Y. FUKUMA
KODAI MATH. J.
21 (1998), 153–178

ON SECTIONAL GENUS OF k-VERY AMPLE LINE BUNDLES ON SMOOTH SURFACES WITH NON-NEGATIVE KODAIRA DIMENSION

YOSHIAKI FUKUMA*

Abstract

Let (X, L) be a polarized surface over the complex number field. Assume that L is k-very ample. In this paper, we study the relation between the sectional genus $g(L)$ and the irregularity $q(X)$. In particular we prove $g(L) \geq (k+2)q(X)$ if X has the Kodaira dimension $\kappa(X) = 0$, 1, or (X, L) is some special cases with $\kappa(X) = 2$. Moreover we classify (X, L) with $g(L) = (k+2)q(X)$ when $\kappa(X) = 0$ or 1.

§ 0. Introduction

Let X be a smooth projective manifold over the complex number field with $\dim X = n \geq 2$ and let L be an ample line bundle on X. Then we call (X, L) a polarized manifold. The sectional genus of (X, L) is defined by the following formula:

$$g(L) := 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1},$$

where K_X is the canonical divisor of X.

In [Fk1] and [Fk2], we studied the relation between the sectional genus and the irregularity of X. In particular, we considered the following Conjecture in the case of $\dim X = 2$.

CONJECTURE 1. *Let (X, L) be a polarized manifold. Then $g(L) \geq q(X)$, where $q(X)$ is the irregularity of X.*

It is not known whether this Conjecture is true or not even if $\dim X = 2$. But if L is ample and spanned, this Conjecture is true.

In this paper, we consider the case in which X is a smooth projective surface with $\kappa(X) \geq 0$ and L is k-very ample (see Definition 1.1). In this case, we propose the following Conjecture about sectional genus:

1991 *Mathematics Subject Classification.* Primary 14C20.
Key words and phrases. Polarized surface, sectional genus, k-very ample.
*Research Fellow of the Japan Society for the Promotion of Science
Received October 7, 1997; revised February 16, 1998.

CONJECTURE 2. *Let* (X, L) *be a polarized surface. Assume that* L *is* k*-very ample and* $\kappa(X) \geq 0$. *Then* $g(L) \geq (k+2)q(X)$.

If L is ample but not spanned, then we put $k := -1$. Then Conjecture 2 is considered as a generalization of Conjecture 1 when X is a surface with $\kappa(X) \geq 0$. In this paper, we consider the case in which $k \geq 0$ and we will prove Conjecture 2 if (X, L) is one of the following cases:

(1) The case in which $\kappa(X) = 0$ and $k \geq 0$ (see Section 2),

(2) The case in which $\kappa(X) = 1$ and $k \geq 0$ (see Section 3),

(3) The case in which (X, L) is one of the special cases with $\kappa(X) = 2$ (see Section 4).

Furthermore if $\kappa(X) = 0$ or 1, then we will classify (X, L) with $g(L) = (k+2)q(X)$.

In general, the inequality $g(L) \geq (k+2)q(X)$ is not true if $\kappa(X) = -\infty$ and L is k-very ample. In Appendix, we consider a lower bound for sectional genus of k-very ample line bundle with $\kappa(X) = -\infty$.

In this paper we work over the complex number field and we use the customary notation in algebraic geometry.

The author would like to express his gratitude to the referee for giving him many valuable comments and suggestions.

§ 1. Preliminaries

DENITION 1.1 (See [BeSo1] or [BeSo2]). Let (X, L) be a polarized surface. Then L is called k-very ample for a nonnegative integer k if for any 0-dimensional subscheme $(Z, \mathcal{O}_Z)$ with length $\mathcal{O}_Z \leq k+1$, the map

$$\Gamma(L) \to \Gamma(L \otimes \mathcal{O}_Z)$$

is surjective.

THEOREM 1.2 (Fujita). *Let* (X, L) *be a polarized manifold with* $\dim X = n \geq 2$ *and let* $\Delta(L) := n + L^n - h^0(L)$ *be the delta genus of* (X, L).

(1) $\Delta(L) \geq 0$. *If* $\Delta(L) = 0$, *then* $\kappa(X) = -\infty$ *and* $q(X) = 0$.

(2) *If* $\mathrm{Bs}|L| = \emptyset$, $g(L) \geq \Delta(L)$, *and* $L^n \geq 2\Delta(L) + 1$, *then* $g(L) = \Delta(L)$, $\kappa(X) = -\infty$, L *is very ample, and* $q(X) = 0$.

(3) *If* X *is a smooth surface with* $\kappa(X) \geq 1$, $\mathrm{Bs}|L| = \emptyset$, $g(L) > \Delta(L)$, *and* $L^2 = 2\Delta(L)$, *then* $q(X) = 0$.

Proof. (1) See (1.4.2) and (1.5.10) in [Fj2].

(2) See (1.3.5) in [Fj2].

(3) By assumption and Theorem 1.4 in [Fj1], (X, L) is a hyperelliptic polarized manifold. So by (6.1) in [Fj1] we get $q(X) = 0$. □

PROPOSITION 1.3. (1) *Let* X *be a smooth projective variety with* $\dim X = n \geq 2$ *and let* $\pi : X \to \boldsymbol{P}^n$ *be a double covering. Then* $q(X) = 0$.

(2) *Let X be a smooth projective surface and let L be an ample and spanned line bundle on X. If $L^2 \leq 2$, then $q(X) = 0$.*

Proof. (1) See Theorem 1 in [La].

(2) Since L is ample and spanned, we get $h^0(L) \geq 3$. Hence we get $\Delta(L) \leq 1$. If $\Delta(L) = 1$, then $L^2 = 2$ and $h^0(L) = 3$. Therefore there exists a double covering $\pi : X \to \boldsymbol{P}^2$ defined by $|L|$. By (1), we get $q(X) = 0$. If $\Delta(L) = 0$, then by Theorem 1.2 we have $q(X) = 0$. □

Proposition 1.4. *Let X be a smooth projective surface and let $\pi : X \to \boldsymbol{P}^2$ be a triple covering defined by an ample and spanned line bundle L. Let $\mathscr{E}$ be a vector bundle of rank two on $\boldsymbol{P}^2$ such that $\pi_*(\mathcal{O}_X) = \mathcal{O}_{\boldsymbol{P}^2} \oplus \mathscr{E}$ and let $c_2 := c_2(\mathscr{E})$. Then the following hold:*

(1) $\chi(\mathcal{O}_X) = (1/2)g(L)(g(L) + 1) + 2 - c_2$,

(2) $K_X^2 = 2g(L)^2 - 4g(L) + 11 - 3c_2$.

Proof. See Lemma 3.2 in [Bes]. □

Lemma 1.5. *Let (X, L) be a polarized surface with $\kappa(X) \geq 0$. Assume that L is spanned and $g(L) \leq 2$. Then $q(X) = 0$.*

Proof. Since $\kappa(X) \geq 0$, we get $L^2 \leq 2$. By Proposition 1.3 (2), we get $q(X) = 0$. □

Proposition 1.6. *Let X be a smooth projective surface which is embedded by a very ample line bundle L in $\boldsymbol{P}^4$. Then*

$$L^2(L^2 - 5) - 10(g(L) - 1) + 12\chi(\mathcal{O}_X) = 2K_X^2.$$

Proof. See p. 434 in [Ha]. □

Theorem 1.7 (Di Rocco). *Let (X, L) be a polarized surface with $\kappa(X) \geq 0$. If L is a k-very ample line bundle with $L^2 \leq 4k + 4$ and $k \geq 2$, then X is a minimal $K3$-surface or a minimal Enriques surface.*

Proof. By using the same argument as in Section 6 in [Di], it is sufficient to prove the following Claim.

Claim 1.7.1. *Let L be a k-very ample line bundle on X with $k \geq 2$, $L^2 \leq 4k + 4$, and $g(L) \leq 3k + 1$. If $\kappa(X) \neq -\infty$, then X is either a minimal $K3$ surface or a minimal Enriques surface.*

Proof. Since $g(L) \leq 3k + 1$, we get $h^1(L_C) \leq 1$ by Proposition 2.5 in [Di], where $C \in |L|$ is a smooth irreducible curve. Assume that $\kappa(X) \geq 1$. Then we remark that $K_X L > 0$.

If $h^1(L_C) = 1$, then by Theorem 2.5 in [BaSo] and the assumption we get $L^2 \geq 2k + g(L)$ because $K_C \neq L_C$. Since

$$2k + 2 \geq \frac{1}{2}L^2 \leq 2k + \frac{1}{2}K_X L + 1,$$

we get $K_X L \leq 2$. By Theorem 4.4 in [BaSo], we get $K_X L \geq (k+2)/2$. Hence $k = 2$ and $K_X L = 2$. On the other hand by Corollary 2.6 in [Di] we get $K_X L \leq k - 1$ and this is a contradiction.
If $h^1(L_C) = 0$, then $L^2 \geq 2k + g(L) + 1$ or $L^2 \geq k + 2g(L)$ by Lemma 2.10 in [BaSo]. If $L^2 \geq 2k + g(L) + 1$, then

$$2k + 2 \geq \frac{1}{2}L^2 \geq 2k + 2 + \frac{1}{2}K_X L$$

and so we get $K_X L \leq 0$. This is a contradiction. If $L^2 \geq k + 2g(L)$, then $-2 - k \geq K_X L$ and this is a contradiction.
Therefore $\kappa(X) = 0$. By Corollary 2.6 in [Di], we get $K_X L \leq k - 1$. So by Theorem 4.4 in [BaSo], X is minimal. Assume that $q(X) \geq 1$. Then $\chi(\mathcal{O}_X) = 0$ by the classification theory of surfaces and we get $h^0(L) = L^2/2$. On the other hand, by Lemma 2.8 in [BaSo], we get that $h^0(L) \geq 2k + 3$ and $L^2 \geq 4k + 6$. This is a contradiction by assumption. Therefore $q(X) = 0$ and X is a minimal K3 surface or a minimal Enriques surface. □

LEMMA 1.8. *Let (X, L) be a polarized surface such that $\kappa(X) = 0$ and X is not minimal. Let $\mu : X \to S$ be the minimalization of X and let $A := \mu_*(L)$ in the sense of cycle theory.*

(1) *Assume that $(S, A) \cong (E_1 \times E_2, p_1^* D_1 + p_2^* D_2)$, where E_i is a smooth elliptic curve, p_i is the i-th projection, and $D_i \in \mathrm{Pic}(E_i)$ for $i = 1$ and 2 with $\deg D_1 = 1$ and $\deg D_2 \geq 1$. Then $\mathrm{Bs}\,|L| \neq \emptyset$.*

(2) *If $\mathrm{Bs}|L| = \emptyset$ and $A^2 \geq 6$, then $\mathrm{Bs}|A| = \emptyset$.*

Proof. (1) Let $f_2 = p_2 \circ \mu$ and let F_2 be a general fiber of f_2. Then F_2 is a smooth elliptic curve. Since $LF_2 = \mu^*(A)F_2 = 1$, we get $\mathrm{Bs}\,|L| \neq \emptyset$.

