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ON LOG HODGE STRUCTURES OF HIGHER DIRECT IMAGES

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1. Introduction

Let Y be an analytic space endowed with an fs log structure $\mathscr{M}_Y$ in the sense of Fontaine-Illusie. The pair $(Y, \mathscr{M}_Y)$ is called an fs log analytic space (cf. [KN]). For an fs log analytic space $(Y, \mathscr{M}_Y)$, K. Kato and C. Nakayama construct in [KN] a ringed space $(Y^{\log}, \mathscr{O}_Y^{\log})$ endowed with a continuous surjective map $\tau : Y^{\log} \to Y$. In this paper, we mainly treat an object on Y called a log Hodge structure which is defined by K. Kato in [Ka2]. It consists of the following triplet that satisfies certain conditions (See 5.3):

- A sheaf of $\boldsymbol{Q}$-modules $\mathscr{H}_{\boldsymbol{Q}}$ on $Y^{\log}$.
- A sheaf of $\mathscr{O}_Y$-modules $\mathscr{H}_{\mathscr{O}}$ on Y endowed with a descending filtration.
- An isomorphism of $\mathscr{O}_Y^{\log}$-modules $\iota : \mathscr{H}_{\boldsymbol{Q}} \otimes_{\boldsymbol{Q}} \mathscr{O}_Y^{\log} \cong \tau^* \mathscr{H}_{\mathscr{O}}$.

Let $f : (X, \mathscr{M}_X) \to (Y, \mathscr{M}_Y)$ be a morphism of fs log analytic spaces satisfying the following condition:

(∗) Locally on X and on Y,

(i) There exists a chart $P := \boldsymbol{N} \to \mathscr{M}_Y$, and a morphism of monoids

$$P \to Q := \boldsymbol{N}^r; \quad 1 \mapsto (1, \dots, 1),$$

for some $r \geq 1$, and

(ii) X is isomorphic to an open subspace of $Y \times_{\operatorname{Spec} \boldsymbol{C}[P]_{an}} \operatorname{Spec} \boldsymbol{C}[Q]_{an}$, where Spec $\boldsymbol{C}[P]$ and Spec $\boldsymbol{C}[Q]$ are endowed with the log structures associated to $P \to \boldsymbol{C}[P]$ and $Q \to \boldsymbol{C}[Q]$, respectively.

First, we prove two basic properties.

THEOREM A. *We have a quasi-isomorphism*

$$(f^{\log})^{-1} \mathscr{O}_Y^{\log} \xrightarrow{\sim} \omega_{X/Y}^{\bullet \log},$$

where $\omega_{X/Y}^{1 \log} = \omega_{X/Y}^1 \otimes_{\mathscr{O}_X} \mathscr{O}_X^{\log}$ *and* $\omega_{X/Y}^{\bullet \log}$ *is its exterior algebra.*

THEOREM B. *Assume moreover f is proper. Let $\tau : Y^{\log} \to Y$ be the canonical map. Then we have an isomorphism of $\mathscr{O}_Y^{\log}$-modules*

$$\iota : R^m f_*^{\log} \boldsymbol{Q} \otimes \mathscr{O}_Y^{\log} \cong \tau^* R^m f_* \omega_{X/Y}^{\bullet}$$

for each m.

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(**) For example, let $Y := \{z \in \boldsymbol{C} \mid |z| < 1\}$ be the unit disk, and $f : X \to Y$ a projective surjective morphism of complex manifolds. We assume that f is smooth over the punctured disk $Y^* = Y - \{0\}$ and that $X_0 = f^{-1}(0)$ is a reduced divisor with normal crossings. Let $P \in X_0$. We assume that there exists a coordinate neighborhood U of P with coordinates $(z_0, \dots, z_n)$ and an integer r with $1 \le r \le n$ such that $P = (0, \dots, 0)$ and $f|U(z_1, \dots, z_n) = z_1 \cdots z_r = z$. Let $\mathscr{M}_Y$ (resp. $\mathscr{M}_X$) be a sheaf of holomorphic functions on Y (resp. X) which are invertible outside the origin (resp. X_0). Then we have a morphism $f : (X, \mathscr{M}_X) \to (Y, \mathscr{M}_Y)$ of fs log analytic spaces, which satisfies the condition (*).

If $f : X \to Y$ is a proper smooth morphism of complex manifolds, it is well known, as relative Poincaré lemma that $\Omega^\bullet_{X/Y}$ is a resolution of the sheaf $f^{-1}\mathcal{O}_Y$. Using this, it is easy to construct an isomorphism of $\mathcal{O}_Y$-modules $R^m f_* \boldsymbol{Q} \otimes \mathcal{O}_Y \xrightarrow{\sim} R^m f_* \Omega^\bullet_{X/Y}$. Theorem A and Theorem B correspond to these facts. As for a log Hodge structure, we have

THEOREM C. *Let $f : X \to Y$ be as in (**). Let $\mathscr{H}_{\boldsymbol{Q}} = R^m f^{\log}_* \boldsymbol{Q}$, $\mathscr{H}_{\mathcal{O}} = R^m f_* \omega^\bullet_{X/Y}$ endowed with a filtration $R^m f_* \omega^{\bullet \ge i}_{X/Y}$ and $\imath$ the isomorphism as in Theorem B. Then the triplet $(\mathscr{H}_{\boldsymbol{Q}}, \mathscr{H}_{\mathcal{O}}, \imath)$ is a log Hodge structure on Y.*

Here is some backgrounds. Let $Y := \{z \in \boldsymbol{C} \mid |z| < 1\}$ be the unit disk, and $f : X \to Y$ a projective surjective morphism of complex manifolds. We assume that f is smooth over the punctured disk $Y^* = Y - \{0\}$ and that $X_0 = f^{-1}(0)$ is a divisor with normal crossings. We can consider a family of the polarized Hodge structures over Y^*. We can consider it as a holomorphic map from Y^* to the classifying space of polarized Hodge structures modulo monodromy. This map is called the period map. W. Schmid has proved in [Sch] that the period map can be approximated by the associated nilpotent orbit. It is a holomorphic map from Y to the compact dual of the classifying space of polarized Hodge structures, for which the origin of Y is mapped to a polarized mixed Hodge structure. On the other hand, log geometry works well with varieties with normal crossings. The aim of Theorem C is to treat the above fact from a viewpoint of log geometry. In the proof of Theorem C, we see that this log Hodge structure amounts to W. Schmid's nilpotent orbit theorem. We expect that log Hodge structures give a construction of compactification of some moduli space.

Remark 1.1. Related topics are studied by some people. S. Usui obtains a theorem corresponding to our Theorem B in [Usu] independently. His method is quite different from ours and he obtains a more general result. F. Kato also obtains Theorem A and Theorem B in [FKa]. His method is similar to ours.

In Section 2, we recall basic notions of a log geometry. In Section 3, we prove Theorem A, a "log version" of relative Poincaré lemma. In Section 4, we prove Theorem B using the log Poincaré lemma and some inductions. In Section 5, we define the log Hodge structure and prove Theorem C.

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2. The ringed space $(X^{\log}, \mathcal{O}_X^{\log})$ associated to a log scheme X

In this section, we recall some notions in log geometry, which will be used in the later sections. For more systematic descriptions, see [Ka1], [KN].

Definition 2.1. Let $\overset{\circ}{X}$ be an analytic space and $\mathcal{O}_{\overset{\circ}{X}}$ the sheaf of holomorphic functions on $\overset{\circ}{X}$. A pre-log structure on $\overset{\circ}{X}$ is a sheaf of monoids $\mathcal{M}$ on $\overset{\circ}{X}$ endowed with a homomorphism of sheaves of monoids $\alpha : \mathcal{M} \to \mathcal{O}_{\overset{\circ}{X}}$ with respect to the multiplication on $\mathcal{O}_{\overset{\circ}{X}}$. It is denoted by $(\mathcal{M}, \alpha)$, or simply $\mathcal{M}$. A pre-log structure is said to be a log structure if $\alpha^{-1}(\mathcal{O}_{\overset{\circ}{X}}^*) \to \mathcal{O}_{\overset{\circ}{X}}^*$ is an isomorphism via α.

