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DECOMPOSITION OF A k-COVECTOR WITH RESPECT TO A VECTOR AND COMPUTING ITS COMASS

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I. Introduction

A smooth differential form Φ on a Riemannian manifold which is closed and has comass one is called a calibration. Corresponding to a calibration is a geometry of minimal surfaces (cf. [H], [HL]).

The constant coefficient calibrations have been studied deeply by R. Harvey, B. Lawson, F. Morgan, J. Dadok ... Many beautiful constant coefficient calibrations and corresponding geometries were constructed in the nice paper of R. Harvey and B. Lawson [HL], for example Special Lagrangian, Associative, Coassociative, Cayley calibrations ... Computing the comass of a k-differential form is quite difficult, even in the simplest cases, the cases of k-covectors viewed as paralell differential forms. The known calibrations are not much, especially the calibrations of high degree. The such well-known calibrations are only Complex Line, Special Lagrangian, power of Kähler forms (see [DHM], [HL]).

The Associative and Coassociative calibrations (see [HL]) on $\boldsymbol{R}^7$ have many beautiful properties, and between them there is a relationship

$$*\varphi = \psi,$$

where φ is Associative calibration, and ψ is Coassociative calibration on $\boldsymbol{R}^7$.

Moreover,

$$\varphi(\eta)^2 + \bar{\psi}(\eta)^2 = 1 \quad \text{for all } \eta \in G(3, \boldsymbol{R}^7),$$

and hence

$$G(\varphi) = G_0(\bar{\psi}).$$

This paper gives a method to compute the comass of some classes of k-covectors, describes the set of all 3-covectors have comass one on $\boldsymbol{R}^8$, whose faces contain a $SLAG$ face (this set is denoted by $F^*(SLAG)$), and constructs new calibrations on $\boldsymbol{R}^{4n-1}$: General Associative and General Coassociative calibrations. The method bases on the decomposition of a covector Φ with respect to a vector $e \in \operatorname{span} \Phi^*$,

$$\Phi = e^* \wedge \varphi + \psi,$$

where $\varphi \in \bigwedge^{k-1}(e^\perp)$; $\psi \in \bigwedge^k(e^\perp)$.

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Theorem 2.4 shows that

$$\|\Phi\|^* = \max_{\eta \in G(k-1, e^{\perp})} \sqrt{\varphi(\eta)^2 + \bar{\psi}(\eta)^2} = A,$$

and

$$G(\Phi) = \{(\cos\alpha e + \sin\alpha f) \wedge \eta / \eta \in G(k-1, e^{\perp})\},$$

where

i) $\varphi(\eta)^2 + \bar{\psi}(\eta)^2 = A^2$,

ii) $f = \dfrac{\bar{\psi}(\eta)}{\|\bar{\psi}(\eta)\|}$,

iii) $\cos\alpha = \dfrac{\varphi(\eta)}{A}$, $\sin\alpha = \dfrac{\|\bar{\psi}(\eta)\|}{A}$.

By using this theorem and Theorems 6.11 and 6.16 in [HL], we prove that $\Phi_{G.ASSOC}$ and $\Phi_{G.COASSOC}$ are calibrations and the relationship $*\Phi_{G.ASSOC} = \Phi_{G.COASSOC}$ holds.

II. Decomposition of a covector with respect to a vector

Let Φ be a k-covector on $\boldsymbol{R}^n$ $(k < n)$, and suppose that $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of $\boldsymbol{R}^n$. Denote

$$\Phi_{e_i} = e_i \lrcorner \Phi, \quad i = 1, 2, \ldots, n,$$
$$(\text{i.e.} \quad \Phi_{e_i}(\eta) = \Phi(e_i \wedge \eta)),$$

and

$$\bar{\Phi} = (\Phi_{e_1}, \Phi_{e_2}, \ldots, \Phi_{e_n}),$$
$$(\text{i.e.} \quad \bar{\Phi}(\eta) = \sum \Phi_{e_i}(\eta) e_i).$$

Remark. Each Φ_{e_i} is a $(k-1)$-covector on $e_i{}^{\perp}$, where $e_i{}^{\perp}$ denote the subspace of $\boldsymbol{R}^n$ containing all vectors orthogonal to e_i.

LEMMA 2.1. 1) *$\bar{\Phi}$ is a linear mapping from $\bigwedge^{k-1}(\boldsymbol{R}^n)$ to $\boldsymbol{R}^n$.*
2) *$\Phi(x \wedge \eta) = \langle x, \bar{\Phi}(\eta)\rangle$, for all $x \in \boldsymbol{R}^n$; $\eta \in \bigwedge^{k-1}(\boldsymbol{R}^n)$.*
3) *$\bar{\Phi}(\eta) \in \operatorname{span}(\eta)^{\perp}$, for all $\eta \in \bigwedge^{k-1}(\boldsymbol{R}^n)$, i.e. $\bar{\Phi}(\eta)$ is orthogonal to* $\operatorname{span}(\eta)$.

