_1)_* \boldsymbol{F}_1)$. By comparison of long exact hypercohomology sequences we obtain

$$\varinjlim_{l} \boldsymbol{H}^k(V - A, U_l - A, \boldsymbol{F}_1) = \boldsymbol{H}^k(V, (j_1)_! j_1^*(i_1)_* \boldsymbol{F}_1) = \boldsymbol{H}^k(V, (j_1)_! i_* \boldsymbol{F})$$

because $j_1^*(i_1)_* \boldsymbol{F}_1 = i_* j^* \boldsymbol{F}_1 = i_* \boldsymbol{F}$, where j is the inclusion of $V - (A \cup W)$ into $V - W$.

Now, Theorem 2.2.3, Lemma 4.1.2 and Lemma 4.3.1 give another vanishing theorem for the hypercohomology:

PROPOSITION 4.3.2. *Let V be a compact complex analytic space and A and W be complex analytic closed subspaces of V. Suppose that $V - W$ satisfies condition $(\overline{TC}_q)$ with respect to $A - W$. Let $(U_l)_{l \in L}$ be a fundamental system of neighbourhoods of W in V. Let i be the inclusion of $V - A \cup W$ into $V - W$. Let $\boldsymbol{F}$ be a complex in ${}^{1/2}D^{\geq 0}(V - A \cup W, R)$ and $\boldsymbol{F}_1$ an extension to $V - A$. Assume that $i_! \boldsymbol{F}$ is weakly constructible. Then we have*

$$\varinjlim_{l} \boldsymbol{H}^s(V - A, U_l - A, \boldsymbol{F}_1) = 0, \quad \textit{for any } s < -q.$$

Of course, it would be interesting to have the vanishing for a particular neighbourhood U. It is quite easy to specify such a U if $V - W$ is q-complete:

PROPOSITION 4.3.3. *Let V be a compact complex analytic space and A and W be complex analytic closed subspaces of V. Suppose that $V - W$ is q-complete. Let $\varphi : V - W \to \boldsymbol{R}^+$ be a proper q-convex function. For $c > 0$ let $U_c := W \cup \{x \in V - W \mid \varphi(x) > c\}$. Let i be the inclusion of $V - A \cup W$ into $V - W$. Let $\boldsymbol{F}$ be a complex in ${}^{1/2}D^{\geq 0}(V - A \cup W, R)$ and $\boldsymbol{F}_1$ an extension to $V - A$. Assume that $i_! \boldsymbol{F}$ is weakly constructible. Then we have for any $c > 0$*

$$\boldsymbol{H}^s(V - A, U_c - A, \boldsymbol{F}_1) = 0, \quad \textit{for any } s \leq -q - 1.$$

Proof. As the proof of Lemma 4.3.1 shows, we have

$$\boldsymbol{H}^s(V - A, U_c - A, \boldsymbol{F}_1) = \boldsymbol{H}^s(V, U_c, (i_1)_* \boldsymbol{F}_1)$$

where i_1 is the inclusion of $V - A$ into V. Let $\bar{U}_c$ be the closure of U_c in V. It is sufficient to show that $\boldsymbol{H}^s(V, \bar{U}_c, (i_1)_* \boldsymbol{F}_1) = 0$, for any $s \leq -q - 1$. But $\boldsymbol{H}^s(V, \bar{U}_c, (i_1)_* \boldsymbol{F}_1) = \boldsymbol{H}^s_c(V - \bar{U}_c, (i_1)_* \boldsymbol{F}_1 | (V - \bar{U}_c))$ and $V - \bar{U}_c$ is q-complete, so that we can conclude using Theorem 2.2.3.

In practice, it is useful to have the concept of good neighbourhoods (see [P]). These neighbourhoods will be used to formulate Zariski-Lefschetz type theorems, when the base change condition used above does not hold:

DENITION 4.3.4. Let V be a complex analytic space and W be a closed complex analytic subspace of V. Let us fix some Whitney stratification $(V_k)_{k \in K}$ of V. Let $T(W)$ be an open neighbourhood of W in V. $T(W)$ is called a good neighbourhood of W with respect to $(V_k)_{k \in K}$ if $T(W)$ is a member of a fundamental system $(T_l(W))_{l \in L}$ of open neighbourhoods of W in V, such that for any l, $m \in L$ such that $\bar{T}_l(W) \subset T_m(W)$ there is a deformation retraction of $T_m(W)$ and of $\bar{T}_m(W)$ onto $\bar{T}_l(W)$ which respects the strata V_k. In this case, $\bar{T}(W)$ (the closure of $T(W)$) is called a good neighbourhood as well.

Remark 4.3.5. Suppose furthermore that W is compact. Then there is always such a good neighbourhood which is relatively compact in V. There are several ways of construction: using a triangulation of V which refines the stratification, using a non-negative subanalytic function on V whose locus is W, or using a suitable Riemannian metric on a real analytic manifold (e.g. $\boldsymbol{R}^N$) into which V has been embedded.