2.2. A log analytic space X is a pair of an analytic space $\overset{\circ}{X}$ and a log structure $\mathcal{M}_X$ on $\overset{\circ}{X}$. It is denoted by $X := (\overset{\circ}{X}, \mathcal{M}_X)$, or simply by $(X, \mathcal{M}_X)$. A morphism $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ of log analytic spaces is defined to be a pair of a morphism of analytic spaces $f : X \to Y$ and a homomorphism $h : f^{-1}(\mathcal{M}_Y) \to \mathcal{M}_X$ such that the diagram

$$\begin{array}{ccc} f^{-1}(\mathcal{M}_Y) & \xrightarrow{h} & \mathcal{M}_X \\ \downarrow & & \downarrow \\ f^{-1}(\mathcal{O}_Y) & \xrightarrow{f} & \mathcal{O}_X \end{array}$$

is commutative. It is denoted by (f, h), or simply by f.

2.3. For a pre-log structure $(\mathcal{M}, \alpha)$ on X, its associated log structure $\mathcal{M}^a$ is defined to be the push out of

$$\begin{array}{ccc} \alpha^{-1}\mathcal{O}_X^* & \longrightarrow & \mathcal{M} \\ \downarrow & & \\ \mathcal{O}_X^* & & \end{array}$$

in the category of sheaves of monoids, endowed with the homomorphism

$$\mathcal{M}^a \to \mathcal{O}_X; \quad (a, b) \mapsto \alpha(a)b \quad (a \in \mathcal{M}, b \in \mathcal{O}_X^*).$$

2.4. A monoid P is said to be an fs monoid if it satisfies the following three conditions:

(i) P is finitely generated.

(ii) If a, b, $c \in P$ and $ab = ac$, then $b = c$.

(iii) If $a \in P^{gp}$ and $a^n \in P$ for some $n \neq 1$, then $a \in P$. Here P^{gp} is the group associated to P.

2.5. A log analytic space $(X, \mathscr{M}_X)$ is said to be an fs log analytic space if locally there exists a constant sheaf P of fs monoids and a homomorphism $P \to \mathscr{O}_X$ such that the log structure $\mathscr{M}_X$ is isomorphic to the log structure associated to the pre-log structure defined by P. A pair of P and the canonical map $P \to \mathscr{M}_X$ is called a chart. By definition, a chart exists locally.

Definition 2.6. Let $X := (\mathring{X}, \mathscr{M}_X)$ be an fs log analytic space. We define the associated topological space $X^{\log}$ in the following way. Let $\mathring{T}$ be the analytic space Spec $\boldsymbol{C}$ endowed with log structure $\mathscr{M}_T$ given by

$$\Gamma(\mathring{T}, \mathscr{M}_T) = \boldsymbol{R}_{\geq 0} \times \mathbf{S}^1,$$

where

$$\boldsymbol{R}_{\geq 0} = \{x \in \boldsymbol{R}; x \geq 0\} \quad \text{and} \quad \mathbf{S}^1 = \{x \in \boldsymbol{C}; |x| = 1\}$$

are considered as the multiplicative semi-groups and the morphism $\mathscr{M}_T \to \mathscr{O}_T$ is given by

$$\boldsymbol{R}_{\geq 0} \times \mathbf{S}^1 \longrightarrow \boldsymbol{C}; \quad (x, y) \mapsto xy.$$

Let T be the log analytic space $(\mathring{T}, \mathscr{M}_T)$. As a set, we define $X^{\log}$ to be the set of all morphisms $T \to X$ of log analytic spaces over $\boldsymbol{C}$. We have the canonical surjective map $\tau : X^{\log} \to \mathring{X}$. We define the topology of $X^{\log}$ as follows. Working on locally on X, let $\alpha : P \to M_X$ be a chart of $\mathscr{M}_X$. Then, by using the homomorphism $P^{gp} \to \mathscr{M}_X^{gp}$, $X^{\log}$ is identified with a closed subset of $X \times \mathrm{Hom}(P^{gp}, \mathbf{S}^1)$. The topology of $X^{\log}$ is given by this identification.

Lemma 2.7 (KN, (1.3)). (i) *The map* $\tau : X^{\log} \to \mathring{X}$ *is continuous. Furthermore it is proper, that is, for any compact subset* C *of* $\mathring{X}$*, the subspace* $\tau^{-1}(C)$ *of* $X^{\log}$ *is compact.*

(ii) *For* $x \in \mathring{X}$*,* $\tau^{-1}(x)$ *is homeomorphic to the product of* r *copies of* $\mathbf{S}^1$ *where* r *is the rank of* $\mathscr{M}_{X,x}^{gp}/\mathscr{O}_{X,x}^*$.

(iii) *Let* $X := (\mathring{X}, \mathscr{M}_X)$ *and* $Y := (\mathring{Y}, \mathscr{M}_Y)$ *be fs log analytic spaces, respectively. Let* $f : X \to Y$ *be a morphism of log analytic spaces. Assume* $f^{-1}\mathscr{M}_Y \xrightarrow{\sim} \mathscr{M}_X$. *Then the diagram of topological spaces*

$$\begin{array}{ccc} X^{\log} & \longrightarrow & Y^{\log} \\ \downarrow & & \downarrow \\ \mathring{X} & \longrightarrow & \mathring{Y} \end{array}$$

is cartesian.

2.8. Let $(X, \mathcal{M}_X)$ be an fs log analytic space and $\tau : X^{\log} \to X$ the canonical map. For a topological space A, we denote by $\mathrm{Cont}(\ ,A)$ the sheaf of continuous functions on $X^{\log}$ with values in A. Let $\tau^{-1}(\mathcal{M}_X^{\mathrm{gp}}) \to \mathrm{Cont}(\ ,\boldsymbol{S}^1)$ be the natural map. Let $\mathrm{Cont}(\ ,i\boldsymbol{R}) \to \mathrm{Cont}(\ ,\boldsymbol{S}^1)$ be the map given by composition with exp. We define a sheaf $\mathcal{L}$ of abelian groups on $X^{\log}$ to be the fibre product of $\mathrm{Cont}(\ ,i\boldsymbol{R})$ and $\tau^{-1}\mathcal{M}_X^{\mathrm{gp}}$ over $\mathrm{Cont}(\ ,\boldsymbol{S}^1)$. Let $h : \tau^{-1}\mathcal{O}_X \to \mathcal{L}$ be the map induced by the map $\tau^{-1}\mathcal{O}_X \to \mathrm{Cont}(\ ,i\boldsymbol{R})$; $f \mapsto f - \mathrm{Re}(f)$. Then we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \boldsymbol{Z}(1) & \longrightarrow & \tau^{-1}\mathcal{O}_X & \xrightarrow{\exp} & \tau^{-1}\mathcal{O}_X^{*} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle h} & & \downarrow & & \\
0 & \longrightarrow & \boldsymbol{Z}(1) & \longrightarrow & \mathcal{L} & \longrightarrow & \tau^{-1}\mathcal{M}_X^{\mathrm{gp}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \theta} & & \downarrow & & \\
0 & \longrightarrow & \boldsymbol{Z}(1) & \longrightarrow & \mathrm{Cont}(\ ,i\boldsymbol{R}) & \longrightarrow & \mathrm{Cont}(\ ,\mathrm{S}^1) & \longrightarrow & 0
\end{array}$$