(2) By assumption and (1), $(S, A) \not\cong (E_1 \times E_2, p_1^* D_1 + p_2^* D_2)$. Since $A^2 \geq 6$, we get that $\mathrm{Bs}\,|A| = \emptyset$ by Theorem 2.1 in [Fk5] (see also Chapter 10, § 1 in [LB]). □

LEMMA 1.9. *Let (X, L) be a polarized surface. Assume that L is k-very ample with $k \geq 0$. Then*

(1) *$LC \geq \max\{k, 1\}$ for any irreducible curve C.*

(2) *$LC \geq k + 2$ for any irreducible curve C with $C \not\cong \boldsymbol{P}^1$.*

(3) *$LC \geq k + 3$ for any irreducible curve C with $g(C) \geq 2$ and $k \geq 1$.*

Proof. (1) Since L is ample, we get $LC \geq 1$. Hence we obtain (1) by Corollary 1.3 in [BeSo1].

(2) If $C \not\cong \boldsymbol{P}^1$ and $k = 0$ (resp. 1), then $LC \geq 2$ (resp. ≥ 3) since L is ample. If $k \geq 2$, then (2) is obtained by Proposition 1.4 in [BeSo1].

(3) This can be easily obtained by (2) and Proposition 1.4 in [BeSo1]. □

Lemma 1.10. *Let (X, L) be a polarized surface with $\kappa(X) \geq 0$ and let $f : X \to C$ be a fiber space with $g(F) \geq 1$ for a general fiber F of f, where C is a smooth projective curve. (For the definition of a fiber space, see Definition 1.14 below.) Assume that f is not relatively minimal and any fiber of the relatively minimal model of f is smooth. If L is k-very ample with $k \geq 0$, then $LF \geq \max\{2k+2, k+3\}$.*

Proof. By assumption there exists a fiber F of f such that $F = F_s + \sum_i a_i E_\iota$, where F_s is a smooth irreducible curve with $g(F_s) \geq 1$ and E_ι is a smooth rational curve. Since L is ample and k-very ample, we get $LF_s \geq k+2$ and $LE_\iota \geq \max\{k, 1\}$ by Lemma 1.9. Hence $LF \geq \max\{2k+2, k+3\}$. □

Lemma 1.11. *Let (X, L) be a polarized surface with $\kappa(X) = 1$ and let $f : X \to C$ be an elliptic fibration, where C is a smooth projective curve. Assume that $q(X) = g(C) + 1$ and the relatively minimal model of f has a multiple fiber. If L is a k-very ample line bundle with $k \geq 0$, then $LF \geq 2(k+2)$, where F is a general fiber of f.*

Proof. Let $f' : X' \to C$ be the relatively minimal model of f and let $\mu : X \to X'$ be its birational morphism. Let $F_1 = mF_r$ be a multiple fiber of f'. Since $q(X') = q(X) = g(C) + 1$, F_r is a smooth elliptic curve. Let $(F_r)_s$ be the strict transform of F_r via μ. Then $L(F_r)_s \geq k+2$ by Lemma 1.9. Hence $LF = L\mu^*(F_1) \geq 2(k+2)$. □

Lemma 1.12. *Let (X, L) be a polarized surface with $\kappa(X) = 1$ and let $f : X \to C$ be an elliptic fibration, where C is a smooth curve. Let $f' : X' \to C$ be the relatively minimal model of f and let $\mu : X \to X'$ be its birational morphism. Assume that $q(X) = g(C) + 1$, L is k-very ample with $k \geq 0$, and f' has a multiple fiber. Then $K_X L \geq 2(k+2)(2q(X) - 4) + 2(k+2)$.*

Proof. By assumption f' has at least 2 multiple fibers (see Proposition 1.3 in [Se]). Let $m_i F_i$ be a multiple fiber of f'. Then F_i is a smooth elliptic curve because $q(X') = g(C) + 1$. Then $L'(m_i - 1)F_i = L\mu^*((m_i - 1)F_i) \geq k+2$ by Lemma 1.9, where $L' = \mu_*(L)$ in the sense of cycle theory. Hence by the canonical bundle formula and Lemma 1.11 we get $K_X L \geq K_{X'} L' \geq 2(k+2)(2q(X) - 4) + 2(k+2)$. □

Proposition 1.13. *Let X be a smooth projective surface of general type. Assume that X is minimal and $q(X) \geq 1$. Then $K_X^2 \geq 2p_g \geq 2q(X)$.*

Proof. See Théorém 6.1 in [De]. □

DENITION 1.14. Let X be a smooth projective surface, let C be a smooth projective curve, and let $f : X \to C$ be a surjective morphism with connected fibers. Then (f, X, C) is called a fiber space with $\dim X = 2$. If L is an ample line bundle, then we call (f, X, C, L) a polarized fiber space.

LEMMA 1.15. *Let (f, X, C, L) be a polarized fiber space with* $\dim X = 2$ *and $g(F) \geq 2$ for a general fiber F of f. Then*

(1) *If f is relatively minimal, then $K_{X/C}$ is nef, where $K_{X/C} := K_X - f^*(K_C)$ is the relative canonical divisor. Furthermore if $K_{X/C}^2 = 0$ then (f, X, C) is locally trivial.*

(2) $K_{X/C}L \geq 0$.

(3) $K_{X/C} + L$ *is nef if* $\kappa(X) \geq 0$.

Proof. (1) See [Bea].

(2) See Claim 5.6 in [Fk1].

(3) See Lemma 2.5 in [Fk3]. □

LEMMA 1.16. *Let (f, X, C) be a fiber space with* $\dim X = 2$. *Then $q(X) \leq g(F) + g(C)$, where F is a general fiber of f. If $g(F) \geq 2$ and $q(X) = g(F) + g(C)$, then $X \sim_{\mathrm{bir}} F \times C$.*

Proof. See Lemme in [Bea]. □

LEMMA 1.17. *Let (f, X, C, L) be a polarized fiber space with* $\dim X = 2$ *and $\kappa(X) = 2$. Assume that f is locally trivial and L is k-very ample with $k \geq 0$. Then $g(L) \geq (k+2)q(X)$.*

Proof. By assumption there exist a smooth projective surface S, a smooth projective curve B, etale coverings $\pi : S \to X$ and $\varepsilon : B \to C$, and a fiber space $p : S \to B$ such that $S \cong B \times F$, p is the first projective, and $\varepsilon \circ p = f \circ \pi$. By Lemma 1.12 in [Fk1], we get

$$q(X) \leq g(C) + \frac{1}{\deg \pi}(g(F) - 1) + 1$$

for a fiber F of f. Since $\kappa(X) = 2$, we remark that $g(F) \geq 2$ and $g(C) \geq 2$. We calculate $K_{X/C}L$;

$$\begin{aligned} K_{X/C}L &= (K_{B\times F/B}\pi^*(L)) \times \frac{1}{\deg \pi} \\ &= (2g(F) - 2)(\pi^*(L)B) \times \frac{1}{\deg \pi}. \end{aligned}$$

(I) The case in which $k = 0$.
Since $\pi^*(L)B = L\pi_*(B) \geq 2$ by Lemma 1.9, we get

$$g(L) = g(C) + \frac{1}{2}(K_{X/C} + L)L + (LF - 1)(g(C) - 1)$$
$$\geq g(C) + 2(g(F) - 1) \times \frac{1}{\deg \pi} + \frac{1}{2}L^2 + (LF - 1)(g(C) - 1).$$

If $LF \geq 3$, then

$$g(L) \geq g(C) + 2(g(F) - 1) \times \frac{1}{\deg \pi} + \frac{1}{2}L^2 + 2(g(C) - 1)$$
$$= 2\left(g(C) + \frac{1}{\deg \pi}(g(F) - 1) + 1\right) + \frac{1}{2}L^2 + g(C) - 4.$$

Then $g(L) \geq 2q(X)$ if $L^2 \geq 3$ because $g(C) \geq 2$.
If $L^2 = 2$, then by Proposition 1.3 we get $q(X) = 0$ and $g(L) > 2q(X)$.
If $LF = 2$, then

$$g(L) \geq g(C) + 2(g(F) - 1) \times \frac{1}{\deg \pi} + \frac{1}{2}L^2 + (g(C) - 1)$$
$$= 2\left(g(C) + \frac{1}{\deg \pi}(g(F) - 1) + 1\right) + \frac{1}{2}L^2 - 3.$$

Hence $g(L) \geq 2q(X)$ if $L^2 \geq 5$. So we may assume that $L^2 \leq 4$. Since $\mathrm{Bs}\,|L_F| = \emptyset$ for any fiber of F, the natural map

$$f^* \circ f_*\mathcal{O}(L) \to \mathcal{O}(L)$$

is surjective. We put $\mathcal{E} := f_*\mathcal{O}(L)$. Then since $\mathrm{Bs}\,|L_F| = \emptyset$, $g(F) \geq 2$, and $LF = 2$, we obtain that $\mathcal{E}$ is a locally free sheaf of rank two on C and there exists a double covering $\rho : X \to \boldsymbol{P}(\mathcal{E})$ such that $f = p \circ \rho$, where $\boldsymbol{P}(\mathcal{E})$ is the projective bundle of $\mathcal{E}$ on C and $p : \boldsymbol{P}(\mathcal{E}) \to C$ is the bundle map. Let B be the branch locus of ρ. Then there exists $Z \in \mathrm{Pic}(\boldsymbol{P}(\mathcal{E}))$ such that $B \in |2Z|$. Then $K_X = \rho^*(K_{\boldsymbol{P}(\mathcal{E})} + Z)$. (See e.g. [Pe].) By construction $L = \rho^*(H(\mathcal{E}))$, where $H(\mathcal{E})$ is the tautological line bundle of $\boldsymbol{P}(\mathcal{E})$. Then $H(\mathcal{E})$ is ample and $H(\mathcal{E})^2 \leq 2$ because $L^2 \leq 4$. On the other hand, $h^0(L) = h^0(H(\mathcal{E})) + h^0(H(\mathcal{E}) - Z)$. Since $K_X F > 0$, we get that $ZF_p > 2$ for a fiber F_p of p. Hence $h^0(H(\mathcal{E}) - Z) = 0$ because $(H(\mathcal{E}) - Z)F_p < 0$. So we get $h^0(L) = h^0(H(\mathcal{E}))$ and $H(\mathcal{E})$ is spanned. But by Proposition 1.3 (2), we get that $g(C) = q(\boldsymbol{P}(\mathcal{E})) = 0$. This is a contradiction.