Proof. 1) $\bar{\Phi}$ is linear since each Φ_{e_i} is linear.
2) Suppose that $x = \sum x_i e_i$, we have

$$\Phi\left(\sum x_i e_i \wedge \eta\right) = \sum x_i \Phi(e_i \wedge \eta) = \sum x_i \Phi_{e_i}(\eta)$$
$$= \langle x, \bar{\Phi}(\eta)\rangle.$$

3) For each $x \in \operatorname{span}(\eta)$, we have

$$\langle x, \bar{\Phi}(\eta)\rangle = \Phi(x \wedge \eta) = \Phi(0) = 0.$$

Thus $\bar{\Phi}(\eta) \perp x$.

The proof of the lemma is completed.

As usual, we define the norm of $\bar{\Phi}$ (denote $\|\bar{\Phi}\|$) is the maximum value of $\|\bar{\Phi}\|$ attain on $G(k-1, \boldsymbol{R}^n)$, $G(\bar{\Phi})$ is the set of all $\eta \in G(k-1, \boldsymbol{R}^n)$, where $\|\bar{\Phi}\|$ attains its maximum, and $G_0(\bar{\Phi})$ is the set of all $\eta \in G(k-1, \boldsymbol{R}^n)$ such that $\|\bar{\Phi}(\eta)\| = 0$.

$$\|\bar{\Phi}\| = \max_{\eta \in G(k-1, \boldsymbol{R}^n)} \|\bar{\Phi}(\eta)\|,$$

$$G(\bar{\Phi}) = \{\eta \in G(k-1, \boldsymbol{R}^n) / \|\bar{\Phi}(\eta)\| = \|\bar{\Phi}\|\},$$

$$G_0(\bar{\Phi}) = \{\eta \in G(k-1, \boldsymbol{R}^n) / \|\bar{\Phi}(\eta)\| = 0\}.$$

LEMMA 2.2. 1) $\|\Phi\|^* = \|\bar{\Phi}\|$.
2) *If* $\|\Phi\|^* = 1$, *then* $G(\Phi) = \{\bar{\Phi}(\eta) \wedge \eta / \eta \in G(\bar{\Phi})\}$.

Proof. 1) For each $\xi \in G(k, \boldsymbol{R}^n)$, let e be a unit vector of $\operatorname{span}(\xi)$, thus $\xi = e \wedge \eta$, where $\eta \in G(k-1, \boldsymbol{R}^n)$. Then

$$\begin{aligned}\Phi(\xi) = \Phi(e \wedge \eta) &= \langle e, \bar{\Phi}(\eta)\rangle \\ &\leq \|e\| \cdot \|\bar{\Phi}(\eta)\| = \|\bar{\Phi}(\eta)\|.\end{aligned}$$

Thus,

$$(*) \qquad \|\Phi\|^* \leq \|\bar{\Phi}\|.$$

Now suppose $\eta \in G(\bar{\Phi})$. Let $e = \bar{\Phi}(\eta)/\|\bar{\Phi}(\eta)\|$, and $\xi = e \wedge \eta$. Then

$$\begin{aligned}\Phi(\xi) = \Phi(e \wedge \eta) &= \frac{1}{\|\bar{\Phi}(\eta)\|}\langle \bar{\Phi}(\eta), \bar{\Phi}(\eta)\rangle \\ &= \|\bar{\Phi}(\eta)\| = \|\bar{\Phi}\|.\end{aligned}$$

Thus,

$$(**) \qquad \|\Phi\|^* \geq \|\bar{\Phi}\|.$$

$(*)$ and $(**)$ show that

$$\|\Phi\|^* = \|\bar{\Phi}\|.$$

2) The proof of the second part is clear.

Suppose Φ is a k-covector on $\boldsymbol{R}^n$ with $\operatorname{span}(\Phi)^* = \boldsymbol{R}^n$, $(\operatorname{span}(\Phi)^* = \{v \in \boldsymbol{R}^n / v \lrcorner \Phi = 0\}^{\perp})$. Let e be a unit vector on $\boldsymbol{R}^n$, and

$$\varphi = e \lrcorner \Phi,$$
$$\psi = \Phi - e^* \wedge \varphi.$$

We have the decomposition of Φ with respect to e

$$\Phi = e^* \wedge \varphi + \psi,$$

where φ and ψ are respectively $(k-1)$-covector and k-covector on $e^{\perp}$.

The following lemma gives a relationship between $\|\Phi\|^*$, $\|\varphi\|^*$ and $\|\psi\|^*$.

LEMMA 2.3.

$$\max\{\|\varphi\|^*, \|\psi\|^*\} \leq \|\Phi\|^* \leq \sqrt{\|\varphi\|^{*2} + \|\psi\|^{*2}}.$$

Proof. Suppose Φ has the decomposition with respect to e

$$\Phi = e^* \wedge \varphi + \psi.$$

Let $\eta \in G(\varphi)$ and let $\xi = e \wedge \eta$. Then

$$\Phi(\xi) = e^* \wedge \varphi(\xi) + \psi(\xi) = (e^* \wedge \varphi)(\xi) = \varphi(\eta) = \|\varphi\|^*.$$

Thus,

$$(*) \qquad \|\Phi\|^* \geq \|\varphi\|^*.$$

Let $\xi \in G(\psi)$, then $(e^* \wedge \varphi)(\xi) = 0$, and hence

$$\Phi(\xi) = \psi(\xi) = \|\psi\|^*.$$

Thus,

$$(**) \qquad \|\Phi\|^* \geq \|\psi\|^*.$$

$(*)$ and $(**)$ prove the first inequality.