Definition 2.9. Let $\mathrm{Sym}_{\boldsymbol{Z}}(\mathcal{L})$ be the symmetric algebras of $\mathcal{L}$ over $\boldsymbol{Z}$. We define a sheaf $\mathcal{O}_X^{\log}$ of $\tau^{-1}\mathcal{O}_X$-algebras on $X^{\log}$ as follows:

$$\mathcal{O}_X^{\log} = (\tau^{-1}\mathcal{O}_X \otimes_{\boldsymbol{Z}} \mathrm{Sym}_{\boldsymbol{Z}}(\mathcal{L}))/\mathfrak{a}$$

where $\mathfrak{a}$ is the ideal of $\tau^{-1}\mathcal{O}_X \otimes_{\boldsymbol{Z}} \mathrm{Sym}_{\boldsymbol{Z}}(\mathcal{L})$ generated by local sections of the form

$$f \otimes 1 - 1 \otimes h(f) \quad \text{for} \quad f \in \tau^{-1}\mathcal{O}_X.$$

For $r \in \boldsymbol{Z}$, we define a filtration $\mathrm{fil}_r(\mathcal{O}_X^{\log})$ of $\mathcal{O}_X^{\log}$ to be the image of $\tau^{-1}\mathcal{O}_X \otimes_{\boldsymbol{Z}} \oplus_{i=0}^{r} \mathrm{Sym}^i_{\boldsymbol{Z}}\mathcal{L}$ in $\mathcal{O}_X^{\log}$, where $\mathrm{Sym}^i_{\boldsymbol{Z}}\mathcal{L}$ denotes the i-th symmetric power over $\boldsymbol{Z}$.

Lemma 2.10 (KN, (3.3)). *Let x be a point of X, y a point of $\tau^{-1}(x) \subseteq X^{\log}$ and $(t_i)_{1 \le i \le n}$ a family of elements of the stalk $\mathcal{L}_y$ whose image under* exp *is a $\boldsymbol{Z}$-basis of $(\mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^{*})_x$. Then the $\mathcal{O}_{X,x}$-algebra homomorphism*

$$\mathcal{O}_{X,x}[T_1, \dots, T_n] \xrightarrow{\sim} \mathcal{O}_{X,y}^{\log}; \quad T_i \mapsto t_i$$

is an isomorphism.

Lemma 2.11 (KN, 3.4)). (i) $\mathrm{fil}_0(\mathcal{O}_X^{\log}) = \tau^{-1}\mathcal{O}_X$.

(ii) *The canonical homomorphism $\tau^{-1}\mathcal{M}_X^{\mathrm{gp}} \cong \mathcal{L}/\boldsymbol{Z}(1) \to \mathrm{fil}_1(\mathcal{O}_X^{\log})/\tau^{-1}\mathcal{O}_X$ induces an isomorphism*

$$\tau^{-1}\mathcal{O}_X \otimes_{\boldsymbol{Z}} \tau^{-1}(\mathrm{Sym}^r_{\boldsymbol{Z}}(\mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^{*})) \cong \mathrm{fil}_r(\mathcal{O}_X^{\log})/\mathrm{fil}_{r-1}(\mathcal{O}_X^{\log})$$

for any $r \ge 0$.

3. Logarithmic relative Poincaré lemma

The aim of this section is to prove Theorem A.

PROPOSITION 3.1 (Relative Poincaré lemma). *Let $f : X \to Y$ be a smooth holomorphic map of complex manifolds. Then*

$$f^{-1}\mathcal{O}_Y \to \Omega^{\bullet}_{X/Y}$$

is a quasi-isomorphism.

3.2. Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a morphism of fs log analytic spaces satisfying the following conditions:
Locally on X and on Y,
(i) there exists a chart $P := \boldsymbol{N} \to \mathcal{M}_Y$, and a morphism of monoids

$$P \to Q := \boldsymbol{N}^r; \quad 1 \mapsto (1, \dots, 1),$$

for some $r \geq 1$, and
(ii) X is isomorphic to an open subspace of $Y \times_{\operatorname{Spec} \boldsymbol{C}[P]_{an}} \operatorname{Spec} \boldsymbol{C}[Q]_{an}$, where $\operatorname{Spec} \boldsymbol{C}[P]$ and $\operatorname{Spec} \boldsymbol{C}[Q]$ are endowed with log structures associated to $P \to \boldsymbol{C}[P]$ and $Q \to \boldsymbol{C}[Q]$, respectively.

PROPOSITION 3.3. *Let Q be the monoid $\boldsymbol{N}$ and P the monoid $\boldsymbol{N}^r$ for $r \in \boldsymbol{N}$. We denote i-th basis of P as e_i. Let X be the analytic space $\boldsymbol{C}^r$ and Y the analytic space $\boldsymbol{C}$. Let $(t_1, \dots, t_r)$ (resp. z) be a coordinate of X (resp. Y). Let $f : X \to Y$ be a morphism defined by $(t_1, \dots, t_r) \mapsto t_1 \cdots t_r$. Let α (resp. β) be the morphism of monoids $P \to \Gamma(X, \mathcal{O}_X)$ (resp. $Q \to \Gamma(Y, \mathcal{O}_Y)$) defined by $n \cdot e_i \mapsto t_i^n$ (resp. $n \mapsto z^n$). Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be the associated log analytic spaces, respectively. Then we have an isomorphism of $f^{-1}\mathcal{O}_Y$-modules*

$$\xi_1 : f^{-1}\mathcal{O}_Y \otimes_{\boldsymbol{Z}} \bigwedge^q \frac{(\mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^*)}{f^{-1}(\mathcal{M}_Y^{\mathrm{gp}}/\mathcal{O}_Y^*)} \xrightarrow{\sim} \mathcal{H}^q(\omega^{\bullet}_{X/Y}) \qquad \text{for all } q \geq 0. \tag{3.4}$$

Proof. If $\mathcal{Z}^q \subset \omega^q_{X/Y}$ is a sheaf of sections of cocycles, we have a morphism

$$f^{-1}\mathcal{O}_Y \otimes \bigwedge^q (\mathcal{M}_X^{\mathrm{gp}}/f^{-1}\mathcal{M}_Y^{\mathrm{gp}}) \to \mathcal{Z}^q; \quad a \otimes \wedge_\iota b_i \mapsto a \wedge_\iota d\log(b_i).$$

Let z be a point of X. If $b_i \in \mathcal{O}^*_{X,z}$ for some i, then a branch of $\log b_i$ is in $\mathcal{O}^*_{X,z}$ and hence the image of $a \otimes \wedge_\iota b_i$ is a coboundary. Hence we have a well-defined morphism of sheaves

$$\xi_1 : f^{-1}\mathcal{O}_Y \otimes \bigwedge^q \frac{\mathcal{M}_X^{\mathrm{gp}}/\mathcal{O}_X^*}{f^{-1}(\mathcal{M}_Y^{\mathrm{gp}}/\mathcal{O}_Y^*)} \to \mathcal{H}^q(\omega^{\bullet}_{X/Y}).$$

It is enough to prove that ξ_1 is an isomorphism at each stalks.

CASE 1: Let $x = (t_1, \dots, t_r)$ be a point of X such that $t_1 \cdots t_r \neq 0$. Since its log structure is trivial on a neighborhood of x, the stalk at x of the right hand side of (3.4) is $f^{-1}\mathcal{O}_{Y,x}$ (resp. 0) if $q=0$ (resp. if $q\neq 0$). Hence we obtain the desired isomorphism in this case by 3.1.