Now, the following result will show that this concept is useful:

PROPOSITION 4.3.6. *Let V be a complex analytic space and A and W be complex analytic closed subspaces of V. Let $\boldsymbol{F}$ be a weakly constructible complex on $V - A \cup W$, $\boldsymbol{F}_1$ an extension to $V - A$. Let $j_1 : V - W \to V$, $i : V - A \cup W \to V - W$ and $i_1 : V - A \to V$ be the inclusions. Assume that $(i_1)_! \boldsymbol{F}_1$ is weakly constructible. Let $(V_k)_{k \in K}$ be a Whitney stratification of V adapted to A and to $(i_1)_! \boldsymbol{F}_1$, and let $T(W)$ be a good neighbourhood of W with respect to $(V_k)_{k \in K}$. Then we have*

$$\boldsymbol{H}^s(V - A, \bar{T}(W) - A, \boldsymbol{F}_1) = \boldsymbol{H}^s(V - A, T(W) - A, \boldsymbol{F}_1) = \boldsymbol{H}^s(V, (j_1)_! i_* \boldsymbol{F}).$$

In particular, $\boldsymbol{H}^s(V - A, T(W) - A, \boldsymbol{F}_1)$ does not depend on the choice of the good neighbourhood.

Proof. Let $(T_l(W))_{l \in L}$ be chosen as in Definition 4.3.4. Let l, $m \in L$ such that $\bar{T}_l(W) \subset T_m(W)$. Because of Theorem [H2] 2.9, we have:

$$\boldsymbol{H}^s(\bar{T}_m(W) - A, \boldsymbol{F}_1) = \boldsymbol{H}^s(T_m(W) - A, \boldsymbol{F}_1) = \boldsymbol{H}^s(\bar{T}_l(W) - A, \boldsymbol{F}_1)$$

Therefore, $\boldsymbol{H}^s(T_l(W) - A, \boldsymbol{F}_1)$ and $\boldsymbol{H}^s(V - A, T_l(W) - A, \boldsymbol{F}_1)$ are independent of l. The rest follows from Lemma 4.3.1.

Finally, when the base change property is not satisfied, using the notion of good neighbourhoods, we can state another vanishing theorem of relative hypercohomology:

THEOREM 4.3.7. *Let V be a compact complex analytic space and A and W be complex analytic closed subspaces of V. Suppose that $V - W$ satisfies condition $(\overline{TC}_q)$ with respect to $A - W$. Let $i : V - A \cup W \to V - W$ and $i_1 : V - A \to V$ be the inclusions. Let $\boldsymbol{F}$ be a complex in ${}^{1/2}D^{\geq 0}(V - A \cup W, R)$ and $\boldsymbol{F}_1$ any extension to $V - A$. Assume that $(i_1)_!\boldsymbol{F}_1$ is weakly constructible. Then we have:*

$$\boldsymbol{H}^s(V - A, T(W) - A, \boldsymbol{F}_1) = 0, \quad \textit{for any } s < -q$$

where $T(W)$ is a good neighbourhood of W with respect to any Whitney stratification of V which is compatible with A and $(i_1)_!\boldsymbol{F}_1$.

Proof. This theorem follows from the preceding proposition, Lemma 4.1.2 and Theorem 2.2.3.

Of course, Theorem 4.3.7 gives a Theorem of Lefschetz type in the case the hyperplane section is not general, where, following the idea of Deligne (see [D1]), we consider a good neighbourhood of the hyperplane section:

COROLLARY 4.3.8. *Let V be a complex projective subvariety of $\boldsymbol{P}^N$ and A be a subvariety of V. Let $\boldsymbol{F}$ be a complex in ${}^{1/2}D^{\geq 0}(V - A, R)$. Let i_1 be the inclusion of $V - A$ into V and assume $i_{1!}\boldsymbol{F}$ is weakly constructible. For any projective hyperplane H and any good neighbourhood $T(H)$ of H with respect to some Whitney stratification of V compatible to A and $i_{1!}\boldsymbol{F}$, we have*

$$\boldsymbol{H}^s(V - A, V \cap T(H) - A, \boldsymbol{F}) = 0, \quad \textit{for any } s < 0.$$

4.4. Now, of course, there are also the corresponding local versions of the preceding theorems. As we have already observed in [H-L4], in this local case the bound for the vanishing of the cohomology is one unit less. Since we are working locally, we may look at locally closed complex analytic subsets of $\boldsymbol{C}^N$ instead of complex analytic spaces.

Let $B_\varepsilon := \{z \in \boldsymbol{C}^N \mid \|z\| < \varepsilon\}$, S_ε the boundary of $\bar{B}_\varepsilon$. Let $\varepsilon_0 > 0$ and let X be a closed complex analytic subset of B_{ε_0} which contains 0. First we need the following lemma (where we omit to denote the restriction of complexes):

LEMMA 4.4.1. *If $\boldsymbol{F}$ is a weakly constructible complex on X and if $\varepsilon > 0$ is sufficiently small, $\varepsilon < \varepsilon_0$, we have*

$$\boldsymbol{H}^k(\bar{B}_\varepsilon \cap X, \boldsymbol{F}) = \boldsymbol{H}^k(B_\varepsilon \cap X, \boldsymbol{F}) = H^k(\boldsymbol{F}_0),$$
$$\boldsymbol{H}^k(S_\varepsilon \cap X, \boldsymbol{F}) = \boldsymbol{H}^k((\bar{B}_\varepsilon - \{0\}) \cap X, \boldsymbol{F}).$$

Proof. The first statement follows from [H2] Theorem 2.9 (for complexes instead of sheaves). Let us choose a complex analytic Whitney stratification of X. Then there is a compatible deformation retraction of $(\bar{B}_\varepsilon - \{0\}) \cap X$ onto $S_\varepsilon \cap X$ provided that ε is small enough, see [B-V], so the second statement follows from [H2], loc. cit.