To prove the second inequality, we use the canonical form of a simple vector with respect to a subspace (see [HL], Lemma 7.5).

Let $\xi \in \bigwedge^k(\boldsymbol{R}^n)$, then ξ has the canonical form with respect to the subspace $\operatorname{span}(e) = \{r.e / r \in \boldsymbol{R}\}$

$$\xi = (\cos\alpha e + \sin\alpha f) \wedge \eta,$$

where e, f are orthonormal vectors, $\eta \in G(k-1, e^{\perp})$; $e \in \operatorname{span}(\eta)^{\perp}$; $f \in \operatorname{span}(\eta)^{\perp}$. Then

$$\begin{aligned}
\Phi(\eta) &= \cos\alpha\varphi(\eta) + \sin\alpha\psi(f \wedge \eta) \\
&\leq \sqrt{\cos^2\alpha + \sin^2\alpha} \cdot \sqrt{\varphi(\eta)^2 + \psi(f \wedge \eta)^2} \\
&\leq \sqrt{\|\varphi\|^{*2} + \|\psi\|^{*2}}.
\end{aligned}$$

The proof of the Lemma 2.3 is completed.

More exactly, we have the following theorem.

THEOREM 2.4. 1) $\|\Phi\|^* = \max_{\eta \in G(k-1,e^\perp)} \left\{\sqrt{\varphi(\eta)^2 + \bar{\psi}(\eta)^2}\right\} = A.$
2) $G(\Phi) = \{(\cos\alpha e + \sin\alpha f) \wedge \eta / \eta \in G(k-1, e^\perp)\}$, *where*
i) $\varphi(\eta)^2 + \bar{\psi}(\eta)^2 = A^2$,
ii) $f = \dfrac{\bar{\psi}(\eta)}{\|\bar{\psi}(\eta)\|}$,
iii) $\cos\alpha = \dfrac{\varphi(\eta)}{A}$, $\sin\alpha = \dfrac{\|\bar{\psi}(\eta)\|}{A}$.

Proof. 1) By the virtue of the proof of Lemma 2.3, we imply that

$$\|\Phi\|^* \leq \max\left\{\sqrt{\varphi(\eta)^2 + \bar{\psi}(\eta)^2}\right\} = A.$$

Now suppose the equality $\sqrt{\varphi(\eta)^2 + \bar{\psi}(\eta)^2} = A$ holds for some $\eta \in G(k-1, e^\perp)$.
Let $f = \bar{\psi}(\eta)/\|\bar{\psi}(\eta)\|$; $\cos\alpha = \varphi(\eta)/A$; $\sin\alpha = \|\bar{\psi}(\eta)\|/A$, and

$$\xi = (\cos\alpha e + \sin\alpha f) \wedge \eta,$$

then

$$\begin{aligned}\Phi(\xi) &= \cos\alpha\varphi(\eta) + \sin\alpha\|\bar{\psi}(\eta)\| \\ &= \frac{\varphi(\eta)^2 + \bar{\psi}(\eta)^2}{A} = \frac{A^2}{A} = A.\end{aligned}$$

The first part is proved.
2) The proof of the second part is clear.

Now we can give a criterion which can be used to check the comass one of some classes of covectors, and construct new examples of calibrations.

COROLLARY 2.5. *Suppose Φ has the decomposition with respect to e, $\|e\| = 1$*

$$\Phi = e^* \wedge \varphi + \psi,$$

where $\|\varphi\|^* = \|\psi\|^* = 1$. Then we have
1) $\|\Phi\|^* = 1$ *if and only if* $\sqrt{\varphi(\eta)^2 + \bar{\psi}(\eta)^2} \leq 1$ *for all* $\eta \in G(k-1, \boldsymbol{R}^n)$.
2) *If* $\|\Phi\|^* = 1$, *then*

$$G(\Phi) = \{\xi = (\cos\alpha e + \sin\alpha f) \wedge \eta / \eta \in G(k-1, e^\perp)\},$$

where
i) $\varphi(\eta)^2 + \bar{\psi}(\eta)^2 = 1$.
ii) $f = \dfrac{\bar{\psi}(\eta)}{\|\bar{\psi}(\eta)\|}$.
iii) $\cos\alpha = \varphi(\eta)$, $\sin\alpha = |\bar{\psi}(\eta)|$.

The proof of the corollary is implied directly from the proof of the Lemma 2.4.

III. $F^*(SLAG)$ on $\boldsymbol{R}^8$

Denote $F^*(SLAG)$ be the set of all calibrations on $\boldsymbol{R}^n$, whose faces contain a Special Lagrangian face. The first cousin principle shows that such calibrations must be in the form

$$\Phi(\lambda, a) = \Phi_{SLAG} + \lambda(e_{14}^* + e_{25}^* + e_{36}^*) \wedge u^* + a.v^* \wedge e_{78}^*,$$

where u is a unit vector in $\mathrm{span}\{e_7, e_8\}$ and v is a unit vector in $\mathrm{span}\{e_1, e_2, \ldots, e_6\}$. Here we used the notation e_{ijk}^* instead of $e_i^* \wedge e_j^* \wedge e_k^*$. By using the action of $SO(2)$ on $\mathrm{span}\{e_7, e_8\}$, we can assume that $u = e_7$ and by using the action of $SU(3)$ on $\mathrm{span}\{e_1, e_2, \ldots, e_6\}$, we can assume that $v = e_1$.