(II) The case in which $k \geq 1$.
Since $LF \geq k + 3$ and $\pi^*(L)B = L\pi_*(B) \geq k + 3$ by Lemma 1.9, we get

$$\begin{aligned} g(L) &= g(C) + \frac{1}{2}(K_{X/C} + L)L + (LF - 1)(g(C) - 1) \\ &\geq g(C) + (k+3)(g(F) - 1) \times \frac{1}{\deg \pi} + \frac{1}{2}L^2 + (k+2)(g(C) - 1) \\ &> (k+2)\left(g(C) + \frac{1}{\deg \pi}(g(F) - 1) + 1\right) + \frac{1}{2}L^2 - 2k - 2 + (g(C) - 2). \end{aligned}$$

If $k \geq 2$, then $L^2 \geq 4k + 5$ by Theorem 1.7. Hence $g(L) > (k+2)q(X)$.
If $k = 1$ and $L^2 \geq 6$, then $g(L) \geq 3q(X)$ is obtained.
If $k = 1$, $L^2 \leq 5$, and $h^0(L) \geq 5$, then $L^2 \geq 2\Delta(L) + 1$. Since $\kappa(X) = 2$ and L is very ample, we get $L^2 \geq 2$ and $g(L) \geq 3$. Hence $g(L) \geq 3 > \Delta(L)$. By Theorem 1.2, this is impossible.

If $k = 1$, $L^2 \leq 5$, and $h^0(L) = 4$, then X is a hypersurface in $\boldsymbol{P}^3$. So we get that $q(X) = 0$ and $g(L) > 3q(X)$. □

LEMMA 1.18. *Let (X, L) be a polarized surface with $\kappa(X) = 2$. Assume that L is k-very ample with $k \geq 0$ and $X \sim_{\mathrm{bir}} F \times C$, where $\sim_{\mathrm{bir}}$ denotes birational equivalence, and F and C are smooth projective curves with $g(F) \geq 2$ and $g(C) \geq 2$. Then $g(L) \geq (k+2)q(X)$.*

Proof. Let $\mu : X \to F \times C$ be the minimalization of X and $p : F \times C \to C$ the second projection. Let $f := p \circ \mu$. Then $K_X \equiv \mu^*((2g(F) - 2)C + (2g(C) - 2)F) + E_\mu$, where E_μ is a μ-exceptional effective divisor. So we get

$$\begin{aligned} K_X L &\geq (2g(F) - 2)L\mu^*(C) + (2g(C) - 2)L\mu^*(F) \\ &\geq (k+2)(2g(F) + 2g(C) - 4) \end{aligned}$$

by Lemma 1.9.

(I) The case in which $k \geq 2$.
By Theorem 1.7, we get $L^2 \geq 4k + 5$. Hence

$$\begin{aligned} g(L) &\geq 1 + (k+2)(g(F) + g(C) - 2) + 2k + \frac{5}{2} \\ &= (k+2)q(X) - \frac{1}{2}. \end{aligned}$$

So we obtain $g(L) \geq (k+2)q(X)$.

(II) The case in which $k = 1$.
If $L^2 \geq 9$, then by the same argument as in the case (I) we get $g(L) \geq 3q(X)$. So we may assume that $L^2 \leq 8$. We remark that $L^2 \geq 2$ since $\kappa(X) = 2$. Hence $g(L) \geq 3$.
If $h^0(L) \geq 6$ and $L^2 \leq 7$, then $L^2 \geq 2\Delta(L) + 1$ and $g(L) \geq 3 \geq \Delta(L)$. Hence by Theorem 1.2 this is impossible.
If $h^0(L) \geq 6$ and $L^2 = 8$, then $\Delta(L) \leq 4$. Since $L^2 = 8$ we get $g(L) \geq 6 > \Delta(L)$. If $\Delta(L) = 4$, then by Theorem 1.2 we get $q(X) = 0$. But this is

a contradiction because $q(X) = g(F) + g(C) \geq 4$. If $\Delta(L) \leq 3$, then $L^2 \geq 2\Delta(L) + 1$ and this is impossible by Theorem 1.2.
If $h^0(L) = 5$ and $L^2 \leq 5$, then $g(L) > 2 \geq \Delta(L)$ and $L^2 \geq 2\Delta(L) + 1$. But by Theorem 1.2, this is impossible.
If $h^0(L) = 5$ and $L^2 = 6$, then $g(L) \geq 5 > 3 = \Delta(L)$ and $L^2 = 2\Delta(L)$. Hence by Theorem 1.2 we get $q(X) = 0$. But this is impossible because $q(X) = g(C) + g(F) \geq 4$.
If $h^0(L) = 5$ and $L^2 = 7$, then by Proposition 1.6 we get

$$10(g(L) - 1) = 14 - 4(g(F) - 1)(g(C) - 1) + 2a,$$

where $a := 8(g(F) - 1)(g(C) - 1) - K_X^2$.
We remark that a is nonnegative integer and $K_X L \geq 3(2g(F) + 2g(C) - 4) + a$. Since $g(F) \geq 2$ and $g(C) \geq 2$, we get that $14 - 4(g(F) - 1)(g(C) - 1) \leq 10$. So we get $a \geq 15$ because $g(L) \geq 5$. Hence $K_X L \geq 3(2q(X) - 4) + 15$ and we get $g(L) > 3q(X)$.
If $h^0(L) = 5$ and $L^2 = 8$, then by Proposition 1.6 we get

$$10(g(L) - 1) = 24 - 4(g(F) - 1)(g(C) - 1) + 2a.$$

(We use the same notation as above.) Since $g(F) \geq 2$ and $g(C) \geq 2$, we get that $24 - 4(g(F) - 1)(g(C) - 1) \leq 20$. So we get $a \geq 15$ because $g(L) \geq 6$. Hence $K_X L \geq 3(2q(X) - 4) + 15$ and we get $g(L) > 3q(X)$.

(III) The case in which $k = 0$.
If $L^2 \geq 5$, then by the same argument as in the case (I) we get $g(L) \geq 2q(X)$. So we may assume $L^2 \leq 4$.

(III-1) The case in which $L^2 = 4$.
If X is not minimal, then

$$\begin{aligned} K_X L &\geq 4(g(F) + g(C) - 2) + 1 \\ &= 4q(X) - 7. \end{aligned}$$

Hence we get $g(L) \geq 2q(X)$. So we may assume that X is minimal. But then by Lemma 1.17 we get $g(L) \geq 2q(X)$.

(III-2) The case in which $L^2 = 3$.
By the same argument as in the case (I) we get $g(L) \geq 2q(X) - 1$. Assume that $g(L) = 2q(X) - 1$. Then $K_X L = 4q(X) - 7$. In particular, μ is a simple blowing up of $F \times C$, and $LF = 2$ for a general fiber F of f. Let $F_e := F_1 + E$ be a fiber of f, where F_1 is a smooth curve of genus $g(F_1) \geq 2$ and E is the (-1)-curve of μ. Since L is ample and $LF = 2$, we get $LF_1 = LE = 1$. But this is impossible because $\mathrm{Bs}\,|L| = \emptyset$ and $g(F_1) \neq 0$.

(III-3) The case in which $L^2 \leq 2$.
Then by Proposition 1.3 (2), we get $q(X) = 0$ and this is a contradiction because $q(X) = g(F) + g(C) \geq 4$.

This completes the proof of Lemma 1.18. □

PROPOSITION 1.19 (Castelnuovo's bound). *Let (X,L) be a polarized surface. Assume that L is very ample with $N = h^0(L) - 2$ and $d = L^2$. Then*

$$g(L) \leq \left\lfloor \frac{d-2}{N-1} \right\rfloor \left(d - N - \left(\left\lfloor \frac{d-2}{N-2} \right\rfloor - 1 \right) \frac{N-1}{2} \right).$$

Proof. See [ACGH]. □

§2. The case in which $\kappa(X) = 0$

THEOREM 2.1. *Let (X,L) be a polarized surface with $\kappa(X) = 0$. Assume that L is k-very ample with $k \geq 0$. Then $g(L) \geq (k+2)q(X)$. Furthermore if $g(L) = (k+2)q(X)$, then (X,L) is one of the following;*

(1) (X,L) is a polarized abelian surface with $L^2 = 4k+6$,

(2) $k = 0$, X is a one point blowing up of S, and $L = \mu^*(A) - 2E$, where S is an abelian surface, A is an ample line bundle with $A^2 = 8$, $\mu : X \to S$ is its blowing up, and E is a (-1)-curve of μ.

Proof. (I) The case in which $k = 0$.

(I-A) The proof of $g(L) \geq 2q(X)$.

By the classification theory of surfaces, we get $q(X) \leq 2$.

If $q(X) \leq 1$, then $g(L) \geq 2 \geq 2q(X)$.

If $q(X) = 2$ and $g(L) \geq 4$, then $g(L) \geq 2q(X)$.

If $q(X) = 2$ and $g(L) \leq 3$, then $L^2 \leq 4$. If $g(L) \leq 2$, then $L^2 \leq 2$ and by Proposition 1.3 (2) we get $q(X) = 0$ and this is impossible. If $g(L) = 3$ and $L^2 = 4$, then X is an abelian surface. But then $h^0(L) = 2$ and this is impossible. If $g(L) = 3$ and $L^2 \leq 2$, then by Proposition 1.3 (2) we get $q(X) = 0$ and this is a contradiction. If $g(L) = 3$, $L^2 = 3$, and $h^0(L) \geq 4$, then $L^2 \geq 2\Delta(L) + 1$ and $g(L) > \Delta(L)$. But by Theorem 1.2 this is impossible. If $g(L) = 3$, $L^2 = 3$, and $h^0(L) = 3$, then there exists a triple covering $\varphi_{|L|} : X \to \boldsymbol{P}^2$ defined by $|L|$. Since $K_X L = 1$, we get that $K_X^2 = -1$. But by Proposition 1.4, this is impossible because $\chi(\mathcal{O}_X) = 0$.

Therefore we get $g(L) \geq 2q(X)$.

(I-B) The classification of (X,L) with $g(L) = 2q(X)$.