CASE 2: Let $x=(0,0,\dots,0)$. We can compute the right hand side of (3.4) as $f^{-1}\mathcal{O}_{Y,x}\otimes_{\mathbf{Z}}\bigwedge^q \mathbf{Z}^r/\mathbf{Z}$. We will prove in three steps that

$$\mathscr{H}^q(\omega^\bullet_{X/Y})_x=\left\{\sum_{1\le i_1<\cdots<i_q\le r} a_{i_1,\dots,i_q} f_{i_1}\wedge\cdots\wedge f_{i_q}; a_{i_1,\dots,i_q}\in f^{-1}\mathcal{O}_{Y,x}\right\}/I,$$

where $f_i=dt_i/t_i$ $(1\le i\le r)$ and I is the submodule generated by $f_1+\cdots+f_r$. We have

$$\omega^1_{X/Y,x}=\frac{\mathcal{O}_{X,x}f_1\oplus\cdots\oplus\mathcal{O}_{X,x}f_r}{\mathcal{O}_{X,x}(f_1\oplus\cdots\oplus f_r)}.$$

We can write an element of $\omega^q_{X/Y,x}$ as

$$\sum_{e_1,\dots,e_r}\left(\sum_{2\le i_1<\cdots<i_q\le r} a^{e_1,\dots,e_r}_{i_1,\dots,i_q} f_{i_1}\wedge\cdots\wedge f_{i_q}\right)t_1^{e_1}\cdots t_r^{e_r},\quad a^{e_1,\dots,e_r}_{i_1,\dots,i_q}\in\mathbf{C}.$$

Let $M^q_{e_1,\dots,e_r}$ be the submodule

$$\left\{\left(\sum_{2\le i_1<\cdots<i_q\le r} a_{i_1,\dots,i_q} f_{i_1}\wedge\cdots\wedge f_{i_q}\right)t_1^{e_1}\cdots t_r^{e_r}; a_{i_1,\dots,i_q}\in\mathbf{C}\right\}$$

of $\omega^q_{X/Y,x}$. Then we have

$$\omega^q_{X/Y,x}=\left\{\phi\in\sum_{e_1,\dots,e_r}M^q_{e_1,\dots,e_r}; \phi \text{ converges}\right\}.$$

STEP 1. For $\phi\in M^q_{e_1,\dots,e_r}$, we will prove that $d\phi\in M^{q+1}_{e_1,\dots,e_r}$.
We write ϕ as follows:

$$\phi=\left(\sum_{2\le i_1<\cdots<i_q\le r} a_{i_1,\dots,i_q} f_{i_1}\wedge\cdots\wedge f_{i_q}\right)t_1^{e_1}\cdots t_r^{e_r}.$$

We have

$$\begin{aligned}
d\phi&=t_1^{e_1}\cdots t_r^{e_r}\sum_{2\le i_1<\cdots<i_q\le r}\ \sum_{j\notin\{i_1,\dots,i_q\}} a_{i_1,\dots,i_q}e_j f_j\wedge f_{i_1}\wedge\cdots\wedge f_{i_q}\\
&=t_1^{e_1}\cdots t_r^{e_r}\sum_{2\le i_1<\cdots<i_q\le r}\ \sum_{j\notin\{1,i_1,\dots,i_q\}} a_{i_1,\dots,i_q}(e_j-e_1) f_j\wedge f_{i_1}\wedge\cdots\wedge f_{i_q}\\
&=t_1^{e_1}\cdots t_r^{e_r}\sum_{2\le i_1<\cdots<i_{q+1}\le r}\left\{\sum_{k=1}^{q+1}(-1)^{k-1}a_{i_1,\dots,\hat{i_k},\dots,i_{q+1}}(e_{i_k}-e_1)\right\}f_{i_1}\wedge\cdots\wedge f_{i_{q+1}}.
\end{aligned}$$

STEP 2. Let $\phi \in M^q_{e_1,\dots,e_r} \cap \ker d$. We will prove that $\phi \notin \operatorname{im} d$ if and only if $e_1 = \cdots = e_r$ and $\phi \neq 0$.

Let

$$\psi = \left(\sum_{2 \le i_1 < \cdots < i_{q-1} \le r} \tilde{a}_{i_1,\dots,i_{q-1}} f_{i_1} \wedge \cdots \wedge f_{i_{q-1}} \right) t_1^{e_1} \cdots t_r^{e_r}$$

be an element of $M^{q-1}_{e_1,\dots,e_r}$. Then $\phi \in \operatorname{im} d$ if and only if there exists a complex vector $(\tilde{a}_{i_1,\dots,i_{q-1}})$ such that $\phi = d\psi$. This is translated as what the simultaneous linear equations in the $\tilde{a}_{i_1,\dots,\hat{i}_k,\dots i_q}$

$$\text{(3.5)} \qquad \begin{cases} \displaystyle\sum_{k=1}^{q} (-1)^{k-1} \tilde{a}_{i_1,\dots,\hat{i}_k,\dots,i_q} (e_{i_k} - e_1) = a_{i_1,\dots,i_q} \\ \qquad\qquad (2 \le i_1 < \cdots < i_q \le r) \end{cases}$$

has a solution. "If part" is clear. In order to prove "only if part", we may assume that $e_1 \neq e_2$ without loss of generality. Let $\tilde{a}$ (resp. a) be the vector $(\tilde{a}_{i_1,\dots,\hat{i}_k,\dots,i_q})$ (resp. $(a_{i_1,\dots,i_q})$) and A the matrix whose entries consist of coefficients of the simultaneous equations (3.5). For $I = \{i_1, \dots, i_q\}$, we call the equation

$$\sum_{k=1}^{q} (-1)^{k-1} \tilde{a}_{i_1,\dots,\hat{i}_k,\dots,i_q} (e_{i_k} - e_1) = a_{i_1,\dots,i_q}$$

as an I-th equation. We call the row of the matrix A corresponding to the I-th equation as I-th row. Then there exists a solution of (3.5) if and only if rank A = rank (A, a). Let I be a set $\{i_1, \dots, i_q\}$ such that $2 < i_1 < \cdots < i_q \le r$. For each $k = 1, \dots, q$, we denote $J_k := \{2\} \cup (I - \{i_k\})$. We will make an elementary transformation as follows. Multiply the I-th row of A by $e_2 - e_1$, and add $(-1)^k (e_{i_k} - e_1)$ times the J_k-th row of A to it for all k. Then we have

$$\begin{aligned} &(I\text{-th equation}) \times (e_2 - e_1) + \sum_{k=1}^{q} ((-1)^k (J_k\text{-th equation})) \times (e_{i_k} - e_1) \\ &= \sum_{k=1}^{q} (-1)^{k-1} \tilde{a}_{i_1,\dots,\hat{i}_k,\dots,i_{q+1}} (e_{i_k} - e_1)(e_2 - e_1) \\ &\quad + \sum_{k=1}^{q} \Bigg\{ (-1)^k \tilde{a}_{i_1,\dots,\hat{i}_k,\dots,i_{q+1}} (e_2 - e_1)(e_{i_k} - e_1) \\ &\quad + \sum_{l=1}^{k-1} (-1)^{l+k} \tilde{a}_{2,i_1,\dots,i_{l-1},\hat{i}_l,i_{l+1},\dots,i_{k-1},\hat{i}_k,i_{k+1},\dots,i_q} (e_{i_l} - e_1)(e_{i_k} - e_1) \\ &\quad + \sum_{l=k+1}^{q} (-1)^{l+k-1} \tilde{a}_{2,i_1,\dots,i_{k-1},\hat{i}_k,i_{k+1},\dots,i_{l-1},\hat{i}_l,i_{l+1},\dots,i_q} (e_{i_l} - e_1)(e_{i_k} - e_1) \Bigg\} \end{aligned}$$

$$= \sum_{1 \le l < k \le q} (-1)^{l+k} \tilde{a}_{2,\iota_1,\ldots,\iota_{l-1},\hat{\iota}_l,\iota_{l+1},\ldots,\iota_{k-1},\hat{\iota}_k,\iota_{k+1},\ldots,\iota_q}(e_{i_l} - e_1)(e_{i_k} - e_1)$$
$$+ \sum_{1 \le l < k \le q} (-1)^{l+k-1} \tilde{a}_{2,\iota_1,\ldots,\iota_{l-1},\hat{\iota}_l,\iota_{l+1},\ldots,\iota_{k-1},\hat{\iota}_k,\iota_{k+1},\ldots,\iota_q}(e_{i_l} - e_1)(e_{i_k} - e_1)$$
$$= 0.$$