LEMMA 3.1. *Let $\Phi \in \bigwedge^3(\boldsymbol{R}^8)^*$ be in the form*

$$\Phi(\lambda, a) = \Phi_{SLAG} + \lambda(e_{14}^* + e_{25}^* + e_{36}^*) \wedge e_7^* + ae_1^* \wedge e_{78}^*,$$

then $\Phi(\lambda, a) \in F^*(SLAG)$ iff

$$a^2 + \lambda^2 \leq 1.$$

Proof. We write $\Phi(\lambda, a)$ in the form

$$\Phi(\lambda, a) = e_8^* \wedge (ae_{17}^*) + \Phi_{SLAG} + \lambda(e_{14}^* + e_{25}^* + e_{36}^*) \wedge e_7^*.$$

Then by virtue of the Theorem 2.4

$$\|\Phi(\lambda, a)\|^* = 1 \quad \text{iff} \quad \|ae_{17}^*(\eta)\|^2 + \|\overline{\Phi_{SLAG} + \lambda(e_{14}^* + e_{25}^* + e_{36}^*) \wedge e_7^*(\eta)}\|^2 \leq 1$$

for all $\eta \in G(2, e_8^\perp) \subset \bigwedge^2(\mathrm{span}\{e_1, e_2, \ldots, e_7\}) \cong \bigwedge^2(\boldsymbol{R}^7)$.

Let $\psi = \Phi_{SLAG} + \lambda(e_{14}^* + e_{25}^* + e_{36}^*) \wedge e_7^* \in \bigwedge^3(\boldsymbol{R}^7)$, we have

$$\psi_{e_1} = e_{23}^* - e_{56}^* + \lambda e_{47}^*,$$

$$\psi_{e_2} = -e_{13}^* + e_{46}^* + \lambda e_{57}^*,$$

$$\psi_{e_3} = e_{12}^* - e_{45}^* + \lambda e_{67}^*,$$

$$\psi_{e_4} = -e_{26}^* + e_{35}^* - \lambda e_{17}^*,$$

$$\psi_{e_5} = e_{16}^* - e_{34}^* - \lambda e_{27}^*,$$

$$\psi_{e_6} = -e_{15}^* + e_{24}^* - \lambda e_{37}^*,$$

$$\psi_{e_7} = \lambda(e_{14}^* + e_{25}^* + e_{36}^*).$$

(By simplicity we shall use the notation e_{ij} for $e_{ij}^*(\eta)$ and $e_{ij}e_{kl}$ for $e_{ij}^*(\eta)e_{kl}^*(\eta)$.) We have

$$
\begin{aligned}
ae_{17}^*(\eta)^2 + \bar{\psi}(\eta)^2 &= \|ae_{17}^*(\eta)\|^2 + \sum \|\psi_{e_i}(\eta)\|^2 \\
&= a^2e_{17}^2 + e_{23}^2 + e_{56}^2 + \lambda^2 e_{47}^2 - e_{23}e_{56} + \lambda e_{23}e_{47} - \lambda e_{56}e_{47} \\
&\quad + e_{13}^2 + e_{46}^2 + \lambda^2 e_{57}^2 - e_{13}e_{46} - \lambda e_{13}e_{57} + \lambda e_{46}e_{57} \\
&\quad + e_{12}^2 + e_{45}^2 + \lambda^2 e_{67}^2 - e_{12}e_{45} + \lambda e_{12}e_{67} - \lambda e_{45}e_{67} \\
&\quad + e_{26}^2 + e_{35}^2 + \lambda^2 e_{17}^2 - e_{26}e_{35} + \lambda e_{26}e_{17} - \lambda e_{35}e_{17} \\
&\quad + e_{16}^2 + e_{34}^2 + \lambda^2 e_{27}^2 - e_{16}e_{34} - \lambda e_{16}e_{27} + \lambda e_{34}e_{27} \\
&\quad + e_{15}^2 + e_{24}^2 + \lambda^2 e_{37}^2 - e_{15}e_{24} + \lambda e_{15}e_{37} - \lambda e_{24}e_{37} \\
&\quad + \lambda^2 e_{14}^2 + \lambda^2 e_{25}^2 + \lambda^2 e_{36}^2 + \lambda^2 e_{14}e_{25} + \lambda^2 e_{14}e_{36} + \lambda^2 e_{25}e_{36} \\
&= (e_{12}^2 + e_{13}^2 + e_{14}^2 + e_{15}^2 + e_{16}^2 + e_{17}^2 + e_{23}^2 + e_{24}^2 + e_{25}^2 + e_{26}^2 \\
&\quad + e_{27}^2 + e_{34}^2 + e_{35}^2 + e_{36}^2 + e_{37}^2 + e_{45}^2 + e_{46}^2 + e_{47}^2 + e_{56}^2 + e_{57}^2 \\
&\quad + (\lambda^2 - 1)(e_{47}^2 + e_{57}^2 + e_{67}^2 + e_{27}^2 + e_{37}^2) \\
&\quad + (\lambda^2 - 1)(e_{14} + e_{25} + e_{36})^2 + (a^2 + \lambda^2 - 1)e_{17}^2 \\
&= 1 + (\lambda^2 - 1)(e_{27}^2 + e_{37}^2 + e_{47}^2 + e_{57}^2 + e_{67}^2) \\
&\quad + (\lambda^2 - 1)(e_{14} + e_{25} + e_{36})^2 + (a^2 + \lambda^2 - 1)e_{17}^2.
\end{aligned}
$$