Similarly to Theorem 4.1.5 we have

THEOREM 4.4.2. *Let Y be complex analytic subset of X containing* 0, *and suppose that the inclusion $j : X - Y \to X$ is dually topologically q-complete with respect to the empty set. Let $\boldsymbol{F}$ be a complex in ${}^{1/2}D^{\geq 0}(X - Y, R)$ such that $j_!\boldsymbol{F}$ is weakly constructible, and let $\boldsymbol{F}_1$ be any extension of $\boldsymbol{F}$ to a weakly constructible complex on X* (e.g. $\boldsymbol{F}_1 = j_!\boldsymbol{F}$). *If $\varepsilon > 0$ is sufficiently small, $\varepsilon < \varepsilon_0$, we have*

$$\boldsymbol{H}^s(B_\varepsilon \cap X - \{0\}, B_\varepsilon \cap Y - \{0\}, \boldsymbol{F}_1) = 0, \quad \textit{for any } s < -q - 1.$$

Proof. Because of Theorem 3.2.2, the direct image with compact supports $j_!\boldsymbol{F}$ is in ${}^{1/2}D^{\geq -q}(X)$. Let $k : \{0\} \to X$ and $l : X - \{0\} \to X$ be the inclusions, then

$$h^s((k^!j_!\boldsymbol{F})_0) = 0, \quad \text{for any } s < -q.$$

Let $\varepsilon > 0$ be sufficiently small, $\varepsilon < \varepsilon_0$, let $U := B_\varepsilon \cap X$. Because of Lemma 4.4.1, $0 = h^s((j_!\boldsymbol{F})_0) = \boldsymbol{H}^s(U, j_!\boldsymbol{F})$ and $h^s((l_*l^*j_!\boldsymbol{F})_0) = \boldsymbol{H}^s(U, l_*l^*j_!\boldsymbol{F}) = \boldsymbol{H}^s(U - \{0\}, j_!\boldsymbol{F})$. By [B-B-D], we have the distinguished triangle

$$\to k_!k^!j_!\boldsymbol{F} \to j_!\boldsymbol{F} \to l_*l^*j_!\boldsymbol{F} \xrightarrow{+1} .$$

By taking the long exact cohomology sequence at 0, we obtain

$$h^s((k^!j_!\boldsymbol{F})_0) = \boldsymbol{H}^{s-1}(U - \{0\}, j_!\boldsymbol{F}) = \boldsymbol{H}^{s-1}(U - \{0\}, j_!j^*\boldsymbol{F}_1).$$

By definition, $\boldsymbol{H}^{s-1}(U - \{0\}, j_!j^*\boldsymbol{F}_1) = \boldsymbol{H}^{s-1}(U - \{0\}, (U - \{0\}) \cap Y, \boldsymbol{F}_1)$, which gives the desired result.

Theorem 4.4.2 gives a local Theorem of Lefschetz type (compare to [H3]):

COROLLARY 4.4.3. *Let Y be a complex analytic subset of X containing* 0 *and defined by one equation. Let j be the inclusion of $X - Y$ into X. Let $\boldsymbol{F}$ be a complex in ${}^{1/2}D^{\geq 0}(X - Y, R)$ such that $j_!\boldsymbol{F}$ is weakly constructible, and let $\boldsymbol{F}_1$ be any extension of $\boldsymbol{F}$ to a weakly constructible complex on X (e.g. $\boldsymbol{F}_1 = j_!\boldsymbol{F}$). If $\varepsilon > 0$ is sufficiently small, $\varepsilon < \varepsilon_0$, we have*

$$\boldsymbol{H}^s(B_\varepsilon \cap X - \{0\}, B_\varepsilon \cap Y - \{0\}, \boldsymbol{F}_1) = 0, \quad \textit{for any } s < -1.$$

There is also a local version of Theorem 4.2.2:

THEOREM 4.4.4. *Let Y and A be closed complex analytic subsets of X which contain* 0. *Let i and j be the inclusions of $X - A \cup Y$ into $X - Y$ and into $X - A$,* *and let i_1 and j_1 be the inclusions of $X - A$ and of $X - Y$ into $X - \{0\}$. Suppose that the inclusion j_2 of $X - Y$ into X is dually topologically q-complete with respect to $A - Y$. Let $\boldsymbol{F}$ be a complex in ${}^{1/2}D^{\geq 0}(X - A \cup Y, R)$ such that $(j_2)_! i_! \boldsymbol{F}$ is weakly constructible, and let $\boldsymbol{F}_1$ be any extension of $\boldsymbol{F}$ to $X - A$* (e.g. $\boldsymbol{F}_1 = j_! \boldsymbol{F}$). *Suppose that we have the base change condition*

$$(j_1)_! i_* \boldsymbol{F} = (i_1)_* j_! \boldsymbol{F}.$$

If $\varepsilon > 0$ is sufficiently small, $\varepsilon < \varepsilon_0$, we have

$$\boldsymbol{H}^s(B_\varepsilon \cap X - A, B_\varepsilon \cap Y - A, \boldsymbol{F}_1) = 0, \quad \textit{for any } s < -q - 1.$$