First we assume that $q(X) \leq 1$. Since $\kappa(X) = 0$, we get $q(X) = 1$ and $g(L) = 2$. But by Lemma 1.5 this is impossible. So we assume that $q(X) = 2$. Then $g(L) = 2q(X) = 4$ and $L^2 \leq 6$.

(I-B-1) The case in which $L^2 \leq 2$.

Then by Proposition 1.3 (2) this is impossible.

(I-B-2) The case in which $L^2 = 3$.

If $h^0(L) \geq 4$, then $\Delta(L) \leq 1$. By Theorem 1.2 (1), we get that $\Delta(L) = 1$ because $\kappa(X) = 0$. Then $L^2 > 2\Delta(L)$ and $g(L) > \Delta(L)$. Hence this is impossible by Theorem 1.2. So we may assume that $h^0(L) = 3$. Then there exists a triple covering $\pi : X \to \boldsymbol{P}^2$ defined by $|L|$. Here we use Proposition 1.4. Since $\chi(\mathcal{O}_X) = 0$ and $g(L) = 4$, we get $c_2 = 12$. On the other hand,

$$\begin{aligned} K_X^2 &= 2g(L)^2 - 4g(L) + 11 - 3c_2 \\ &= -9. \end{aligned}$$

But since $K_X L = 3$, this is a contradiction.

(I-B-3) The case in which $L^2 = 4$.

In this case $K_X L = 2$. In particular X is not minimal. Let $\mu : X \to S$ be the minimalization of X. Then S is an abelian surface. Let $A := \mu_*(L)$. Then $A^2 = 8$ or 6.

(I-B-3-1) The case in which $A^2 = 6$.

Then μ is a composition of two blowing ups. By Lemma 1.8 (2) we get $\mathrm{Bs}\,|A| = \emptyset$. But since $h^0(A) = 3$ and $h^0(L) < h^0(A)$, this is impossible.

(I-B-3-2) The case in which $A^2 = 8$.

Then μ is one point blowing up and $L = \mu^*(A) - 2E$, where E is a (-1)-curve of μ. This is the type (2) is Theorem 2.1.

(I-B-4) The case in which $L^2 = 5$.

In this case, $K_X L = 1$. Then $L = \mu^*(A) - E$ and $A^2 = 6$ and $h^0(A) = 3$. By Lemma 1.8 (2) we get $\mathrm{Bs}\,|A| = \emptyset$. But then $h^0(L) < h^0(A) = 3$ and this is impossible.

(I-B-5) The case in which $L^2 = 6$.

Then this is the type (1) in Theorem 2.1.

(II) The case in which $k = 1$.

(II-A) The proof of $g(L) \geq 3q(X)$.

If $q(X) \leq 1$ and $g(L) \geq 3$, then $g(L) \geq 3q(X)$.

If $q(X) \leq 1$ and $g(L) \leq 2$, then $L^2 \leq 2$. Since L is very ample with $\kappa(X) = 0$, we get that $h^0(L) \geq 4$. But then $\Delta(L) \leq 0$ and $\kappa(X) = -\infty$, a contradiction.

If $q(X) = 2$ and $g(L) \geq 6$, then $g(L) \geq 3q(X)$.

If $q(X) = 2$ and $g(L) \leq 5$, then $L^2 \leq 8$. We remark that $h^0(L) \geq 5$ in this case. (By the above we get $h^0(L) \geq 4$. If $h^0(L) = 4$, then X is a hypersurface of $\boldsymbol{P}^3$. But then $q(X) = 0$ and this is a contradiction.) Hence $\Delta(L) \leq L^2 - 3$.

(II-A-1) The case in which $L^2 \leq 5$.

Then $L^2 \geq 2\Delta(L) + 1$ and $g(L) \geq 2 \geq \Delta(L)$. But by Theorem 1.2 this is impossible because $\kappa(X) = 0$.

(II-A-2) The case in which $L^2 = 8$.

Then X is an abelian surface and $h^0(L) = 4$. But this is impossible because $h^0(L) \geq 5$.

(II-A-3) The case in which $L^2 = 7$.

Then X is not minimal. Let $\mu : X \to S$ be the minimalization of X. Then S is an abelian surface. Let $A := \mu_*(L)$. Then $L = \mu^*(A) - E$, $A^2 = 8$, and $h^0(A) = 4$ since $g(L) \leq 5$. By Lemma 1.8 (2) we get $\mathrm{Bs}\,|A| = \emptyset$. But then $h^0(L) < h^0(A) = 4$ and this is impossible.

(II-A-4) The case in which $L^2 = 6$.

Then X is not minimal. Let $\mu : X \to S$ be the minimalization of X. Then S is an abelian surface. Let $A := \mu_*(L)$. Then $A^2 = 10$ or 8 because $K_X L \leq 2$.

(II-A-4-1) The case in which $A^2 = 8$.

Then μ is a composition of two simple blowing ups. By Lemma 1.8 (2) we get $\mathrm{Bs}\,|A| = \emptyset$. But $h^0(A) = 4$ and $h^0(L) < h^0(A)$, this is impossible.

(II-A-4-2) The case in which $A^2 = 10$.

Then μ is one point blowing up and $L = \mu^*(A) - 2E$, where E is a (-1)-curve of μ. By Lemma 1.8 (2) we get $\mathrm{Bs}\,|A| = \emptyset$. But $h^0(A) = 5$ and $h^0(L) < h^0(A)$, so this is impossible.

Therefore $g(L) \geq 3q(X)$.

(II-B) The classification of (X, L) with $g(L) = 3q(X)$.

Since $q(X) \leq 2$, we get two possibilities; $(g(L), q(X)) = (3, 1), (6, 2)$.

(II-B-1) The case in which $(g(L), q(X)) = (3, 1)$.

Then $L^2 \leq 4$. Since $q(X) = 1$, we get that $h^0(L) \geq 5$ by the same argument as above. Hence $\Delta(L) \leq 1$. By Theorem 1.2 we get that $\Delta(L) = 1$ and $L^2 = 4$. Hence X is minimal and $h^0(L) = L^2/2 = 2$. This is impossible.

(II-B-2) The case in which $(g(L), q(X)) = (6, 2)$.

Assume that $L^2 \leq 9$. Then X is not minimal. Let $\mu : X \to S$ be the minimalization of X. Then S is an abelian surface. Let $A := \mu_*(L)$. We remark that $h^0(L) \geq 5$.

(II-B-2-1) The case in which $L^2 \leq 5$.

Then $L^2 \geq 2\Delta(L) + 1$ and $g(L) > \Delta(L)$. But this is impossible by Theorem 1.2.

(II-B-2-2) The case in which $L^2 \geq 6$.

Then $K_X L \leq 4$. If $h^0(L) = 5$, then by Proposition 1.6 this is impossible. (We remark that the value of K_X^2 is -1, -2, -3, or -4.) So we may assume that $h^0(L) \geq 6$.

(II-B-2-2-a) The case in which $6 \leq L^2 \leq 7$.

Then $L^2 \geq 2\Delta(L) + 1$ and $g(L) \geq \Delta(L)$. But this is impossible by Theorem 1.2.

(II-B-2-2-b) The case in which $L^2 = 8$.

Then $A^2 = 12$ or 10.

(b-1) The case in which $A^2 = 10$.

Then μ is a composition of two simple blowing ups. By Lemma 1.8 (2) we get $\mathrm{Bs}\,|A| = \emptyset$. But $h^0(A) = 5$ and $h^0(L) < h^0(A)$, this is impossible.

(b-2) The case in which $A^2 = 12$.

Then μ is one point blowing up and $L = \mu^*(A) - 2E$, where E is a (-1)-curve of μ. By Lemma 1.8 (2) we get $\mathrm{Bs}\,|A| = \emptyset$. But $h^0(A) = 6$ and $h^0(L) < h^0(A)$, this is impossible.

(II-B-2-2-c) The case in which $L^2 = 9$.

Then $L = \mu^*(A) - E$ and $A^2 = 10$ and $h^0(A) = 5$. By Lemma 1.8 (2) we get $\mathrm{Bs}\,|A| = \emptyset$. But then $h^0(L) < h^0(A) = 5$ and this is impossible.

Therefore $L^2 = 10$. In this case X is an abelian surface. This is the type (1) in Theorem 2.1.

(III) The case in which $k \geq 2$.

(III-A) The proof of $g(L) \geq (k+2)q(X)$.

If $q(X) = 0$, then $g(L) > (k+2)q(X)$.

If $q(X) = 1$ or 2, then by Theorem 1.7 we get that $L^2 \geq 4k + 5$. Hence

$$g(L) \geq 1 + \left\lceil \frac{4k+5}{2} \right\rceil$$
$$= 2k + 4$$
$$\geq (k+2)q(X).$$

(III-B) The classification of (X, L) with $g(L) = (k+2)q(X)$.
By the above argument, we get $q(X) \neq 0$ and so we get $L^2 \geq 4k + 5$ by Theorem 1.7. Therefore there exist two possibilities; $(L^2, K_X L) = (4k+6, 0)$, $(4k+5, 1)$. If $(L^2, K_X L) = (4k+5, 1)$, then X is one point blowing up of an abelian surface and $L = \mu^*(A) - E$, where $\mu : X \to S$ is the minimalization of X and $A := \mu_*(L)$. But this is impossible because $1 = K_X L = EL \geq k \geq 2$ by Lemma 1.9.
If $(L^2, K_X L) = (4k+6, 0)$, then X is an abelian surface and this is the type (1) in Theorem 2.1.

This completes the proof of Theorem 2.1. □

§3. The case in which $\kappa(X) = 1$

THEOREM 3.1. *Let (X, L) be a polarized surface such that $\kappa(X) = 1$. Assume that L is k-very ample with $k \geq 0$. Then $g(L) \geq (k+2)q(X)$. Furthermore if $g(L) = (k+2)q(X)$, then (X, L) is one of the following:*

(1) $k = 0$, $L^2 = 4$, $q(X) = 3$, X has a locally trivial elliptic fibration $f : X \to C$, and $LF = 3$ for a fiber F of f, where C is a smooth projective curve with $g(C) = 2$.

(2) $k \geq 1$, $L^2 = 4k+6$, $q(X) \geq 3$, X has a locally trivial elliptic fibration $f : X \to C$, and $LF = k+2$ for a fiber F of f, where C is a smooth projective curve with $g(C) = q(X) - 1$.

Proof. Since $\kappa(X) = 1$, there exists an elliptic fibration $f : X \to C$, where C is a smooth projective curve. Then we remark that $q(X) = g(C)$ or $q(X) = g(C) + 1$.

(I) The case in which $k = 0$.