– If $a^2 + \lambda^2 > 1$, let $\eta \in G(2, \boldsymbol{R}^7)$ such that $\mathrm{span}(\eta) = \mathrm{span}\langle e_1, e_7\rangle$, then $\|ae_{17}^*(\eta)\|^2 + \|\bar{\psi}(\eta)\|^2 = a^2 + \lambda^2 > 1$, and hence $\|\Phi\| > 1$.

– If $a^2 + \lambda^2 \le 1$, since

$$
\begin{aligned}
&(\lambda^2 - 1)(e_{47}^2 + e_{57}^2 + e_{67}^2 + e_{27}^2 + e_{37}^2) \le 0, \\
&(\lambda^2 - 1)(e_{14} + e_{25} + e_{36})^2 \le 0, \\
&(a^2 + \lambda^2 - 1)e_{17}^2 \le 0,
\end{aligned}
$$

for all $\eta \in G(2, \boldsymbol{R}^7)$ we imply that $ae_{17}^*(\eta)^2 + \bar{\psi}(\eta)^2 \le a^2 + \lambda^2 \le 1$, and hence $\|\Phi\|^* = 1$.

3.2. Classification of $F^*(SLAG)$ on $\boldsymbol{R}^8$

Now suppose $\|\Phi(\lambda, a)\|^* = 1$, we have the folowing cases:

1. If $\lambda = \pm 1$, then $a = 0$. $\Phi(\lambda, a)$ becomes to Associative-calibration and in this case Φ is an exposed calibration.

2. If $a = \pm 1$, then $\lambda = 0$. $\Phi(\lambda, a)$ is in the form

$$\Phi = \Phi_{SLAG} \pm e_{178}^*.$$

In this case $\Phi(\lambda)$ also an exposed calibration and $G(\Phi(\lambda, a)) = G(\Phi_{SLAG}) \bigcup CP^2$.

3. If $\lambda^2 + a^2 < 1$, then $G(\Phi) = G(\Phi_{SLAG})$. In this case Φ is not a maximal calibration and hence not exposed calibration.

4. If $\lambda^2 + a^2 = 1$ ($\lambda \neq 0$ and $a \neq 0$), η must be a solution of the following system

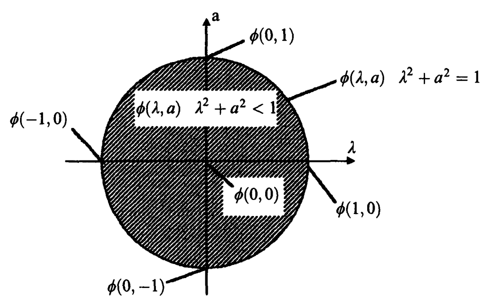
$$\begin{cases} e_{27}^* = e_{37}^* = e_{47}^* = e_{57}^* = e_{67}^* = 0, \\ e_{14}^* + e_{25}^* + e_{36}^* = 0. \quad \text{(Special Lagrangian condition)} \end{cases}$$

i) If $\operatorname{span}(\eta) \subset (e_7^*)^{\perp}$. Then $\xi \in G(\Phi(SLAG))$.

ii) If $\operatorname{span}(\eta) \not\subset (e_7^*)^{\perp}$. From the above system we imply that η must be in the form

$$e_1 \wedge (a_2e_2 + a_3e_3 + a_5e_5 + a_6e_6 + a_7e_7).$$

In this case, each $\Phi(\lambda, a)$ is a maximal calibration.



IV. General associative calibrations, general coassociative calibrations

4.1. Strengthening of the Wirtinger inequality

First we iterate the result of the strengthening of Wirtinger inequality (Theorem 6.11 in [HL]) and another analog result (Remark 6.16 in [HL]), which are used to construct our new calibrations.

THEOREM 4.1.1 (Theorem 6.11 in [HL]). *For each (real) simple $2p$-vector ξ in C^n*

$$|\Omega_p(\xi)|^2 + \cdots + \sum_{|I|=2k}' |dZ^I \wedge \Omega_{p-k}(\xi)|^2 + \cdots = |\xi|^2.$$

If $2p \le n$ the last term on the left-hand side is $\sum'_{|I|=2p} |dZ^I(\xi)|^2$. If $2p > n$ the left-hand side is $\sum'_{|I|=2(n-p)} |dZ^I \wedge \Omega_{p-k}(\xi)|^2$.