Proof. We begin as in the proof of Theorem 4.4.2. Let $\varepsilon > 0$ be sufficiently small, $\varepsilon < \varepsilon_0$, let $U := B_\varepsilon \cap X$. By Theorem 3.2.2, the complex $(j_2)_! i_* \boldsymbol{F}$ is in ${}^{1/2}D^{\geq -q}(X, R)$. Let $k : \{0\} \to X$ and $l : X - \{0\} \to X$ be the inclusions, then

$$h^s((k^!(j_2)_! i_*(\boldsymbol{F})_0) = 0, \quad \text{for any } s \leq -q - 1.$$

By [B-B-D], we have the distinguished triangle

$$\to k_! k^!(j_2)_! i_* \boldsymbol{F} \to (j_2)_! i_* \boldsymbol{F} \to l_* l^*(j_2)_! i_* \boldsymbol{F} \xrightarrow{+1} .$$

By taking the long exact cohomology sequence at 0, we obtain

$$h^s((k^!(j_2)_! i_* \boldsymbol{F})_0) = h^{s-1}((l_* l^*(j_2)_! i_* \boldsymbol{F})_0).$$

But we have

$$l_* l^*(j_2)_! i_* \boldsymbol{F} = l_* l^* l_! (j_1)_! i_* \boldsymbol{F} = l_* (j_1)_! i_* \boldsymbol{F}.$$

By hypothesis, we obtain

$$l_* l^*(j_2)_! i_* \boldsymbol{F} = l_*(j_1)_! i_* \boldsymbol{F} = l_*(i_1)_* j_! \boldsymbol{F} = (i_2)_* j_! F = (i_2)_* j_! j^* \boldsymbol{F}_1$$

where i_2 is the inclusion of $X - A$ into X. Finally,

$$\begin{aligned} h^{s-1}(((i_2)_* j_! j^* \boldsymbol{F}_1)_0) &= \boldsymbol{H}^{s-1}(U, (i_2)_* j_! j^* \boldsymbol{F}_1) \\ &= \boldsymbol{H}^{s-1}(U - A, j_! j^* \boldsymbol{F}_1) = \boldsymbol{H}^{s-1}(U - A, U \cap Y - A, \boldsymbol{F}_1). \end{aligned}$$

Theorem 4.4.4 also gives a local Theorem of Zariski-Lefschetz type (compare to [H-L3]):

COROLLARY 4.4.5. *Let Y and A be closed complex analytic subsets of X which contain 0 and suppose that Y is defined by one equation. Let i and j be the inclusions of $X - A \cup Y$ into $X - Y$ and into $X - A$, let i_1 and j_1 be the inclusions of $X - A$ and of $X - Y$ into $X - \{0\}$, and let j_2 be the inclusion of $X - Y$ into X. Let $\boldsymbol{F}$ be a complex in ${}^{1/2}D^{\geq 0}(X - A \cup Y, R)$ such that $(j_2)_! i_! \boldsymbol{F}$ is weakly constructible, and let $\boldsymbol{F}_1$ be any extension of $\boldsymbol{F}$ to $X - A$ (e.g. $\boldsymbol{F}_1 = j_! \boldsymbol{F}$). Suppose that we have the base change condition*

$$(j_1)_! i_* \boldsymbol{F} = (i_1)_* j_! \boldsymbol{F}.$$

If $\varepsilon > 0$ *is sufficiently small,* $\varepsilon < \varepsilon_0$, *we have*

$$\boldsymbol{H}^s(B_\varepsilon \cap X - A, B_\varepsilon \cap Y - A, \boldsymbol{F}_1) = 0, \quad \textit{for any } s < -1.$$

The base change condition of Theorem 4.4.4 is fulfilled if $Y - \{0\}$ is transverse to $A - \{0\}$, see Appendix. We may weaken this condition:

THEOREM 4.4.6. *Let Y and A be closed complex analytic subsets of X which contain 0. Let i and j be the inclusions of $X - A \cup Y$ into $X - Y$ and into $X - A$, and let i_2 and j_2 be the inclusions of $X - A$ and of $X - Y$ into X. Suppose that j_2 is q-complete. Let $\boldsymbol{F}$ be a complex in ${}^{1/2}D^{\geq 0}(X - A \cup Y, R)$ which has a weakly constructible extension $\boldsymbol{F}'$ to X, and let $\boldsymbol{F}_1$ be any extension of $\boldsymbol{F}$ to $X - A$ (e.g. $\boldsymbol{F}_1 = j_!\boldsymbol{F}$). Suppose that there is a closed analytic subspace S of $Y \cap A$ such that $Y - S$ is transverse to $A - S$ and $\boldsymbol{F}'|X - S$. If $\varepsilon > 0$ is sufficiently small, $\varepsilon < \varepsilon_0$, we have*

$$\boldsymbol{H}^s(B_\varepsilon \cap X - A, B_\varepsilon \cap Y - A, \boldsymbol{F}_1) = 0, \quad \textit{for any } s < -q - \dim_{\boldsymbol{C}} S - 1$$

where $\dim \emptyset := -\infty$.