CLAIM 3.2. $L^2 \geq 2$. *If $L^2 = 2$, then $q(X) = 0$ and $g(L) > 2q(X)$.*

Proof. If $L^2 = 1$, then $\Delta(L) = 0$ and by Theorem 1.2 this is impossible. If $L^2 = 2$, then by Proposition 1.3 we get $q(X) = 0$. In particular $g(L) > 2q(X)$. □

(I-1) The case in which $q(X) = g(C)$.
Then $K_X L \geq (2q(X) - 2)LF$ by the canonical bundle formula, where F is a fiber of f.
If $q(X) = 0$, then $g(L) > 2q(X)$.
If $q(X) \geq 1$, then $K_X L \geq (2q(X) - 2)LF \geq 4(q(X) - 1)$ by Lemma 1.9. So we obtain $g(L) \geq 2q(X) - 1 + (1/2)L^2$. Since $g(L) \in \boldsymbol{Z}$, we get that $g(L) > 2q(X)$ by Claim 3.2.

(I-2) The case in which $q(X) = g(C) + 1$.

(I-2-A) The proof of $g(L) \geq 2q(X)$.
Assume that $q(X) \leq 1$. By Claim 3.2, we get $g(L) \geq 3 > 2q(X)$.
Next we assume that $q(X) \geq 2$. By the canonical bundle formula and Lemma 1.9, we get $K_X L \geq (2q(X) - 4)LF \geq 4(q(X) - 2)$. Hence $g(L) \geq 2q(X) - 3 + (1/2)L^2$. If $L^2 \geq 5$, then $g(L) \geq 2q(X)$. So we may assume that $L^2 = 3$ or 4 by Claim 3.2.

(I-2-A-1) The case in which $L^2 = 4$.
Assume that $g(L) < 2q(X)$. Then if X is not minimal or minimal such that f has a multiple fiber, then by Lemma 1.9 we get $K_X L \geq (2q(X) - 4)LF + 1 \geq 4q(X) - 7$ and we obtain that $g(L) \geq 2q(X)$. So we may assume that X is minimal and f has no multiple fiber. In particular any fiber of f is smooth because $q(X) = g(C) + 1$. Then $K_X \equiv (2q(X) - 4)F$. Since $\kappa(X) = 1$, we get that $q(X) \geq 3$. By assumption and Lemma 1.9, we get that $LF = 2$. Since $LF = 2$ and any fiber of f is smooth, the natural map

$$f^* \circ f_* \mathcal{O}(L) \to \mathcal{O}(L)$$

is surjective. We put $\mathscr{E} := f_* \mathcal{O}(L)$. Then $\mathscr{E}$ is a locally free sheaf of rank two on C and there exists a double covering $\pi : X \to \boldsymbol{P}(\mathscr{E})$ such that $f = p \circ \pi$, where $\boldsymbol{P}(\mathscr{E})$ is the projective bundle of $\mathscr{E}$ on C and $p : \boldsymbol{P}(\mathscr{E}) \to C$ is the bundle map. Let B be the branch locus of π. Then there exists $Z \in \operatorname{Pic}(\boldsymbol{P}(\mathscr{E}))$ such that $B \in |2Z|$. Then $K_X = \pi^*(K_{\boldsymbol{P}(\mathscr{E})} + Z)$. (See e.g. [Pe].) By construction $L = \pi^*(H(\mathscr{E}))$, where $H(\mathscr{E})$ is the tautological line bundle of $\boldsymbol{P}(\mathscr{E})$. Then $H(\mathscr{E})$ is ample and $H(\mathscr{E})^2 = 2$ because $L^2 = 4$. On the other hand, $h^0(L) = h^0(H(\mathscr{E})) + h^0(H(\mathscr{E}) - Z)$. Since $K_X F = 0$, we get that $ZF_p = 2$ for a fiber F_p of p. Hence $h^0(H(\mathscr{E}) - Z) = 0$ because $(H(\mathscr{E}) - Z)F_p < 0$. So we get $h^0(L) = h^0(H(\mathscr{E}))$ and $H(\mathscr{E})$ is spanned. But by Proposition 1.3, we get that $g(C) = q(\boldsymbol{P}(\mathscr{E})) = 0$. Hence $q(X) = g(C) + 1 = 1$ and this is a contradiction.

(I-2-A-2) The case in which $L^2 = 3$.
Assume that $g(L) < 2q(X)$. If the relatively minimal model of f has a multiple fiber, then by Lemma 1.12 we get $K_X L \geq 4(2q(X) - 4) + 4$ and we can prove $g(L) \geq 2q(X)$ since $q(X) \geq 2$. So we may assume that the relatively minimal model of f has no multiple fiber. Then $q(X) \geq 3$ since $\kappa(X) = 1$. Since $L^2 = 3$, we get $K_X L$ is odd. Hence X is not minimal. By Lemma 1.10, we get $LF \geq 3$. But then $K_X L \geq 6(q(X) - 2) + 1$. So we get $g(L) \geq 2q(X)$ and this is a contradiction.

(I-B) The classification of (X, L) with $g(L) = 2q(X)$.

By the above proof, we get $q(X) = g(C) + 1$. First we study the upper bound of L^2. Since $K_X L \geq 2(q(X) - 2)LF \geq 4(q(X) - 2)$, we get $g(L) \geq 2q(X) - 3 + (1/2)L^2$. Hence $L^2 \leq 6$.

(I-B-1) The case in which $L^2 = 6$.
Then X is minimal, $LF = 2$, and $K_X \equiv (2q(X) - 4)F$. In particular any fiber of f is smooth and $q(X) \geq 3$. Since $LF = 2$ and any fiber of f is smooth, the natural map

$$f^* \circ f_* \mathcal{O}(L) \to \mathcal{O}(L)$$

is surjective. We put $\mathscr{E} := f_* \mathcal{O}(L)$. Then $\mathscr{E}$ is a locally free sheaf of rank two on C and there exists a double covering $\pi : X \to \boldsymbol{P}(\mathscr{E})$ such that $f = p \circ \pi$, where $\boldsymbol{P}(\mathscr{E})$ is the projective bundle of $\mathscr{E}$ on C and $p : \boldsymbol{P}(\mathscr{E}) \to C$ is the bundle map. By construction $L = \pi^*(H(\mathscr{E}))$, where $H(\mathscr{E})$ is the tautological line bundle of $\boldsymbol{P}(\mathscr{E})$. Then $H(\mathscr{E})$ is ample and $H(\mathscr{E})^2 = 3$ because $L^2 = 6$. By the same argument as above, we get $h^0(L) = h^0(H(\mathscr{E}))$ and $H(\mathscr{E})$ is spanned.
If $h^0(H(\mathscr{E})) \geq 4$, then $\Delta(H(\mathscr{E})) \leq 1$. If $g(H(\mathscr{E})) = 0$, then $q(\boldsymbol{P}(\mathscr{E})) = 0$ and this is a contradiction. If $g(H(\mathscr{E})) \geq 1$, then $g(H(\mathscr{E})) \geq \Delta(H(\mathscr{E}))$ and $H(\mathscr{E})^2 \geq 2\Delta(H(\mathscr{E})) + 1$. But by Theorem 1.2 $q(\boldsymbol{P}(\mathscr{E})) = 0$ and this is impossible.
If $h^0(H(\mathscr{E})) = 3$, then there exists a triple cover $\varphi_{|H(\mathscr{E})|} : \boldsymbol{P}(\mathscr{E}) \to \boldsymbol{P}^2$ defined by $H(\mathscr{E})$. Since $g(H(\mathscr{E})) = g(C)$, $K^2_{\boldsymbol{P}(\mathscr{E})} = 8(1 - g(C))$ and $\chi(\mathcal{O}_{\boldsymbol{P}(\mathscr{E})}) = 1 - g(C)$, we get that $g(C) = 0$ or 1 by Proposition 1.4. But this is a contradiction because $3 \leq q(X) = g(C) + 1$.

(I-B-2) The case in which $L^2 = 5$.
Since $L^2 = 5$, we obtain $g(L) \geq 4$. Hence $q(X) \geq 2$ because $g(L) = 2q(X)$. Assume that the relatively minimal model of f has a multiple fiber. Then by Lemma 1.12, we get that $g(L) > 2q(X)$ and this is a contradiction. Hence the relatively minimal model of f has no multiple fiber. This fact induces $q(X) \geq 3$. Since L^2 is odd, we get that X is not relatively minimal by the canonical bundle formula. By Lemma 1.10, we get $LF \geq 3$ for a fiber F of f. Hence $K_X L \geq 3(2q(X) - 4) + 1$ and we get $g(L) \geq 2q(X) + q(X) - 2 > 2q(X)$. This is a contradiction.

(I-B-3) The case in which $L^2 = 4$.
By the same argument as in the case (I-B-2), the relatively minimal model of f has no multiple fiber. In particular $q(X) \geq 3$. By Lemma 1.9 (2), we get $LF \geq 2$ for a general fiber F. If $LF = 2$, then f is relatively minimal by Lemma 1.10. Because $K_X \equiv (2q(X) - 4)F$, we get $K_X L = 4q(X) - 8$. Since $L^2 = 4$, we get $g(L) = 2q(X) - 1$. But this is a contradiction because $g(L) = 2q(X)$. So we get $LF \geq 3$. Hence $K_X L \geq 3(2q(X) - 4)$ and we get $g(L) \geq 2q(X) + q(X) - 3$. Since $g(L) = 2q(X)$, we get that f is relatively minimal and $q(X) = 3$. In particular f is a locally trivial fibration. This is the type (1) in Theorem 3.1.

(I-B-4) The case in which $L^2 = 3$.
By the same argument as in the case (I-B-2), the relatively minimal model of f has no multiple fiber. In particular $q(X) \geq 3$. Furthermore X is not minimal because L^2 is odd.

If $h^0(L) \geq 4$, then $\Delta(L) \leq 1$. Since $L^2 = 3$, we obtain $g(L) \geq 3$. Hence $g(L) > \Delta(L)$ and $L^2 \geq 2\Delta(L) + 1$. But by Theorem 1.2 this is impossible.
So we assume that $h^0(L) = 3$. Then there exists a triple covering $\varphi_{|L|} : X \to \boldsymbol{P}^2$ which is defined by $|L|$. We remark that $\chi(\mathcal{O}_X) = 0$ and $K_X^2 < 0$. By Proposition 1.4, we have the following equalities:

$$0 = \frac{1}{2}g(L)(g(L)+1) + 2 - c_2,$$

$$K_X^2 = 2g(L)^2 - 4g(L) + 11 - 3c_2.$$

By these equalities we have $2K_X^2 = (g(L)-1)(g(L)-10)$. Since $K_X^2 < 0$, we get that $1 < g(L) < 10$. By assumption $g(L)$ is even. Hence $g(L) = 2, 4, 6, 8$. By the above, we get $K_X^2 = -4$ (resp. -9, -10, -7) if $g(L) = 2$ (resp. 4, 6, 8). In particular, X is at least four times blowing up of the relatively minimal model of f. By using Lemma 1.10 and the canonical bundle formula, we get $K_X L \geq 3(2q(X) - 4) + 4$ and we obtain $g(L) \geq 2q(X) + q(X) - (3/2) > 2q(X)$ because $q(X) \geq 3$. This is impossible.