This means that, by the above elementary transformation, A is transformed to

$$\left(\begin{array}{ccc|c} e_2 - e_1 & & 0 & \\ & \ddots & & * \\ 0 & & e_2 - e_1 & \\ \hline & 0 & & 0 \end{array}\right).$$

Using the same elementary transformation for the vector a, its I-th row is transformed to

$$\sum_{k=0}^{q} (-1)^k a_{i_0,\ldots,\hat{i}_k,\ldots,i_q}(e_{i_k} - e_1), \quad (i_0 = 2). \tag{3.6}$$

Therefore the condition $\phi \in \ker d$ implies that (3.6) is equal to 0. This means that (A, a) is transformed to

$$\left(\begin{array}{ccc|c|c} e_2 - e_1 & & 0 & & \\ & \ddots & & * & * \\ 0 & & e_2 - e_1 & & \\ \hline & 0 & & 0 & 0 \end{array}\right).$$

Thus we have rank $A =$ rank (A, a) as desired.

STEP 3. Let ϕ be an element of $\omega^q_{X/Y,x} \cap \ker d$ and $\phi^{e_1,\ldots,e_r}$ an element of $M^q_{e_1,\ldots,e_r}$ such that $\sum_{e_1,\ldots,e_r} \phi^{e_1,\ldots,e_r}$ converges in some neighborhood of x and such that this sum is equal to ϕ. We claim that there exists $\psi \in \omega^{q-1}_{X/Y,x}$ such that $d\psi = \phi$ if and only if there exist $\psi^{e_1,\ldots,e_r} \in M^{q-1}_{e_1,\ldots,e_r}$ such that $d\psi^{e_1,\ldots,e_r} = \phi^{e_1,\ldots,e_r}$ for all $(e_1, \ldots, e_r)$.

The "only if part" is clear, hence we will prove the "if part." All what we have to do is to prove $\sum_{e_1,\ldots,e_r} \psi^{e_1,\ldots,e_r}$ also converges in some neighborhood of x. We write $\phi^{e_1,\ldots,e_r}$ (resp. $\psi^{e_1,\ldots,e_r}$) as

$$\left(\sum_{2 \le \iota_1 < \cdots < \iota_q \le r} a^{e_1,\ldots,e_r}_{i_1,\ldots,i_q} f_{i_1} \wedge \cdots \wedge f_{i_q}\right) t_1^{e_1} \cdots t_r^{e_r}$$

$$\left(\text{resp. } \left(\sum_{2 \le \iota_1 < \cdots < \iota_{q-1} \le r} \tilde{a}^{e_1,\ldots,e_r}_{i_1,\ldots,i_{q-1}} f_{i_1} \wedge \cdots \wedge f_{i_{q-1}}\right) t_1^{e_1} \cdots t_r^{e_r}\right).$$

Then it is enough to prove that

$$\sum_{e_1,\dots,e_r} \tilde{a}^{e_1,\dots,e_r}_{i_1,\dots,i_{q-1}} t_1^{e_1}\cdots t_r^{e_r}$$

converges in some neighborhood of x for all $(i_1,\dots,i_{q-1})$. By the argument in Step 2, we can solve the simultaneous linear equations in $\tilde{a}^{e_1,\dots,e_r}_{i_1,\dots,i_{q-1}}$. In fact, assume $\phi \neq 0$ and take an integer k, with $1 \le k \le r$ and $e_k \neq e_1$, then we have, for example, for some $k = k(e_1,\dots,e_r)$ such that $e_k \neq e_1$, we can write as follows:

$$\tilde{a}^{e_1,\dots,e_r}_{i_1,\dots,i_{q-1}} = \begin{cases} (e_k - e_1)^{-1} a^{e_1,\dots,e_r}_{i_1,\dots,k,\dots,i_{q-1}} & k \notin \{i_1,\dots,i_{q-1}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\limsup_{e_1+\cdots+e_r\to\infty} \sqrt[e_1+\cdots+e_r]{|\tilde{a}^{e_1,\dots,e_r}_{i_1,\dots,i_{q-1}}|}$$

is finitely bounded, therefore $\sum \psi^{e_1,\dots,e_r}$ converges in some neighborhood of x.

From Step 2, we have

$$H^q(M^\bullet_{e_1,\dots,e_r}) = \begin{cases} M^q_{e_1,\dots,e_r}, & e_1 = \cdots = e_r, \\ 0, & \text{otherwise.} \end{cases}$$

From Step 1 and Step 3,

$$\mathscr{H}^q(\omega^\bullet_{X/Y})_x = \left\{ \phi = \sum_{e_1,\dots,e_r} \phi^{e_1,\dots,e_r} \in \sum_{e_1,\dots,e_r} H^q(M^\bullet_{e_1,\dots,e_r});\ \phi \text{ converges} \right\}.$$

Thus we have

$$\mathscr{H}^q(\omega^\bullet_{X/Y})_x = \left\{ \sum_{1\le i_1<\cdots<i_q\le r} a_{i_1\cdots i_q} f_{i_1}\wedge\cdots\wedge f_{i_q};\ a_{i_1,\dots,i_q} \in f^{-1}\mathcal{O}_{Y,x} \right\} \Big/ I$$

as desired.

CASE 3: Let x be the point $(0,0,\dots,0,t_{k+1},\dots,t_r)$ such that $t_{k+1}\cdots t_r \neq 0$ $(k < r-1)$, then $x \in X$ has an affine open neighborhood

$$\operatorname{Spec} \boldsymbol{C}[z,t_1,\dots,t_k,t_{k+1}^{\pm 1},\dots,t_r^{\pm 1}]/(z - t_1\cdots t_r).$$

We change coordinate t_1 by $T = t_1 t_{k+1}\cdots t_r$. This change of coordinate induces an isomorphism of fs log analytic spaces. We denote by X' the resulting open set:

$$X' = (\operatorname{Spec} \boldsymbol{C}[z,T,t_2,\dots,t_k,t_{k+1}^{\pm 1},\dots,t_r^{\pm 1}]/(z - Tt_2\cdots t_k)).$$

Then f becomes

$$f' : X' \to Y;\quad (z,T,t_2,\dots,t_r) \mapsto Tt_2\cdots t_k.$$

Now we will compute the cohomology $\mathscr{H}^q(\omega^\bullet_{X'/Y})_x$. We have

$$\omega^1_{X'/Y,x} = \frac{\mathcal{O}_{X,x}f'_1 \oplus \cdots \oplus \mathcal{O}_{X,x}f'_r}{\mathcal{O}_{X,x}(f'_1 + \cdots + f'_k)},$$

where $f'_1 = dT/T$, $f'_i = dt_i/t_i$ for $i > 1$. Similarly as Case 2, we define submodules $M'^q_{e_1,\dots,e_r}$ of $\omega^q_{X'/Y,x}$. For

$$\phi' = \left(\sum_{2 \le \iota_1 < \cdots < \iota_q \le r} a_{\iota_1,\dots,\iota_q} f'_{\iota_1} \wedge \cdots \wedge f'_{\iota_q} \right) t_1^{e_1} \cdots t_r^{e_r} \in M'^q_{e_1,\dots,e_r},$$

we have

$$d\phi' = t_1^{e_1} \cdots t_r^{e_r} \left(\sum_{2 \le \iota_1 < \cdots < \iota_q \le r} \left(\sum_{\substack{j=1 \\ j \notin \{1,\iota_1,\dots,\iota_q\}}}^{k} a_{\iota_1,\dots,\iota_q}(e_j - e_1) f'_j \wedge f'_{\iota_1} \wedge \cdots \wedge f'_{\iota_q} \right.\right.$$
$$\left.\left. + \sum_{\substack{j=k+1 \\ j \notin \{1,\iota_1,\dots,\iota_q\}}}^{r} a_{\iota_1,\dots,\iota_q} e_j f'_j \wedge f'_{\iota_1} \wedge \cdots \wedge f'_{\iota_q} \right)\right).$$