Proof. The case $S = \emptyset$ follows from the appendix, while the case $S = \{0\}$ is covered by Theorem 4.4.4. So let $0 \in S$ and $\dim_{\boldsymbol{C}} S > 0$. Let L be a linear subspace of $\boldsymbol{C}^N$ of codimension $\dim_{\boldsymbol{C}} S$. We may choose L sufficiently general such that $L - \{0\}$ intersects $X - \{0\}$, $A - \{0\}$, $Y - \{0\}$ and $A \cap Y - \{0\}$ transversally. The inclusion of $X - L$ into X is $(\dim_{\boldsymbol{C}} S - 1)$-complete. Furthermore, we may assume without loss of generality that $\boldsymbol{F}_1 = j_!\boldsymbol{F}$. By assumption, j_2 is q-complete, so j too. By Theorem 3.2.2 we have that $j_!\boldsymbol{F}$ is in

$${}^{1/2}D^{\geq -q}(X - A, R).$$

By Theorem 4.4.4, we obtain that

$$\boldsymbol{H}^s(B_\varepsilon \cap X - A, B_\varepsilon \cap X \cap L - A, j_!\boldsymbol{F}) = 0, \quad \text{for any } s \leq -q - \dim_{\boldsymbol{C}} S - 2.$$

By base change, we have that $(j_!\boldsymbol{F})|(X \cap L - A) = (j_L)_!(\boldsymbol{F}|(X \cap L - A \cup Y))$, where j_L is the inclusion of $X \cap L - A \cup Y$ into $X \cap L - A$. So

$$\boldsymbol{H}^s(B_\varepsilon \cap X \cap L - A, j_!\boldsymbol{F}) = \boldsymbol{H}^s(B_\varepsilon \cap X \cap L - A, j_{L!}(\boldsymbol{F}|(X \cap L - A \cup Y))).$$

Since L intersects $X - Y \cup A$ transversally, we can apply (∗∗∗) in the proof of 3.1.5 and we have that the restriction $\boldsymbol{F}|(X \cap L - A \cup Y)$ is in

$${}^{1/2}D^{\geq -\dim_{\boldsymbol{C}} S}(X \cap L - A \cup Y, R).$$

By Theorem 4.4.4 again, we obtain that

$$\boldsymbol{H}^s(B_\varepsilon \cap X \cap L - A, j_{L!}(\boldsymbol{F}|(X \cap L - A \cup Y)) = 0, \quad \text{for any } s \leq -q - \dim_{\boldsymbol{C}} S - 2.$$

Altogether this implies that

$$\boldsymbol{H}^s(B_\varepsilon \cap X - A, j_!\boldsymbol{F}) = 0, \quad \text{for any } s < -q - \dim_{\boldsymbol{C}} S - 1,$$

which is our assertion.

Proof of Theorem 4.2.5. Because of the constructibility of $\boldsymbol{F}$ the assertion of Theorem 4.4.6 can be expressed in the following form:

$$h^s((i_2)_* j_! \boldsymbol{F})_0 = 0, \quad \text{for any } s < -q - \dim_{\boldsymbol{C}} S - 1.$$

This implies the hypothesis to be verified in Theorem 4.2.5.

Note that in the case that V is a subspace of some projective space $\boldsymbol{P}^N$ we could have directly argued similarly to the proof of Theorem 4.4.6, using a general linear subspace of $\boldsymbol{P}^N$.

There is also a local version of Theorem 4.3.7:

THEOREM 4.4.7. *Let Y and A be closed complex analytic subsets of X which contain 0. Let i be the inclusion of $X - A \cup Y$ into $X - Y$. Suppose that the inclusion j_2 of $X - Y$ into X is dually topologically q-complete with respect to $A - Y$. Let $\boldsymbol{F}$ be a complex in ${}^{1/2}D^{\geq 0}(X - A \cup Y, R)$ such that $(j_2)_! i_* \boldsymbol{F}$ is weakly constructible, and let $\boldsymbol{F}_1$ be any extension of $\boldsymbol{F}$ to $X - A$. Let us fix a Whitney stratification of X adapted to A and to $(j_2)_! i_* \boldsymbol{F}$. Let $\varepsilon > 0$ be small enough; then $S_\varepsilon \cap X$ will have an induced Whitney stratification. Let $T(Y)$ be a good neighbourhood of $S_\varepsilon \cap Y$ in $S_\varepsilon \cap X$ with respect to this stratification. Then we have*

$$\boldsymbol{H}^s(S_\varepsilon \cap X - A, T(Y) - A, \boldsymbol{F}_1) = 0, \quad \textit{for any } s < -q - 1.$$

Proof. Let $\varepsilon > 0$ be sufficiently small, $U := B_\varepsilon \cap X$, $\partial U := S_\varepsilon \cap X$. According to Theorem 3.2.2, the direct image $j_{2!} i_* \boldsymbol{F}$ is in ${}^{1/2}D^{\geq -q}(X, R)$. Let $k : \{0\} \to X$ be the inclusion, then the cohomology sheaves $h^s(k^! j_{2!} i_* \boldsymbol{F})$ vanish, for any $s \leq -q - 1$.
On the other hand we have that the stalk $(h^s(k^! j_{2!} i_* \boldsymbol{F}))_0$ is isomorphic to

$$\boldsymbol{H}^s(U, U - \{0\}, j_{2!} i_* \boldsymbol{F}).$$

Moreover

$$\boldsymbol{H}^s(U, j_{2!} i_* \boldsymbol{F}) = (h^s(j_{2!} i_* \boldsymbol{F}))_0 = 0$$