(II) The case in which $k = 1$.

CLAIM 3.3. *$h^0(L) \geq 5$, $L^2 \geq 5$, and $g(L) \geq 4$.*

Proof. Since L is very ample and $\kappa(X) = 1$, we get that $h^0(L) \geq 4$. If $h^0(L) = 4$, then there exists an embedding $X \to \boldsymbol{P}^3$. But since $\kappa(X) = 1$, this is impossible. Therefore $h^0(L) \geq 5$.
If $L^2 \leq 3$, then $\Delta(L) = 0$ and this is impossible by Theorem 1.2. Hence $L^2 \geq 4$. Since $K_X L > 0$, we get $g(L) \geq 4$.
If $L^2 = 4$, then $\Delta(L) \leq 1$. We get $g(L) > \Delta(L)$ and $L^2 > 2\Delta(L) + 1$. But this is a contradiction by Theorem 1.2. □

(II-A) The proof of $g(L) \geq 3q(X)$.

(II-A-1) The case in which $L^2 \geq 9$.

If $q(X) \leq 1$, then $g(L) \geq 4 > 3 \geq 3q(X)$ by Claim 3.3. So we assume that $q(X) \geq 2$. Then by Lemma 1.9 and the canonical bundle formula, we get $K_X L \geq 3(2q(X) - 4)$ since $q(X) = g(C)$ or $q(X) = g(C) + 1$. Hence $g(L) \geq 3q(X) - 5 + (1/2)L^2$ and we get $g(L) \geq 3q(X)$.

(II-A-2) The case in which $L^2 = 7$ or 8.

If $h^0(L) \geq 6$, then $\Delta(L) \leq 4$.
If $\Delta(L) = 4$, then $L^2 = 8$ and $h^0(L) = 6$. In particular, $L^2 = 2\Delta(L)$. On the other hand $g(L) \geq 6 > \Delta(L)$. By Theorem 1.2 we get $g(L) > 3q(X)$.
If $\Delta(L) \leq 3$, then $L^2 \geq 7 \geq 2\Delta(L) + 1$ and $g(L) \geq 5 > \Delta(L)$. Hence by Theorem 1.2 this is a contradiction.
Assume that $h^0(L) = 5$. If $q(X) = g(C)$, then $K_X L \geq 3(2q(X) - 2)$ and $g(L) > 3q(X)$. So we may assume $q(X) = g(C) + 1$. Then $\chi(\mathcal{O}_X) = 0$.
Assume that $L^2 = 8$. Then by Proposition 1.6 we get that $K_X^2 \leq -13$ since $g(L) \geq 6$. Hence $K_X L \geq 3(2q(X) - 4) + 13$ by the canonical bundle formula. So we get $g(L) > 3q(X)$.

Assume that $L^2 = 7$. By Proposition 1.6, we get that $K_X^2 \leq -13$ since $g(L) \geq 5$. Therefore $K_X L \geq 3(2q(X) - 4) + 13$ by the canonical bundle formula. So we get $g(L) > 3q(X)$.

(II-A-3) The case in which $L^2 = 6$.

Then $\Delta(L) \leq 3$. Since $L^2 = 6$, we get that $L^2 \geq 2\Delta(L)$ and $g(L) \geq 5 > \Delta(L)$.

If $L^2 = 2\Delta(L)$, then by Theorem 1.2 we get $g(L) > 3q(X)$.

If $L^2 \geq 2\Delta(L) + 1$, then this is a contradiction because $g(L) > \Delta(L)$.

(II-A-4) The case in which $L^2 = 5$.

Then $\Delta(L) \leq 2 < 4 \leq g(L)$. Since $L^2 \geq 2\Delta(L) + 1$, we get $g(L) = \Delta(L)$ by Theorem 1.2. But this is a contradiction.

(II-B) The classification of (X, L) with $g(L) = 3q(X)$.

By the proof of the above, we get $g(L) > 3q(X)$ if $L^2 \leq 8$. So we get that $L^2 \geq 9$. By Claim 3.3 and the assumption, we get $q(X) \geq 2$. Since $K_X L \geq 3(2q(X) - 4)$, we get that $g(L) \geq 3q(X) - 5 + (L^2)/2$. Hence $L^2 \leq 10$.

(II-B-1) The case in which $L^2 = 10$.

Then f is the relatively minimal elliptic fibration, f has no multiple fiber, and $q(X) = g(C) + 1$. In particular, f is a locally trivial fibration. Since $\kappa(X) = 1$, we get that $q(X) \geq 3$ by the canonical bundle formula. This is the type (2) in Theorem 3.1.

(II-B-2) The case in which $L^2 = 9$.

If $q(X) = g(C)$, then $K_X L \geq 3(2q(X) - 2)$ by Lemma 1.9 and the canonical bundle formula. Hence $g(L) \geq 3q(X) + (5/2) > 3q(X)$. So we get that $q(X) = g(C) + 1$. If the relatively minimal model of f has a multiple fiber, then by Lemma 1.12 we get $K_X L \geq 6(2q(X) - 4) + 6$. So we have $g(L) \geq 3q(X) + 3q(X) - (7/2)$. Since $q(X) \geq 2$, we get that $g(L) > 3q(X)$. Hence the relatively minimal model of f has no multiple fiber. In particular $q(X) \geq 3$ because $\kappa(X) = 1$. Since L^2 is odd, f is not relatively minimal. By Lemma 1.10, we have $LF \geq 4$. Hence $K_X L \geq 4(2q(X) - 4) + 1$ and we get $g(L) \geq 3q(X) + q(X) - 2$. Since $q(X) \geq 3$, $g(L) > 3q(X)$ is obtained and this is a contradiction.

(III) The case in which $k \geq 2$.

By Theorem 1.7, we obtain $L^2 \geq 4k + 5$. By Lemma 1.9, we get $LF \geq k + 2$ for a general fiber F of f.

(III-A) The proof of $g(L) \geq (k + 2)q(X)$.

If $q(X) \leq 1$, then

$$\begin{aligned} g(L) &= 1 + \frac{1}{2}(K_X + L)L \\ &> 1 + \frac{1}{2}(4k + 5) \\ &= 2k + \frac{7}{2} \\ &> (k + 2)q(X). \end{aligned}$$

If $q(X) \geq 2$, then $K_X L \geq (k+2)(2q(X)-4)$ by the canonical bundle formula. Hence $g(L) \geq (k+2)q(X) - (1/2)$. Since $g(L) \in \boldsymbol{Z}$, we get $g(L) \geq (k+2)q(X)$.

(III-B) The classification of (X, L) with $g(L) = (k+2)q(X)$.

By the above proof we get that $q(X) = g(C) + 1$ and $q(X) \geq 2$ in this case. If the relatively minimal model of f has a multiple fiber, then by Lemma 1.12 we get $K_X L \geq 2(k+2)(2q(X)-4) + 2(k+2)$. So we have $g(L) \geq (k+2)q(X) + (k+2)q(X) - k - (5/2)$. Since $q(X) \geq 2$, we get that $g(L) > (k+2)q(X)$. Hence the relatively minimal model of f has no multiple fiber. In particular $q(X) \geq 3$ because $\kappa(X) = 1$. On the other hand since

$$(k+2)q(X) = g(L) \geq 1 + \frac{1}{2}(k+2)(2q(X)-4) + \frac{1}{2}L^2,$$

We get $L^2 \leq 4k+6$. Therefore $L^2 = 4k+5$ or $4k+6$.

(III-B-1) The case in which $L^2 = 4k+6$.

Then f is the relatively minimal elliptic fibration, f has no multiple fiber, and $q(X) = g(C) + 1$. In particular, f is a locally trivial fibration. Since $\kappa(X) = 1$, we get that $q(X) \geq 3$ by the canonical bundle formula. This is the type (2) in Theorem 3.1.

(III-B-2) The case in which $L^2 = 4k+5$.

Since L^2 is odd, f is not relatively minimal. By Lemma 1.10, we have $LF \geq 2k+2$. Hence $K_X L \geq (2k+2)(2q(X)-4) + 1$ and we get $g(L) \geq (k+2)q(X) + kq(X) - 2k$. Since $q(X) \geq 3$, $g(L) > (k+2)q(X)$ is obtained and this is a contradiction.

This completes the proof of Theorem 3.1. □

§4. The case in which $\kappa(X) = 2$

THEOREM 4.1. *Let (X, L) be a polarized surface with $\kappa(X) = 2$. Assume that* $\mathrm{Bs}\,|L| = \emptyset$ *and $h^0(L) \geq 5$. Then $g(L) \geq 2q(X)$.*

Proof. First we prove the following Claim;

CLAIM 4.2. *Let $x_1 \in X$ be a point and let $\psi : X' \to X$ be blowing up at x_1. We put $L_1 := \psi^* L - E$ and $L_2 := L_1 - E$, where E is the (-1)-curve of ψ. Then $h^0(L_2) \geq 2$.*

Proof of Claim 4.2. By the following exact sequence

$$0 \to H^0(L_1) \to H^0(\psi^* L) \to H^0(\mathcal{O}_E),$$

we get that $h^0(L) - h^0(L_1) \leq h^0(\mathcal{O}_E) = 1$.

By the following exact sequence

$$0 \to H^0(L_2) \to H^0(L_1) \to H^0(\mathcal{O}_E(1)),$$

we get that $h^0(L_1) - h^0(L_2) \leq h^0(\mathcal{O}_E(1)) = 2$.

Hence $h^0(L_2) \geq 2$. □

Let $\Lambda_2 \subset |L_2|$ be a linear pencil and let $\Lambda := \psi_* \Lambda_2$. Then $\Lambda \subset |L \otimes m_{x_1}^2|$, where m_{x_1} is the ideal sheaf of x_1. Let Λ_M be the movable part of Λ and let Z be the fixed part of Λ.