By a similar argument in Case 2, we can show that $\mathscr{H}^q(\omega^\bullet_{X/Y})_x$ is isomorphic to the stalk of the right hand side of (3.4) at x. □

THEOREM A (Log relative Poincaré lemma). *Let $f: X \to Y$ be a morphism of fs log analytic spaces satisfying* 3.2. *Let $\omega^{\bullet\log}_{X/Y}$ be $\sigma^*\omega^\bullet_{X/Y}$, here $\sigma: X^{\log} \to X$ is the canonical map. Then the canonical morphism*

$$(f^{\log})^{-1}\mathcal{O}^{\log}_Y \to \omega^{\bullet\log}_{X/Y}$$

is a quasi-isomorphism.

Proof. Let $(P \to \mathcal{M}_X, Q \to \mathcal{M}_Y, Q \to P)$ be a chart of the morphism f. Let S (resp. T) be Spec $\boldsymbol{C}[P]_{\text{an}}$ (resp. Spec $\boldsymbol{C}[Q]_{\text{an}}$). The question being local, we may assume $X \cong Y \times_T S$. Since $\omega^{\bullet\log}_{X/Y} \cong \omega^{\bullet\log}_{S/T} \otimes_{\mathcal{O}_S} \mathcal{O}_X \cong \omega^{\bullet\log}_{S/T} \otimes_{\mathcal{O}_T} \mathcal{O}_Y$, we have

$$\mathscr{H}^q(\omega^{\bullet\log}_{X/Y}) \cong \mathscr{H}^q(\omega^{\bullet\log}_{S/T}) \otimes_{\mathcal{O}_T} \mathcal{O}_Y.$$

On the other hand, as $\mathcal{O}^{\log}_Y \cong \mathcal{O}^{\log}_T \otimes_{\mathcal{O}_T} \mathcal{O}_Y$, we may assume that $X = S$ and $Y = T$. Let $x' \in X^{\log}$, $x = \sigma(x') \in X$ and $y = f(x) \in Y$. Let $(t_i)_{1 \le i \le r}$ be a family of elements of $\mathcal{M}^{\text{gp}}_{X,x}$ whose classes in $\mathcal{M}^{\text{gp}}_{X,x}/\mathcal{O}^*_{X,x}$ is a basis of $\mathcal{M}^{\text{gp}}_{X,x}/\mathcal{O}^*_{X,x}$ over $\boldsymbol{Z}$ and u an element of $\mathcal{M}^{\text{gp}}_{Y,y}$ whose class in $\mathcal{M}^{\text{gp}}_{Y,y}/\mathcal{O}^*_{Y,y}$ is a basis of $\mathcal{M}^{\text{gp}}_{Y,y}/\mathcal{O}^*_{Y,y}$ over $\boldsymbol{Z}$.

STEP 1. Let A (resp. B) be the polynomial ring $\mathcal{O}_{Y,y}[z]$ (resp. $\mathcal{O}_{Y,y}[T_1,\dots,T_r]$). We define a morphism of $\mathcal{O}_{Y,y}$-algebras by

$$A \to B; \quad z \mapsto T_1 + \cdots + T_r$$

and a morphism ϕ of complexes of $\mathcal{O}_{Y,y}$-modules by

$$\phi : \Omega^\bullet_{B/A} \to \omega^{\bullet \log}_{X/Y,x}; \quad dT_i \mapsto \frac{dt_i}{t_i}.$$

Let $A \to \Omega^\bullet_{B/A}$ be the canonical morphism of complexes. Then the diagram

$$\begin{array}{ccc} A & \longrightarrow & \omega^{\bullet \log}_{X/Y,x} \\ \downarrow & & \| \\ \Omega^\bullet_{B/A} & \longrightarrow & \omega^{\bullet \log}_{X/Y,x} \end{array}$$

is commutative.

STEP 2. The morphism $A \to \Omega^\bullet_{B/A}$ is a quasi-isomorphism. This is well known.

STEP 3. We define increasing filtrations F of $\Omega^\bullet_{B/A}$ and G of $\omega^{\bullet \log}_{X/Y,x}$ by

$$F_k(\Omega^q_{B/A}) := \{\Sigma f\eta;\ f \in B,\ \deg f \le k,\ \eta = dT_{i_1} \wedge \cdots \wedge dT_{i_q} \in \Omega^q_{B/A}(i_1 < \cdots < i_q)\},$$

$$G_k(\omega^{q \log}_{X/Y}) := \text{the image of } \mathrm{fil}_k(\mathcal{O}^{\log}_X) \otimes \omega^q_{X/Y} \text{ in } \omega^{q \log}_{X/Y}.$$

Here fil is the filtration introduced after 2.9. Since ϕ respects to filtrations F and G, it induces $\mathrm{Gr}(\phi) : \mathrm{Gr}^F_k(\Omega^\bullet_{B/A}) \to \mathrm{Gr}^G_k(\Omega^{\bullet \log}_{X/Y})_x$. We claim that $\mathrm{Gr}(\phi)$ is a quasi-isomorphism.

(From Step 3, $\phi : \Omega^\bullet_{B/A} \to \omega^{\bullet \log}_{X/Y,x}$ is a quasi-isomorphism, hence $A \to \omega^{\bullet \log}_{X/Y,x}$ is a quasi-isomorphism.)

Now we prove Step 3. By 2.11, there is the canonical isomorphism of complexes

$$\psi : \mathrm{Gr}^G_k(\omega^{\bullet \log}_{X/Y}) \cong \sigma^{-1}(\mathrm{Sym}^k_{\mathbf{Z}}(\mathcal{M}^{\mathrm{gp}}_X/\mathcal{O}^*_X)) \otimes_{\mathbf{Z}} \sigma^{-1}\omega^\bullet_{X/Y}.$$

Let

$$\xi_2 : H^q(\mathrm{Gr}^F_k \Omega^\bullet_{B/A}) \xrightarrow{\sim} \left(f^{-1}\mathcal{O}_Y \otimes \bigwedge^q \frac{\mathcal{M}^{\mathrm{gp}}_X/\mathcal{O}^*_X}{f^{-1}(\mathcal{M}^{\mathrm{gp}}_Y/\mathcal{O}^*_Y)} \otimes \mathrm{Sym}^k_{\mathbf{Z}}(\mathcal{M}^{\mathrm{gp}}_X/\mathcal{O}^*_X)\right)_y$$

be the natural isomorphism. Let ξ_1 be a morphism as in 3.3. Put $\xi = (\xi_1 \otimes \mathrm{id}) \circ \xi_2$. Then ξ makes the following diagram commutative:

$$\begin{array}{ccc} H^q(\mathrm{Gr}^F_k \Omega^\bullet_{B/A}) & \xrightarrow{\xi} & \mathcal{H}^q(\omega^\bullet_{X/Y})_x \otimes \mathrm{Sym}^k_{\mathbf{Z}}(\mathcal{M}^{\mathrm{gp}}_X/\mathcal{O}^*_X)_x \\ {\scriptstyle \mathrm{Gr}(\phi)}\downarrow & & \| \\ \mathcal{H}^q(\mathrm{Gr}^G_k \omega^{\bullet \log}_{X/Y})_x & \longrightarrow & \mathcal{H}^q(\omega^\bullet_{X/Y})_x \otimes \mathrm{Sym}^k_{\mathbf{Z}}(\mathcal{M}^{\mathrm{gp}}_X/\mathcal{O}^*_X)_x \end{array}$$

Thus we have $\mathrm{Gr}(\phi)$ is a quasi-isomorphism as desired. □

4. The morphism $\tau^* Rf_*\omega^\bullet_{X/Y} \to Rf_*{}^{\log}\mathbf{C} \otimes \mathcal{O}^{\log}_Y$

Let $f : X \to Y$ be a proper smooth morphism of complex manifolds. Then we have a quai-isomorphism $Rf_*\mathbf{C} \otimes_{\mathbf{C}} \mathcal{O}_Y \xrightarrow{\sim} Rf_*\Omega^\bullet_{X/Y}$. We construct a similar quasi-isomorphism on fs log analytic spaces satisfying 3.2.