THEOREM 4.1.2 (Remark 6.16 in [HL]). *For each (real) simple $2p+1$-vector ξ in C^n*

$$\sum |dz_j \wedge \Omega_p(\xi)|^2 + \cdots + \sum_{|I|=2k+1}' |dZ^I \wedge \Omega_{p-k}(\xi)|^2 + \cdots = |\xi|^2.$$

If $2p+1 \le n$ *the last term on the left-hand side is* $\sum'_{|I|=2p+1} |dZ^I(\xi)|^2$. *If* $2p > n$ *the left-hand side is* $\sum'_{|I|=2p+1} |dZ^I \wedge \Omega_{2p+1-n}(\xi)|^2$.

4.2. Some notes on associative and coassociative calibrations

As the next step, we give some notes on Associative and Coassociative calibrations on $\mathrm{Im}(O) \cong \boldsymbol{R}^7$. Theorem 1.4 in [HL] shows that the trilinear form $\Phi_{ASSOC}(x, y, z) = \langle x, yx \rangle$ on $\mathrm{Im}(O)$ is alternating and has comass one i.e a calibration (Associative calibration). Lemma 1.12 and Theorem 1.16 in [HL] show that the 4-form $\Phi_{COASSOC}(x, y, z, w) = 1/2\langle x, [y, z, w] \rangle$ on $\mathrm{Im}(O)$ is alternating and has comass one i.e. a calibration (Coassociative calibration).

Let e_1, $e_2 = i$, $e_3 = j$, $e_4 = k$, $e_5 = e$, $e_6 = ie$, $e_7 = je$, $e_8 = ke$ denote the standard basis for the Octonions O. Let $e_1^*, e_2^*, \ldots, e_8^*$ denote the dual basis for O^*. By simplicity we use the notation e_{pqr}^* for $e_p^* \wedge e_q^* \wedge e_r^*$. Then the form Φ_{ASSOC} can be expressed in terms of axis 3-planes as follows

$$\begin{aligned}\Phi_{ASSOC} &= e_{234}^* - e_{278}^* - e_{638}^* - e_{674}^* - e_{265}^* - e_{375}^* - e_{485}^* \\ &= e_2^* \wedge (e_{34}^* + e_{56}^* - e_{78}^*) + (e_{368}^* - e_{467}^* + e_{357}^* + e_{458}^*),\end{aligned}$$

and the form $\Phi_{COASSOC}$ can be expressed in terms axis 4-planes as follows

$$\begin{aligned}\Phi_{COASSOC} &= e_{5678}^* - e_{5634}^* - e_{5274}^* - e_{5238}^* + e_{3478}^* + e_{2468}^* + e_{2367}^* \\ &= e_2^* \wedge (e_{457}^* - e_{358}^* + e_{468}^* + e_{367}^*) + (e_{5678}^* - e_{3456}^* + e_{3478}^*).\end{aligned}$$

Consider on O the complex structure J defined by

$$e_3, e_4 = Je_3, \quad e_5, e_6 = Je_5, \quad e_7, e_8 = -Je_7, \quad e_1, e_2 = Je_1.$$

On $(e_1, Je_1)^\perp$, let Ω denotes the Kähler form, $\Omega_2 = 1/2\Omega^2$ and $\mathrm{Re}(dZ)$ denotes the Special Lagrangian form. Then Φ_{ASSOC} can be expressed as follows

$$\Phi_{ASSOC} = Je_1^* \wedge (\Omega) + \mathrm{Re}\, dZ,$$

and $\Phi_{COASSOC}$ can be expressed as follows

$$\Phi_{COASSOC} = Je_1^* \wedge (\mathrm{Im}\, dZ) + \Omega_2.$$

4.3. General associative calibrations

These are calibrations of degree $(2n-1)$ on $\boldsymbol{R}^{4n-1}$, and Associative calibration is the special case of them when $n = 2$. Let $\{e_1, Je_1, e_2, Je_2, \ldots, e_{2n}, Je_{2n}\}$ be an orthnormal basis on $\boldsymbol{C}^{2n}$ correspond to the complex structure J. The subspace $(e_n, Je_n)^\perp \simeq \boldsymbol{C}^{2n-1}$ inherit the induced complex structure (we also denote by J), and it is easy to see that $\{e_1, Je_1, e_2, Je_2, \ldots, e_{2n-1}, Je_{2n-1}\}$ is an orthonormal basis on it.