therefore $(h^s(k^! j_{2!} i_* \boldsymbol{F}))_0$ is isomorphic to $\boldsymbol{H}^{s-1}(U - \{0\}, j_{2!} i_* \boldsymbol{F})$ which is isomorphic to

$$\boldsymbol{H}^{s-1}(\partial U, \rho_1^* j_{2!} i_* \boldsymbol{F})$$

where ρ_1 is the inclusion of ∂U into X, see Lemma 4.4.1. Let j_ε and ρ be the inclusions of $\partial U - Y$ into ∂U and into $X - Y$. Then $\rho_1^* j_{2!} i_* \boldsymbol{F} = j_{\varepsilon!} \rho^* i_* \boldsymbol{F}$.
Let i_ε and $\tilde{\rho}$ be the inclusions of $\partial U - A \cup Y$ into $\partial U - Y$ and into $X - A \cup Y$. Since S_ε intersects the stratification of X transversally, we have $\rho^* i_* \boldsymbol{F} = (i_\varepsilon)_* \tilde{\rho}^* \boldsymbol{F}$ (see Appendix Lemma A.3).
Let $\hat{j}$ be the inclusion of $\partial U - A \cup Y$ into $\partial U - A$, $\hat{\rho}$ the inclusion of $\partial U - A$ into $X - A$, and j the inclusion of $X - A \cup Y$ into $X - A$. Then $j \circ \tilde{\rho} = \hat{\rho} \circ \hat{j}$, so

$\tilde{\rho}^* F = \tilde{\rho}^* j^* F_1 = \hat{j}^* \hat{\rho}^* F_1$. Altogether we have

$$\rho_1^* j_{2!} i_* F = j_{\varepsilon !} \rho^* i_* F = j_{\varepsilon !}(i_\varepsilon)_* \tilde{\rho}^* F = j_{\varepsilon !}(i_\varepsilon)_* \hat{j}^* \hat{\rho}^* F_1 = j_{\varepsilon !}(i_\varepsilon)_*(F_1|(\partial U - A \cup Y)).$$

Therefore, using a proposition for subanalytic spaces similar to Proposition 4.3.6 with $V = \partial U$, where $\varepsilon_2 < \varepsilon < \varepsilon_1$, we have

$$H^{s-1}(\partial U, \rho_1^* j_{2!} i_* F) = H^{s-1}(\partial U - A, T(Y) - A, F_1 | \partial U - A)$$

which gives the desired result.

Finally Theorem 4.4.7 gives a local Lefschetz theorem relatively to a neighbourhood of a hyperplane section, as proposed by P. Deligne in [D1]:

COROLLARY 4.4.8. *Let Y and A be closed complex analytic subsets of X which contain 0 and suppose that Y is defined by one equation. Let i be the inclusion of $X - A \cup Y$ into $X - Y$. Let j_2 be the inclusion of $X - Y$ into X. Let F be a complex in*

$$^{1/2}D^{\geq 0}(X - A \cup Y, R)$$

such that $(j_2)_! i_* F$ is weakly constructible, and let F_1 be any extension of F to $X - A$. Let us fix a Whitney stratification of X adapted to A and to $(j_2)_! i_* F$. Let $\varepsilon > 0$ be small enough; then $S_\varepsilon \cap X$ will have an induced Whitney stratification. Let $T(Y)$ be a good neighbourhood of $S_\varepsilon \cap Y$ in $S_\varepsilon \cap X$ with respect to this stratification. Then we have

$$H^s(S_\varepsilon \cap X - A, T(Y) - A, F_1) = 0, \quad \textit{for any } s < -1.$$

Appendix

In this appendix we prove the base change theorem that we needed in this paper.

DENITION A.1. Let V be a closed subanalytic subset of a real analytic manifold M. Let A and W_0 be closed subanalytic subsets of V. Let F be a weakly constructible complex on V. Then, W_0 is called transverse to A and F, if there is a closed subanalytic subset W of M, a weakly constructible extension F_1 of F to M and Whitney stratifications $\mathcal{S}$ of V and $\mathcal{T}$ of M adapted to F_1 such that

a) $W_0 = W \cap V$,

b) A is a union of strata of $\mathcal{S}$,

c) W is a union of strata of $\mathcal{T}$,

d) S and T intersect transversally within M, where S and T are arbitrary strata of $\mathcal{S}$ resp. $\mathcal{T}$ contained in A and W, *respectively.*

Note that the condition is not symmetric in A and W_0.

LEMMA A.2. *Let V be a closed subanalytic subset of a real analytic manifold M, and let A and W_0 be closed subanalytic subsets of V. Let F be a complex on*

$V - A \cup W_0$ and let G be a weakly constructible complex on $V - A$ which have weakly constructible extensions F_1 and G_1 to V. Assume that W_0 is transverse to A and the weakly constructible complexes F_1 and G_1. Let i and j be the inclusions of $V - A \cup W_0$ into $V - W_0$ and into $V - A$. Let i_1, j_1 and k_1 be the inclusions of $V - A$, $V - W_0$ and W_0 into V. Let k and i_2 be the inclusions of $W_0 - A$ into $V - A$ and W_0.

$$\begin{array}{ccccc} W_0 - A & \xrightarrow{k} & V - A & \xleftarrow{j} & V - A \cup W_0 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i} \\ W_0 & \xrightarrow{k_1} & V & \xleftarrow{j_1} & V - W_0 \end{array}$$

Then we have the following base change properties:

$$(j_1)_! i_* F = (i_1)_* j_! F,$$
$$(k_1)^* (i_1)_* G = (i_2)_* k^* G.$$

Proof. Let W, $\mathscr{S}$ and $\mathscr{T}$ be chosen as in Definition A.1. To prove the first equality in the lemma, we first show that there is a morphism of $j_{1!} i_*$ to $i_{1*} j_!$.