LEMMA 4.1 (Proper base change theorem). *Let X, Y, Z, W be locally compact Hausdorff topological spaces and $f : X \to Y$, $g : Z \to W$, $\sigma : X \to Z$, $\tau : Y \to W$ continuous maps such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \sigma\downarrow & & \downarrow\tau \\ Z & \xrightarrow{g} & W \end{array}$$

is cartesian. Assume that g is proper (i.e., an inverse image of a compact set is compact). Then for any complex $K^\bullet$ of sheaves of abelian groups on Z, we have a quasi-isomorphism

$$\tau^{-1}Rg_*K^\bullet \xrightarrow{\sim} Rf_*(\sigma^{-1}K^\bullet).$$

See [SGA, p. 39].

LEMMA 4.2. *Let $f : X \to Y$ be a proper continuous map of locally compact Hausdorff topological spaces, $\mathscr{A}$ a sheaf of rings on Y, $\mathscr{F}$ a sheaf of $(f^{-1}\mathscr{A})$-modules on X and $\mathscr{G}$ a sheaf of $\mathscr{A}$-modules such that $\mathscr{G}_y$ is a free $\mathscr{A}_y$-module for each $y \in Y$. Then we have a quasi-isomorphism*

$$Rf_*\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G} \xrightarrow{\sim} Rf_*(\mathscr{F} \otimes_{f^{-1}\mathscr{A}} f^{-1}\mathscr{G}).$$

Proof. First notice that, using 4.1, we may assume that Y is a point. Hence it is enough to prove that

$$\oplus_I H^m(X, \mathscr{F}) \to H^m(X, \oplus_I \mathscr{F})$$

is an isomorphism for all m. If I is a finite set, it is clear. If I is an infinite set, use [Ive, p. 173, Theorem 5.1]. □

4.3. Let X, Y, f be as in Theorem A in section 3 and moreover, assume that f is proper. Let $\mathring{X}$ (resp. $\mathring{Y}$) be the underlying analytic space of X (resp. Y). We have the canonical commutative diagram of topological spaces

$$\begin{array}{ccc} X^{\log} & \xrightarrow{f^{\log}} & Y^{\log} \\ \sigma\downarrow & & \downarrow\tau \\ \mathring{X} & \xrightarrow{\mathring{f}} & \mathring{Y}. \end{array}$$

Let X' be a log analytic space $(\mathring{X}, f^*\mathscr{M}_Y)$. Let $g : X'^{\log} \to Y^{\log}$, $h : X^{\log} \to X'^{\log}$, $\tilde{\sigma} : X'^{\log} \to \mathring{X}$ be the canonical maps, respectively.

$$\begin{array}{ccccc} X^{\log} & \xrightarrow{h} & X'^{\log} & \xrightarrow{g} & Y^{\log} \\ {\scriptstyle\sigma}\downarrow & & {\scriptstyle\tilde{\sigma}}\downarrow & & {\scriptstyle\tau}\downarrow \\ \mathring{X} & = & \mathring{X} & \xrightarrow{f} & \mathring{Y} \end{array}$$

4.4. Let $\mathscr{F}$ be a locally free $\mathcal{O}_X$-module of finite rank. From the natural morphism $\mathscr{F} \to \sigma_*\sigma^*\mathscr{F} \cong \tilde{\sigma}_*h_*\sigma^*\mathscr{F}$, we have an associated morphism $\tilde{\sigma}^*\mathscr{F} \to Rh_*\sigma^*\mathscr{F}$.

LEMMA 4.5. *$\tilde{\sigma}^*\mathscr{F} \to Rh_*\sigma^*\mathscr{F}$ is a quasi-isomorphism.*

Proof. Since taking a cohomology commutes with taking a direct sum, our task is now to show $\tilde{\sigma}^*\mathcal{O}_X \to Rh_*\sigma^*\mathcal{O}_X$ is a quasi-isomorphism. This is equivalent to show that $\mathcal{O}_{X'}^{\log} \to Rh_*\mathcal{O}_X^{\log}$ is a quasi-isomorphism. Let x be a point of $X'^{\log}$. Since h is proper, we have

$$(R^i h_* \mathcal{O}_X^{\log})_x = \boldsymbol{H}^i(h^{-1}(x), \mathcal{O}_X^{\log}|h^{-1}(x)).$$

Let r be $\mathrm{rank}_{\boldsymbol{Z}}\ (\mathscr{M}_{X,x}^{\mathrm{gp}}/\mathcal{O}_{X,x}^*)$. Then we have $h^{-1}(x) \cong (\boldsymbol{S}^1)^{r-1}$. Let X_k be the log analytic space whose base space is $\mathring{X}$ and whose log structure is locally defined by the chart

$$\boldsymbol{N}^k \to \Gamma(X_k, \mathcal{O}_{X_k}) = \Gamma(X, \mathcal{O}_X); \quad (e_1, \dots, e_k) \mapsto t_1^{e_1} \cdots t_{k-1}^{e_{k-1}} t_k^{e_k} \cdots t_r^{e_k}.$$

(Hence we have $X_r = X$ and $X_1 = X'$.) For $1 \le k \le r-1$, let $\psi_k : X_{k+1} \to X_k$ be the morphism of log analytic spaces, that is defined by the morphism of monoids

$$\boldsymbol{N}^k \to \boldsymbol{N}^{k+1}; \quad (e_1, \dots, e_k) \mapsto (e_1, \dots, e_k, e_k).$$

Let $h_k : X_{k+1}^{\log} \to X_k^{\log}$ be the associated morphism to ψ_k. Then $h_k^{-1}(x) \cong \boldsymbol{S}^1$. Therefore, in order to show being a quasi-isomorphism $\mathcal{O}_{X'}^{\log} \xrightarrow{\sim} Rh_*\mathcal{O}_X^{\log}$, it is enough to prove that the following morphism are quasi-isomorphisms:

$$\mathcal{O}_{X_k}^{\log} \xrightarrow{\sim} Rh_{k*}\mathcal{O}_{X_{k+1}}^{\log} \quad (1 \le k \le r-1).$$

We will prove it only in the case $r = 2$, since the argument is the same as in the general case. Set $h = h_1$. We compute the cohomology of $h^{-1}(x) = \boldsymbol{S}^1$ with coefficients in $\mathcal{O}_X^{\log}|h^{-1}(x)$ in the Čech method. We define a coordinate θ on $\boldsymbol{S}^1$ defined by

$$\boldsymbol{S}^1 = \{\exp(\sqrt{-1}\theta)\ ;\ \theta \in \boldsymbol{R}\}.$$

Let $\{U_1, U_2\}$ be an open covering of $\boldsymbol{S}^1$ defined by

$$U_1 = \{\exp(\sqrt{-1}\theta);\ 0 < \theta < 2\pi\},$$
$$U_2 = \{\exp(\sqrt{-1}\theta)\ ;\ \pi < \theta < 3\pi\}.$$