Let

$$\mathrm{Re}(e^{i\theta} dZ) = \mathrm{Re}(e^{i\theta} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{2n-1})$$

denotes the Special Lagrangian calibration (of degree $2(n-1)$) on $(e_n, Je_n)^\perp$ and

$$\Omega_{n-1} = \frac{1}{(n-1)!}\Omega^{n-1},$$

where Ω is the Kähler form on $(e_n, Je_n)^{\perp}$. It is very easy to show that

$$\mathrm{Re}(e^{i\theta}dZ)_{e_i} = \mathrm{Re}(e^{i\theta}dz_1 \wedge dz_2 \wedge \cdots \wedge \widehat{dz_\imath} \wedge \cdots \wedge dz_{2n-1}),$$

$$\mathrm{Re}(e^{i\theta}dZ)_{Je_i} = \mathrm{Im}(e^{i\theta}dz_1 \wedge dz_2 \wedge \cdots \wedge \widehat{dz_\imath} \wedge \cdots \wedge dz_{2n-1}),$$

and hence for all $\eta \in G(2(n-1), (e_n, Je_n)^{\perp})$

$$\begin{aligned}|\overline{\mathrm{Re}(e^{i\theta}dZ)}(\eta)|^2 &= \sum |\mathrm{Re}(e^{i\theta}dz_1 \wedge dz_2 \wedge \cdots \wedge \widehat{dz_\imath} \wedge \cdots \wedge dz_{2n-1})|^2 \\ &\quad + \sum |\mathrm{Im}(e^{i\theta}dz_1 \wedge dz_2 \wedge \cdots \wedge \widehat{dz_\imath} \wedge \cdots \wedge dz_{2n-1})|^2 \\ &= \sum |dz_2 \wedge \cdots \wedge \widehat{dz_\imath} \wedge \cdots \wedge dz_{2n}|^2 \\ &= \sum_{|I|=2(n-1)} |dZ^I(\eta)|^2.\end{aligned}$$

By virtue of the Theorem 3.1.1, we have

$$\text{(4.3.1)} \qquad \begin{aligned}&|\Omega_{n-1}(\eta)|^2 + \sum_{|I|=2(n-1)} |dZ^I(\eta)|^2 \\ &= |\Omega_{n-1}(\eta)|^2 + |\overline{\mathrm{Re}(dZ)}(\eta)|^2 \le |\eta|^2 = 1,\end{aligned}$$

the equality holds if and only if

$$\sum_{k=1}^{p-1} {\sum_{|I|=2k}}' |dZ^I \wedge \Omega_{p-k}(\eta)|^2 = 0.$$

More exactly η is a solution of the following system of equations

$$\text{(4.3.2)} \qquad \begin{cases} |\mathrm{Re}\, dZ^I \wedge \Omega_{p-k}(\xi)| = 0 & (|I| = 2k; k = 1, 2, \ldots, p-1), \\ |\mathrm{Im}\, dZ^I \wedge \Omega_{p-k}(\xi)| = 0 & (|I| = 2k; k = 1, 2, \ldots, p-1). \end{cases}$$

Let $\xi \in G(2(n-1), Je_n \oplus \boldsymbol{C}^{2n-1})$. Suppose that ξ has the canonical form with respect to the subspaces $\mathrm{span}(Je_n)$

$$\xi = (\cos\alpha Je_n + \sin\alpha f) \wedge \eta.$$

Definition 4.3.3. ξ is called G.ASSOCIATIVE if η is a solution of (4.3.2.) and

i) $f = \dfrac{\overline{\mathrm{Re}\, dZ}(\eta)}{\|\overline{\mathrm{Re}\, dZ}(\eta)\|}$.

ii) $\cos\alpha = \Omega_{n-1}(\eta)$, $\sin\alpha = |\overline{\mathrm{Re}\, dZ}(\eta)|$.

Let $\Phi_{G.ASSOC}$ be the $(2n-1)$-covector on $\operatorname{span}(Je_n) \oplus \boldsymbol{C}^{2n-1} \simeq \boldsymbol{R}^{4n-1}$ defined by

$$\Phi_{G.ASSOC(\theta)} = Je_n \wedge \Omega_{n-1} + \operatorname{Re}(e^{i\theta} dZ).$$

We have an S^1-family of G.ASSOC Geometries compatible with the given complex structure. Since these Geometries are equivalent under $SU(2n-1)$, it will suffice to study the one associated to $\theta = 0$.

THEOREM 4.3.4. *The* $(2n-1)$*-covector on* $\operatorname{span}(Je_n) \oplus \boldsymbol{C}^{2n-1} \simeq \boldsymbol{R}^{4n-1}$

$$\Phi_{G.ASSOC} = Je_n^* \wedge \Omega_{n-1} + \operatorname{Re} dZ$$

has comass one, i.e.

$$\Phi_{G.ASSOC}(\xi) \le |\xi| \quad \text{for all } \xi \in G(2n-1, \boldsymbol{R}^{4n-1}),$$

and the equality holds iff ξ *is a G.ASSOCIATIVE, i.e.*

$$G(\Phi_{G.ASSOC}) = \{\xi / \xi \text{ is a G.ASSOCIATIVE}\}.$$

Proof. The proof is implied directly by (4.3.1), (4.3.2) and Theorem 2.4.

4.4. General coassociative calibrations

These are calibrations of degree $2n$ on $\boldsymbol{R}^{4n-1}$, and Coassociative calibration is the special case of them when $n = 2$.