We may identify $i^* i_* F$ with F, so that

$$i_{1*} j_! F = i_{1*} j_! i^* i_* F.$$

By usual base change $j_! i^* \simeq i_1^* j_{1!}$, so

$$i_{1*} j_! i^* i_* F \simeq i_{1*} i_1^* j_{1!} i_* F.$$

We have an adjunction morphism

$$j_{1!} i_* F \to i_{1*} i_1^* j_{1!} i_* F$$

which yields

$$j_{1!} i_* F \xrightarrow{\alpha} i_{1*} j_! F.$$

To prove that α is an isomorphism, we prove that, for any $x \in V$, it induces an isomorphism of the cohomology of the stalks.

For $x \in V - W$, it is enough to prove that the image $j_1^* \alpha$ of α by j_1^* is an isomorphism. In fact, $j_1^* \alpha$ is the composition of $j_1^* j_{1!} i_* F \simeq i_* F \simeq i_* j^* j_! F \simeq j_1^* i_{1*} j_! F$.

For $x \in V - A$, it is enough to prove that the image $i_1^* \alpha$ of α by i_1^* is an isomorphism. Now, $i_1^* \alpha$ is the composition of $i_1^* j_{1!} i_* F \simeq j_! i^* i_* F \simeq j_! F \simeq i_1^* i_{1*} j_! F$.

It remains to prove that, for $x \in A \cap W$, α gives an isomorphism. In fact, we will show that the cohomology sheaves of $j_{1!} i_* F$ and of $i_{1*} j_! F$ vanish at x.

However $h^k(j_{1!} i_* F)_x = 0$, because W_0 is closed in V.

We have an induced stratification $\mathscr{S}'$ of V whose strata are the connected components of $S \cap T$, where S and T are strata of $\mathscr{S}$ and $\mathscr{T}$, respectively. We prove our assertion by induction on $-\dim S'$, where S' is the stratum of $\mathscr{S}'$ which contains x.

Consider $x \in A \cap W$ and an adequate neighbourhood U of x in V such that

$$h^k(i_{1*}j_!\boldsymbol{F})_x \simeq \boldsymbol{H}^k(U, i_{1*}j_!\boldsymbol{F}).$$

So, we have to prove that $(h^k(i_{1*}j_!\boldsymbol{F})_x$ or) $\boldsymbol{H}^k(U, i_{1*}j_!\boldsymbol{F})$ is 0. Leray spectral sequence applied to i_1 gives

$$\boldsymbol{H}^k(U, i_{1*}j_!\boldsymbol{F}) = \boldsymbol{H}^k(U - A, j_!\boldsymbol{F}).$$

Let S, S', and T be the strata of $\mathscr{S}$, $\mathscr{S}'$ and $\mathscr{T}$ which contain x. Then we may suppose that $S' \cap U = S \cap T \cap U$. We may write $U = \tilde{U} \cap V$, $\tilde{U}$ being an open subset of M. We may identify $\tilde{U}$ with an open subset of $\boldsymbol{R}^n$ in such a way that $\tilde{U} \cap S'$ corresponds to the intersection by some linear subspace L. Therefore we have a projection

$$\pi : U \to S' \cap U$$

which corresponds to the orthogonal projection onto L.

Now, let ϕ and ψ denote the distance functions to S and T, defined on U, and d the distance function to x, defined on $S' \cap U$. Let ε, α and β be positive real numbers which are sufficiently small. With a suitable choice of U, the map (π, ϕ, ψ) defines a proper stratified submersion of $U - S \cup T$ onto $(S' \cap U \cap \{d < \varepsilon\}) \times]0, \alpha[\times]0, \beta[$. Also, we have a stratified submersion

$$(\pi, \phi) : U - S \to (S' \cap U \cap \{d < \varepsilon\}) \times]0, \alpha[.$$

Since $(S' \cap U \cap \{d < \varepsilon\}) \times \{\alpha'\}$ is a deformation retract of $(S' \cap U \cap \{d < \varepsilon\}) \times]0, \alpha[$, $0 < \alpha' < \alpha$, we obtain that $U \cap \{\phi = \alpha'\}$ is a deformation retract of $U - S$ in the stratified sense for the given stratifications.