(I) The case in which $Z = 0$.
In this case, $\dim \mathrm{Bs}\,\Lambda \leq 0$. Since $x_1 \in \mathrm{Bs}\,\Lambda$, we get $\mathrm{mult}_{x_1} D \geq 2$ for any $D \in \Lambda$. Here we use the same argument as in the proof of Theorem 3.1 in [Fk3]. Let φ be a rational map $X \to \boldsymbol{P}^1$ associated with Λ_M, let $\mu : X' \to X$ be an elimination of indeterminacy of φ, and let φ' be the morphism $X' \to \boldsymbol{P}^1$. Let $\varphi' = \delta \circ f : X \to C \to \boldsymbol{P}^1$ be its Stein factorization, where $\delta : C \to \boldsymbol{P}^1$ is a finite morphism and f is a fiber space $X \to C$.
We put $a := \deg \delta$, $L' := \mu^* L$, and F_f is a general fiber of f.

Let b be the number of times of blowing up μ.
We put $\mu = \mu_1 \circ \mu_2 \circ \cdots \circ \mu_b : X' = X_b \to X_{b-1} \to \cdots \to X_1 \to X_0 = X$, where μ_i is the blowing down of (-1)-curve E_i and $\mu_1 : X_1 \to X_0$ is the blowing up at $x_1 \in X$. Let $L_0 = L$ and $L_i = \mu_i^* L_{i-1}$ for $i = 1, 2, \ldots, b$. Then $L' = L_b$. We take an element M of Λ_M. We put $M_0 = M$ and $\Lambda_M = \Lambda_0$. Let Λ_i be the movable part of $\mu_i^* \Lambda_{i-1}$. Then we write $\Lambda_i = \mu_i^* \Lambda_{i-1} - n_i E_i$, where $n_i > 0$ for $i = 1, \ldots, b$. Let $M_i = \mu_i^* M_{i-1} - n_i E_i$ and $M' = M_b$. Then $M_i \in \Lambda_i$. We remark that $M' \equiv a F_f$, where $\equiv$ is numerical equivalence.
Then

$$(K_{X'} + L')(L' - M') = (K_X + L)(L - M) - \sum_{i=1}^{b} n_i.$$

Since $M' \equiv a F_f$, then

$$M^2 = \sum_{i=1}^{b} n_i^2.$$

We remark that $K_X + L$ is nef. By construction, $L - M$ is an effective divisor. Hence $(K_X + L)(L - M) \geq 0$. Because $n_i > 0$ and $n_1 \geq 2$, we have

$$2 + \sum_{i=1}^{b} n_i \leq \sum_{i=1}^{b} n_i^2.$$

Therefore by the above

$$\begin{aligned}
(K_{X'} + L')L' &= (K_{X'} + L')M' + (K_X + L)(L - M) - \sum_{i=1}^{b} n_i \\
&\geq (K_{X'} + L')M' - \sum_{i=1}^{b} n_i^2 + 2 \\
&= (K_{X'} + L')M' - M^2 + 2 \\
&= K_{X'} M' + LM - M^2 + 2.
\end{aligned}$$

On the other hand, $LM - M^2 = (L - M)M$. Since $M \in \Lambda_M$, M is a nef divisor on X. So we have $LM - M^2 \geq 0$.
Hence

$$\begin{aligned}(K_{X'} + L')L' &\geq K_{X'}M' + 2 \\ &= 2a(g(F_f) - 1) + 2 \\ &\geq 2g(F_f)\end{aligned}$$

by $a \geq 1$.
Therefore

$$\begin{aligned}g(L') &\geq g(F_f) + 1 \\ &= 2\frac{g(F_f) + 1}{2} \\ &\geq 2q(X')\end{aligned}$$

by Theorem 1 in [X]. Since $g(L) = g(L')$ and $q(X) = q(X')$, we obtain $g(L) \geq 2q(X)$.

(II) The case in which $Z \neq 0$.

Let $M \in \Lambda_M$. We remark that $MZ > 0$ because $M + Z \in |L|$ is 1-connected.

(II-1) $M^2 > 0$ case.

Then M is nef-big and $\dim \mathrm{Bs}\,|M| \leq 0$. So we get

$$\begin{aligned}g(L) &= 1 + \frac{1}{2}(K_X + L)L \\ &\geq 1 + \frac{1}{2}(K_X + L)M \\ &= 1 + \frac{1}{2}(K_X + M)M + \frac{1}{2}MZ \\ &\geq 1 + \frac{1}{2}(K_X + M)M + \frac{1}{2} \\ &= g(M) + \frac{1}{2}\end{aligned}$$

since $MZ > 0$. On the other hand $g(M) \geq 2q(X) - 1$ by Corollary 3.2 in [Fk3]. Since $g(L) \in \boldsymbol{Z}$, we get $g(L) \geq 2q(X)$.

(II-2) $M^2 = 0$ case.

Then $\mathrm{Bs}\,\Lambda_M = \emptyset$. Let $\varphi : X \to \boldsymbol{P}^1$ be a surjective morphism defined by Λ_M. By taking Stein factorization, if necessary, there exists a smooth curve C, a finite morphism $\pi : C \to \boldsymbol{P}^1$, and a surjective morphism with connected fibers $f : X \to C$ such that $\varphi = \pi \circ f$.

(II-2-1) The case in which $g(C) = 0$.

Then we can prove that $g(L) \geq g(M) + (1/2)$ by the same argument as above. On the other hand by construction we have $M \equiv aF$, where a is a natural number. Hence $g(L) \geq g(F) + (1/2) \geq 2q(X) - (1/2)$ by Theorem 1 in [X]. So we get $g(L) \geq 2q(X)$.

(II-2-2) The case in which $g(C) \geq 1$.
Then $a \geq 2$ by construction. We remark that $LF \geq 2$ for a fiber F of f because $\mathrm{Bs}\,|L| = \emptyset$ and $\kappa(X) = 2$. Since $K_{X/C} + L$ is nef by Lemma 1.15 (3),

$$\begin{aligned} g(L) &= g(C) + \frac{1}{2}(K_{X/C} + L)L + (LF - 1)(g(C) - 1) \\ &\geq 2g(C) - 1 + \frac{1}{2}(K_{X/C} + L)(aF) + \frac{1}{2}(K_{X/C} + L)Z \\ &\geq 2g(C) - 1 + 2g(F) - 2 + 2 \\ &= 2(g(C) + g(F)) - 1. \end{aligned}$$

On the other hand by Lemma 1.16 we get $q(X) \leq g(C) + g(F)$.
If $g(C) + g(F) \geq q(X) + 1$, then $g(L) > 2q(X)$. So it is sufficient to consider the case in which $q(X) = g(C) + g(F)$. Then by Lemma 1.16, we get $X \sim_{\mathrm{bir}} F \times C$. By Lemma 1.18, we get $g(L) \geq 2q(X)$. This completes the proof of Theorem 4.1. □

Corollary 4.3. *Let (X, L) be a polarized surface with $\kappa(X) = 2$. If L is very ample, then $g(L) \geq 2q(X)$.*

Proof. Since L is very ample and $\kappa(X) = 2$, we get $h^0(L) \geq 4$.
If $h^0(L) \geq 5$, then $g(L) \geq 2q(X)$ by Theorem 4.1.
If $h^0(L) = 4$, then X is a hypersurface in $\boldsymbol{P}^3$. Hence we get that $q(X) = 0$ and so we have $g(L) > 2q(X)$. □

Theorem 4.4. *Let (f, X, C, L) be a polarized fiber space with $\dim X = 2$ and $\kappa(X) = 2$. If L is k-very ample with $k \geq 0$ and $q(X) \leq g(C) + 1$, then $g(L) \geq (k + 2)q(X)$.*

Proof. (I) The case in which $k = 0$.

(I-1) The case in which $g(C) = 0$.
Then $q(X) \leq 1$. So we get $g(L) \geq 2 \geq 2q(X)$.

(I-2) The case in which $g(C) \geq 1$.
Then

$$\begin{aligned} g(L) &= g(C) + \frac{1}{2}(K_{X/C} + L)L + (LF - 1)(g(C) - 1) \\ &\geq 2g(C) - 1 + \frac{1}{2}(K_{X/C} + L)L \end{aligned}$$

since $LF \geq 2$ by Lemma 1.9, where F is a general fiber of f.

(I-2-1) The case in which $L^2 \leq 3$.

If $L^2 \le 2$, then by Proposition 1.3 we get $q(X) = 0$ and so we have $g(L) > 2q(X)$. If $L^2 = 3$ and $K_{X/C}L \ge 2$, then $g(L) \ge 2g(C) + (3/2)$ and we get $g(L) \ge 2g(C) + 2 \ge 2q(X)$.

If $L^2 = 3$ and $K_{X/C}L \le 1$, then $0 \le K_{X'/C}L' \le K_{X/C}L \le 1$, where $f' : X' \to C$ is the relatively minimal model of f, $\mu : X \to X'$ is its birational morphism, and $L' = \mu_*(L)$ in the sense of cycle theory. Since $(L')^2 \ge 3$, we get $(K_{X'/C})^2 = 0$ by Hodge index Theorem and Lemma 1.15. In particular f' is a locally trivial fibration.

If $K_{X'/C}L' = 0$, then $(K_{X'/C})^2 = 0$ and $K_{X'/C} \equiv 0$. But this is impossible because $\kappa(X) = 2$. If $K_{X'/C}L' = 1$, then $K_{X/C}L = K_{X'/C}L'$ and so we get $X \cong X'$. In particular, f is a locally trivial fibration. By Lemma 1.17 we get $g(L) \ge 2q(X)$.

(I-2-2) The case in which $L^2 \ge 4$.

By Lemma 1.15, we get $K_{X/C}L \ge 0$.

If $K_{X/C}L = 0$, then X is minimal, $(K_{X/C})^2 = 0$, and $K_{X/C} \equiv 0$. But this is impossible because $\kappa(X) = 2$. Hence $K_{X/C}L \ge 1$. So we get $g(L) \ge 2g(C) + (3/2)$. Therefore $g(L) \ge 2g(C) + 2 \ge 2q(X)$.

(II) The case in which $k = 1$.

(II-1) $g(C) = 0$ case.

Then $q(X) \le 1$. Since L is very ample and $\kappa(X) = 2$, we get $L^2 \ge 2$. Hence $g(L) \ge 3$. So we get that $g(L) \ge 3q(X)$.

(II-2) $g(C) \ge 1$ case.

By Lemma 1.9, we get that $LF \ge 4$ because $g(F) \ge 2$.

(II-2-1) The case in which $L^2 \le 8$.

We remark that $L^2 \ge 2$ since $\kappa(X) = 2$. Hence $g(L) \ge 3$.