Consider the Power of Kähler form of degree $2n$ on $(e_n, Je_n)^{\perp}$, with the induced complex structure.

$$\Omega_n = \frac{1}{n!}\Omega^n,$$

where Ω is the Kähler form on $(e_n, Je_n)^{\perp}$. Direct computation shows that

$$(\Omega_n)_{e_i} = Je_i^* \wedge \Omega_{n-1},$$

$$(\Omega_n)_{Je_i} = e_i^* \wedge \Omega_{n-1},$$

and hence for all $\eta \in G(2n-1, \boldsymbol{R}^{4n-1})$

$$\begin{aligned} |\overline{\Omega_n}(\eta)|^2 &= \sum |e_i^* \wedge \Omega_{n-1}(\eta)|^2 + \sum |Je_i^* \wedge \Omega_{n-1}(\eta)|^2 \\ &= \sum |dz_i \wedge \Omega_{n-1}(\eta)|^2. \end{aligned}$$

Let dZ denote the Lagrangian form on $(e_n, Je_n)^{\perp}$. Then by virtue of the Theorem 4.3.2, we have

$$|e^{i\theta} \operatorname{Im} dZ(\eta)|^2 + |\overline{\Omega_n}(\eta)|^2 \le 1. \tag{4.4.1}$$

The equality holds if and only if

1) $|\operatorname{Re} dZ(\eta)|^2 = 0$,
2) $\sum_{k=1}^{2(n-1)} \sum_{I=2k+1} |dZ^I \wedge \Omega_{2k+1}(\eta)|^2 = 0$.

More exactly η is a solution of the following system of equations

$$(4.4.2)\qquad \begin{cases} \operatorname{Re} dZ(\eta) = 0, \\ \operatorname{Re} dZ^I \wedge \Omega_{2k+1}(\eta) = 0 \quad (|I| = 2k+1; k = 1, 2, \dots, 2(n-1)), \\ \operatorname{Im} dZ^I \wedge \Omega_{2k+1}(\eta) = 0 \quad (|I| = 2k+1; k = 1, 2, \dots, 2(n-1)). \end{cases}$$

Let $\xi \in G(2n, Je_n \oplus \mathbf{C}^{2n-1})$. Suppose that ξ has the canonical form with respect to the subspace $\operatorname{span}(Je_n)$

$$\xi = (\cos \alpha Je_n + \sin \alpha f) \wedge \eta,$$

where $f \in \mathbf{C}^{2n-1}$, $\eta \in G(2n-1, Je_n \oplus \mathbf{C}^{2n-1})$.

DEFINITION 4.4.3. ξ is called G.COASSOCIATIVE if η is a solution of (4.4.2) and

i) $f = \dfrac{\overline{\Omega}(\eta)}{\|\overline{\Omega}(\eta)\|}$.

ii) $\cos \alpha = \operatorname{Re} dZ(\eta)$; $\sin \alpha = |\overline{\Omega_{n-1}}(\eta)|$.

Let $\Phi_{G.COASSOC}$ be the $2n$-covectors on $\operatorname{span}(Je_1) \oplus \mathbf{C}^{2n-1} \simeq \mathbf{R}^{4n-1}$ defined by

$$\Phi_{G.COASSOC} = Je_n^* \wedge \operatorname{Im}(e^{i\theta})\, dZ - \Omega_n,$$

we have an S^1-family of G.COASSOC Geometries compatible with the given complex structure. Since these Geometries are equivalent under $SU(2n-1)$, it will suffice to study the one associated to $\theta = 0$.

THEOREM 4.4.4. *The* $2n$*-covector on* $\operatorname{span}(Je_n) \oplus \mathbf{C}^{2n-1} \simeq \mathbf{R}^{4n-1}$

$$\Phi_{G.COASSOC} = (-1)^n Je_n^* \wedge \operatorname{Im} dZ - \Omega_n$$

has comass one, i.e.

$$\Phi_{G.COASSOC}(\xi) \le |\xi| \quad \textit{for all } \xi \in G(2n, \mathbf{R}^{4n-1}),$$

and the equality holds iff ξ *is G.COASSOCIATIVE.*

Proof. The proof is implied directly by Theorem 2.4 and (4.4.1).

4.5. Relationship between G.ASSOC and G.COASSOC calibrations

In this section we give the first relationship between G.ASSOC and G.COASSOC calibrations. This relationship likes the first relationship between ASSOC and COASSOC calibrations

$$*\Phi_{G.ASSOC} = \Phi_{G.COASSOC}.$$

And in [Hi2] the second relationship

$$\Phi_{G.ASSOC}(\eta)^2 + \overline{\Phi_{G.COASSOC}}(\eta)^2 \le 1$$

holds, for all $\eta \in G(2n-1, \mathbf{R}^{4n-1})$.

If $n = 2$ we the following equality holds for all $\eta \in G(3, \boldsymbol{R}^7)$

$$\Phi_{ASSOC}(\eta)^2 + \overline{\Phi_{COASSOC}}(\eta)^2 = 1.$$

LEMMA 3.5.1. *On* $\boldsymbol{R}^{4n-1}$

$$*\Phi_{G.ASSOC} = \Phi_{G.COASSOC}.$$

Proof. It is easily to see that: on $\boldsymbol{R}^{4n-2}$

$$*\mathrm{Re}\, dZ = (-1)^{n-1} \mathrm{Im}\, dZ \tag{4.5.2}$$

$$*\Omega_{2n} = \Omega_{2(n-1)} \tag{4.5.3}$$

The proof is implied directly by 4.5.2 and 4.5.3.

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