Since $S \subset A$ we obtain from [H2] (Theorem 2.9)

$$\boldsymbol{H}^k(U - A, j_!\boldsymbol{F}) = \boldsymbol{H}^k((U - A) \cap \{\phi = \alpha'\}, j_!\boldsymbol{F}).$$

Furthermore, we have that, for $0 < \beta' < \beta$, the space $(S' \cap U \cap \{d < \varepsilon\}) \times \{\alpha'\} \times \{\beta'\}$ is a deformation retract of $(S' \cap U \cap \{d < \varepsilon\}) \times \{\alpha'\} \times [\beta', \beta[$, which implies that

$$U \cap \{\phi = \alpha'\} \cap \{\psi = \beta'\}$$

is a deformation retract of $U \cap \{\phi = \alpha'\} \cap \{\psi \geq \beta'\}$ in the stratified sense for the given stratifications. So

$$0 = \boldsymbol{H}^k((U - A) \cap \{\phi = \alpha'\} \cap \{\psi \geq \beta'\}, (U - A) \cap \{\phi = \alpha'\} \cap \{\psi = \beta'\}, j_!\boldsymbol{F})$$
$$= \boldsymbol{H}^k((U - A) \cap \{\phi = \alpha'\}, (U - A) \cap \{\phi = \alpha'\} \cap \{\psi \leq \beta'\}, j_!\boldsymbol{F})$$

which means that

$$\boldsymbol{H}^k((U - A) \cap \{\phi = \alpha'\}, j_!\boldsymbol{F}) = \boldsymbol{H}^k(U \cap \{\phi = \alpha'\} \cap \{\psi \leq \beta'\}, i_{1*}j_!\boldsymbol{F}).$$

Now we can pass to the direct limit:

$$\varinjlim_{\beta'} \boldsymbol{H}^k(U \cap \{\phi = \alpha'\} \cap \{\psi \leq \beta'\}, i_{1*}j_!\boldsymbol{F}) = \boldsymbol{H}^k(U \cap T \cap \{\phi = \alpha'\}, i_{1*}j_!\boldsymbol{F}).$$

By induction, we have

$$H^k(U \cap T \cap \{\phi = \alpha'\}, i_{1*}j_!F) = H^k(U \cap T \cap \{\phi = \alpha'\}, j_{1!}i_*F) = 0.$$

for any k, because the cohomology sheaves of $j_{1!}i_*F$ vanish along $V \cap T \subset V \cap W$. This implies

$$0 = \varinjlim_{\beta'} H^k(U \cap \{\phi = \alpha'\} \cap \{\psi \leq \beta'\}, i_{1*}j_!F) = H^k((U - A) \cap \{\phi = \alpha'\}, j_!F)$$
$$= H^k(U - A, j_!F) = H^k(U, i_{1*}j_!F)$$

which finishes the proof of the first equality. It remains to prove the second one.

We prove that there is a natural morphism

$$(k_1)^*(i_1)_*G \xrightarrow{\beta} (i_2)_*k^*G$$

which is defined from the adjunction morphism

$$(k_1)^*(i_1)_*G \to (i_2)_*(i_2)^*(k_1)^*(i_1)_*G$$

and the equalities

$$(i_2)^*(k_1)^* = k^*(i_1)^*,$$
$$(i_1)^*(i_1)_*G = G.$$

We shall prove that β is an isomorphism. To do so, we make use of the distinguished triangle

$$\to j_!j^*G \to G \to k_*k^*G \xrightarrow{+1}.$$

Then, it is enough to check that β induces isomorphisms for $j_!j^*G$ and k_*k^*G. For k_*k^*G, β induces

$$(k_1)^*(i_1)_*k_*k^*G \xrightarrow{\beta} (i_2)_*k^*k_*k^*G$$

which is the identity, since $(i_1)_*k_* = (k_1)_*(i_2)_*$ and $k^*k_* = (k_1)^*(k_1)_* = \mathrm{Id}$.

For $j_!j^*G$, β induces

$$(k_1)^*(i_1)_*j_!j^*G \xrightarrow{\beta} (i_2)_*k^*j_!j^*G$$

which is the trivial isomorphism, since $k^*j_! = (k_1)^*(j_1)_! = 0$ and $(j_1)_!i_* = (i_1)_*j_!$ by the first equality of the lemma proved above. This ends the proof of Lemma A.2.

In particular, we have the following consequence:

Lemma A.3. *Let M be a real analytic manifold, A and V closed subanalytic subsets of M, $A \subset V$. Let $\mathscr{S}$ be a Whitney stratification of (V, A). Let $\phi : M \to \mathbf{R}$ be a real analytic function, and let R be a regular value for the restrictions of ϕ to the strata of V, $W = \{\phi = R\}$ (or: $W = \{\phi \geq R\}$). Let i and j be the inclusions of $V - A \cup W$ into $V - W$ and into $V - A$, let i_1, j_1 and k_1 be* *the inclusions of $V - A$, $V - W$ and W into V. Let k and i_2 be the inclusions of $W - A$ into $V - A$ and W.*

$$\begin{array}{ccccc} W - A & \xrightarrow{k} & V - A & \xleftarrow{j} & V - A \cup W \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i} \\ W & \xrightarrow{k_1} & V & \xleftarrow{j_1} & V - W \end{array}$$

Let $\boldsymbol{F}$ be a complex on $V - A \cup W$ and let $\boldsymbol{G}$ be a weakly constructible complex on $V - A$ which have weakly constructible extensions to V to which the Whitney stratification $\mathcal{S}$ is adapted. Then we have the following base change properties:

$$(j_1)_! i_* \boldsymbol{F} = (i_1)_* j_! \boldsymbol{F},$$
$$(k_1)^*(i_1)_* \boldsymbol{G} = (i_2)_* k^* \boldsymbol{G}.$$

Proof. This follows from Lemma A.2: first, we may assume the R is a regular value of ϕ, then, the stratification $\mathcal{T}$ consists of the strata $\{\phi < R\}$, $\{\phi = R\}$ and $\{\phi > R\}$.

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