If $h^0(L) = 4$, then X is a hypersurface in $\boldsymbol{P}^3$. Hence $q(X) = 0$ and we get $g(L) > 3q(X)$.

If $h^0(L) \ge 6$ and $L^2 \le 7$, then $L^2 \ge 2\Delta(L) + 1$ and $g(L) \ge 3 \ge \Delta(L)$. Hence by Theorem 1.2 this is impossible.

If $h^0(L) \ge 6$ and $L^2 = 8$, then $\Delta(L) \le 4$. Since $L^2 = 8$, we get $g(L) \ge 6 > \Delta(L)$. If $\Delta(L) = 4$, then by Theorem 1.2 we get $g(L) > 3q(X)$. If $\Delta(L) \le 3$, then $L^2 \ge 2\Delta(L) + 1$ and this is impossible by Theorem 1.2.

If $h^0(L) = 5$ and $L^2 \le 5$, then $g(L) \ge 2 \ge \Delta(L)$ and $L^2 \ge 2\Delta(L) + 1$. But by Theorem 1.2, this is impossible.

If $h^0(L) = 5$ and $L^2 = 6$, then $g(L) \ge 5 > 3 = \Delta(L)$ and $L^2 = 2\Delta(L)$. Hence by Theorem 1.2 we get $g(L) > 3q(X)$.

If $h^0(L) = 5$ and $L^2 = 7$, then by Proposition 1.19 we get $g(L) \le 6$. Hence $K_X L \le 3$. Let $\mu : X \to X'$ be the minimal of X and $L' := \mu_*(L)$. Then $3 \ge K_X L \ge K_{X'}L'$ and $(L')^2 \ge L^2 = 7$. Hence $K_{X'}^2 \le 1$ by Hodge index Theorem. By Proposition 1.13, we get $q(X) = 0$ and $g(L) > 3q(X)$.

If $h^0(L) = 5$ and $L^2 = 8$, then by Proposition 1.19 we get $g(L) \le 9$. Hence $K_X L \le 8$. Let $\mu : X \to X'$ be the minimal of X and $L' := \mu_*(L)$. Then we remark that $K_X L \ge K_{X'}L'$ and $(L')^2 \ge L^2$. Since $K_{X'}^2 > 0$ and L^2 is even, we get $K_X L \ge 4$ by Hodge index Theorem.

If $K_X L = 4$, then $K_{X'}^2 \le 2$ and $q(X) \le 1$ by Proposition 1.13. Hence $g(L) = 7 > 3q(X)$.

If $K_XL = 6$, then $K_{X'}^2 \leq 4$ and $q(X) \leq 2$ by Proposition 1.13. Hence $g(L) = 8 > 3q(X)$.
If $K_XL = 8$, then $K_{X'}^2 \leq 8$ and $q(X) \leq 4$ by Proposition 1.13. If $q(X) \leq 3$, then $g(L) = 9 \geq 3q(X)$. So we may assume $q(X) = 4$. In this case we get $K_{X'}^2 = 8$ and so we obtain $X = X'$ and $K_X \equiv L$ by Hodge index Theorem. By Proposition 1.6 we get $\chi(\mathcal{O}_X) = 6$. Therefore $p_g = 9$. But by Proposition 1.13 this is a contradiction because $K_X^2 = 8$.

(II-2-2) The case in which $L^2 \geq 9$.
Then

$$\begin{aligned} g(L) &= g(C) + \frac{1}{2}(K_{X/C} + L)L + (LF - 1)(g(C) - 1) \\ &\geq 3g(C) + g(C) + \frac{3}{2} \\ &\geq 3g(C) + \frac{5}{2}. \end{aligned}$$

Hence $g(L) \geq 3g(C) + 3 \geq 3q(X)$.

(III) The case in which $k \geq 2$.
Then by Theorem 1.7, we get $L^2 \geq 4k + 5$.
If $g(C) \geq 1$, then we get

$$\begin{aligned} g(L) &= g(C) + \frac{1}{2}(K_{X/C} + L)L + (LF - 1)(g(C) - 1) \\ &\geq (k+3)g(C) + k + \frac{1}{2} \\ &= (k+2)(g(C) + 1) + g(C) - \frac{3}{2} \end{aligned}$$

since $LF \geq k + 3$ and $K_{X/C}L \geq 0$ by Lemma 1.9 and Lemma 1.15.
Hence $g(L) \geq (k+2)(g(C) + 1) + g(C) - 1 \geq (k+2)q(X)$.
If $g(C) = 0$, then $q(X) \leq 1$. So we get

$$\begin{aligned} g(L) &\geq 1 + \frac{1}{2}(1 + 4k + 5) \\ &= 2k + 4 \\ &> (k+2)q(X). \end{aligned}$$

This completes the proof of Theorem 4.4. □

COROLLARY 4.5. *Let (f, X, C, L) be a polarized fiber space with* $\dim X = 2$ *and $\kappa(X) = 2$. Assume that $g(F) = 2$ for a general fiber F of f and L is k-very ample with $k \geq 0$. Then $g(L) \geq (k+2)q(X)$.*

Proof. By Lemma 1.16, we get $q(X) \leq g(C) + 2$.
If $q(X) = g(C) + 2$, then by Lemma 1.16 and Lemma 1.18 we get the assertion.

If $q(X) \leq g(C)+1$, then by Theorem 4.4 we get the assertion. □

Appendix. Let (X,L) be a polarized surface with $\kappa(X)=-\infty$. Assume that L is k-very ample. In this appendix, we consider a lower bound for sectional genus with $\kappa(X)=-\infty$.
If $q(X)=0$, then $g(L) \geq (k+2)q(X)$. So we assume that $q(X) \geq 1$.
If (X,L) is not a scroll over a smooth curve C, then we can prove that $g(L) \geq 2q(X)$ for any polarized surface with $\kappa(X)=-\infty$.
Here we consider the case in which $k \geq 2$.

LEMMA A.1. *Let (X,L) be a polarized surface. Assume that $\kappa(X)=-\infty$, L is k-very ample with $k \geq 2$, and $L^2 \leq 4k+4$. Then $g(L) \geq (k+2)q(X)$ unless*

($*$) $k=2$, X is a $\boldsymbol{P}^1$-bundle over a smooth curve C of genus two, and $L \equiv 2C_0+2F$ with $C_0^2=2$, where C_0 is a minimal section of the projection map $X \to C$ and F is its fiber.

Proof. By the classification of (X,L) with $L^2 \leq 4k+4$ by Di Rocco [Di], we obtain the assertion. □

We remark that if (X,L) is ($*$), then $g(L)=7$.

THEOREM A.2. *Let (X,L) be a polarized surface with $\kappa(X)=-\infty$ and $q(X) \geq 1$. Assume that X is relatively minimal, and L is k-very ample with $k \geq 2$. Then $g(L) \geq kq(X)$.*

Proof. Let $f: X \to C$ be the $\boldsymbol{P}^1$-bundle. Let $\mathscr{E}$ be a normalized vector bundle of rank two on C such that $X=\boldsymbol{P}(\mathscr{E})$, and let C_0 be a minimal section of f. We can write $L \equiv aC_0+bF$, where F is a fiber of f. Let $e:=-C_0^2$. Then

$$g(L)=ag(C)+(a-1)\left(b-\frac{1}{2}ae-1\right).$$

If $e \geq 0$, then by Proposition 2.20 in [Ha], we get $b-ae>0$. Hence $b-(1/2)ae-1>(1/2)ae-1 \geq -1$. On the other hand, $a \geq k$ by Lemma 1.9. Therefore we get $g(L) \geq kg(C)=kq(X)$.
If $e<0$, then by Proposition 2.21 in [Ha], we get $b-(1/2)ae>0$. If $b-(1/2)ae \geq 1$, then $g(L) \geq kq(X)$ by the same argument as above. If $b-(1/2)ae=1/2$, then $L^2=a(2b-ae)=a$. By Lemma A.1 we may assume that $a=L^2 \geq 4k+5$. Hence

$$\begin{aligned} g(L) &= \frac{a+1}{2}g(C)+\frac{a-1}{2}g(C)-\frac{a-1}{2} \\ &\geq (2k+3)g(C)+\frac{a-1}{2}-\frac{a-1}{2} \\ &\geq (2k+3)q(X). \end{aligned}$$

This completes the proof of Theorem A.2. □

THEOREM A.3. *Let (X, L) be a polarized surface with $\kappa(X) = -\infty$ and $q(X) \geq 1$. Assume that X is not relatively minimal, and L is k-very ample with $k \geq 2$. Then $g(L) \geq (k+2)q(X)$.*

Proof. Let $f : X \to C$ be the Albanese fibration, where C is a smooth curve of genus $g(C) = q(X) \geq 1$. We remark that

$$g(L) = g(C) + \frac{1}{2}(K_{X/C} + L)L + (LF - 1)(g(C) - 1),$$

where F is a general fiber of f.
By assumption $LF \geq 2k$ by Lemma 1.9. So we can prove that $\kappa(K_F + (1/2)L_F) \geq 0$ for a general fiber F of f. By Lemma 0.1 in [Fk4], we get $(K_{X/C} + (1/2)L)L \geq 0$. Therefore

$$\begin{aligned} g(L) &\geq g(C) + \frac{1}{4}L^2 + (2k-1)(g(C) - 1) \\ &= 2kg(C) + \frac{1}{4}L^2 - (2k-1) \\ &= (k+2)g(C) + (k-2)g(C) + \frac{1}{4}L^2 - (2k-1). \end{aligned}$$

By Lemma A.1, we may assume that $L^2 \geq 4k + 5$. Hence

$$\begin{aligned} g(L) &\geq (k+2)g(C) + (k-2) + k + \frac{5}{4} - (2k-1) \\ &= (k+2)g(C) + \frac{1}{4}. \end{aligned}$$

So we obtain the assertion. □

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Department of Mathematics
Faculty of Science
Tokyo Institute of Technology
Oh-okayama, Meguro-ku, Tokyo 152-8551
Japan
E-mail: fukuma@math.titech.ac.jp

Current address:
Department of Mathematics
College of Education
Naruto University of Education
Takashima, Naruto-cho, Naruto-shi 772-8502
Japan
E-mail: fukuma@naruto-u.ac.jp

Note added in proof. After the acceptance of this paper for publication, Dr. Kazuyoshi Takahashi pointed out to the author that the type $(*)$ in Lemma A.1 is excluded because $L^2 = 16 > 4k+5$ and this contradicts the assumption of Lemma A.1. But we need not to change the assertion of Theorem A.2 and Theorem A.3. The author would like to thank Dr. K. Takahashi for pointing out these.