Let V_1 (resp. V_2) be an open set $\{\exp(\sqrt{-1}\theta)\ ;\ 0 < \theta < \pi\}$ (resp. $\{\exp(\sqrt{-1}\theta);\ \pi < \theta < 2\pi\}$) of $\boldsymbol{S}^1$. If V be the intersection of U_1 and U_2, then V is a disjoint union of V_1 and V_2. Since $\mathcal{O}_X^{\log}|U_1, \mathcal{O}_X^{\log}|U_2$ and $\mathcal{O}_X^{\log}|V$ are constant sheaves, we have

$$\boldsymbol{H}^k(U_i, \mathcal{O}_X^{\log}|U_i) = \boldsymbol{H}^k(V, \mathcal{O}_X^{\log}|V) = 0$$

for $k > 0$. Hence, we can compute the Čech cohomology of $\mathcal{O}_X^{\log}|h^{-1}(x)$ by the open covering $\{U_1, U_2\}$. Let $y = \tilde{\sigma}(x) \in X$. We denote the restriction of $\mathcal{O}_X^{\log}$ to U_1 (resp. U_2) by $\mathcal{O}_X^{\log}|U_1 = \mathcal{O}_{X,y}[T_1, T_2]$ (resp. $\mathcal{O}_X^{\log}|U_2 = \mathcal{O}_{X,y}[T_1', T_2']$) where T_i, T_i' are variables such that the difference of T_i and T_i' on V is in $2\pi\sqrt{-1}\boldsymbol{Z}$. From the assumption of f, we have $T_1 + T_2 = T_1' + T_2'$. Therefore we may assume that

$$T_1' = T_1 + 2\pi\sqrt{-1}, \quad T_2' = T_2 - 2\pi\sqrt{-1}.$$

Thus we have the following Čech complex $C^\bullet$

$$C^0 = \mathcal{O}_{X,y}[T_1, T_2] \oplus \mathcal{O}_{X,y}[T_1', T_2'],$$
$$C^1 = \mathcal{O}_{X,y}[T_1, T_2] \oplus \mathcal{O}_{X,y}[T_1, T_2],$$
$$C^i = 0 \quad (i \geq 2),$$
$$d : C^0 \to C^1; (p(T_1, T_2), q(T_1', T_2'))$$
$$\mapsto (p(T_1, T_2) - q(T_1, T_2), p(T_1, T_2) - q(T_1 + 2\pi\sqrt{-1}, T_2 - 2\pi\sqrt{-1})).$$

Hence we have

$$\begin{aligned}\boldsymbol{H}^0(C^\bullet) &= \ker d \\ &= \{p(T_1, T_2) \in \mathcal{O}_{X,y}[T_1, T_2];\ p(T_1, T_2) = p(T_1 + 2\pi\sqrt{-1}, T_2 - 2\pi\sqrt{-1})\} \\ &= \mathcal{O}_{X,y}[T_1 + T_2] \cong \mathcal{O}_{X',x}^{\log}.\end{aligned}$$

It is clear that $\boldsymbol{H}^1(C^\bullet) = 0$. This completes the proof. □

Similarly, we have the following proposition.

PROPOSITION 4.6. *Let X be an fs log analytic space, $\mathcal{F}$ a locally free $\mathcal{O}_X$-module of finite rank and $\tau : X^{\log} \to X$ the canonical continuous map. Then we have a quasi-isomorphism*

$$\mathcal{F} \xrightarrow{\sim} R\tau_*\tau^*\mathcal{F}.$$

Hence

$$\tau_*\mathcal{O}_X^{\log} \cong \mathcal{O}_X, \ \textit{and}\ R^i\tau_*\mathcal{O}_X^{\log} = 0, \quad \textit{for } i \geq 1.$$

PROPOSITION 4.7. *Let X, Y, f, τ be as in* 4.3. *We have a quasi-isomorphism*

$$\tau^* Rf_* \omega^\bullet_{X/Y} \xrightarrow{\sim} Rf^{\log}_* \omega^{\bullet\log}_{X/Y}.$$

Proof. We have the notation in 4.3. From 2.7 (iii), the following diagram of topological spaces is catesian.

$$\begin{array}{ccc} X'^{\log} & \xrightarrow{g} & Y^{\log} \\ {\scriptstyle\tilde{\sigma}}\Big\downarrow & & \Big\downarrow{\scriptstyle\tau} \\ \mathring{X} & \xrightarrow{f} & \mathring{Y} \end{array}$$

By 4.1, we have a quasi-isomorphism $\tau^{-1} Rf_* \omega^\bullet_{X/Y} \xrightarrow{\sim} Rg_* \tilde{\sigma}^{-1} \omega^\bullet_{X/Y}$. Thus using 4.2, we have quasi-isomorphisms

$$\begin{aligned} \tau^* Rf_* \omega^\bullet_{X/Y} &= (\tau^{-1} Rf_* \omega^\bullet_{X/Y}) \otimes_{\tau^{-1}\mathcal{O}_Y} \mathcal{O}^{\log}_Y \\ &\xrightarrow{\sim} Rg_*(\tilde{\sigma}^{-1} \omega^\bullet_{X/Y}) \otimes_{\tau^{-1}\mathcal{O}_Y} \mathcal{O}^{\log}_Y \\ &\xrightarrow{\sim} Rg_*(\tilde{\sigma}^{-1} \omega^\bullet_{X/Y} \otimes_{(\tau g)^{-1}\mathcal{O}_Y} g^{-1} \mathcal{O}_Y). \end{aligned}$$

Since $\mathcal{O}^{\log}_{X'} \cong \tilde{\sigma}^{-1} \mathcal{O}_X \otimes_{(\tau g)^{-1}} g^{-1} \mathcal{O}^{\log}_Y$, we have a quasi-isomorphism

$$\tau^* Rf_* \omega^\bullet_{X/Y} \xrightarrow{\sim} Rg_* \tilde{\sigma}^* \omega^\bullet_{X/Y}. \tag{4.8}$$

By 4.5, we have a quasi-isomorphism $\tilde{\sigma}^* \omega^\bullet_{X/Y} \xrightarrow{\sim} Rh_* \sigma^* \omega^\bullet_{X/Y}$. Since $f^{\log} = gh$, we obtain quasi-isomorphisms

$$Rg_* \tilde{\sigma}^* \omega^\bullet_{X/Y} \xrightarrow{\sim} Rg_* Rh_* \sigma^* \omega^\bullet_{X/Y} \tag{4.9}$$

$$\xrightarrow{\sim} Rf^{\log}_* \sigma^* \omega^\bullet_{X/Y}. \tag{4.10}$$

By 4.8–4.10, we obtain the desired quasi-isomorphism. □

THEOREM B. *Let $f : X \to Y$ be a proper morphism of fs log analytic spaces that satisfies* 3.2. *Then we have a quasi-isomorphism*

$$Rf^{\log}_* \boldsymbol{C} \otimes_{\boldsymbol{C}} \mathcal{O}^{\log}_Y \cong \tau^* Rf_* \omega^\bullet_{X/Y}.$$

Proof. By Theorem A, we have a quasi-isomorphism

$$\boldsymbol{C} \otimes_{\boldsymbol{C}} f^{\log -1} \mathcal{O}^{\log}_Y \xrightarrow{\sim} \omega^{\bullet\log}_{X/Y}.$$

Hence, using 4.2, we have quasi-isomorphisms

$$\begin{aligned} Rf^{\log}_* \boldsymbol{C} \otimes_{\boldsymbol{C}} \mathcal{O}^{\log}_Y &\xrightarrow{\sim} Rf^{\log}_* (\boldsymbol{C} \otimes_{\boldsymbol{C}} f^{\log -1} \mathcal{O}^{\log}_Y) \\ &\xrightarrow{\sim} Rf^{\log}_* (\omega^{\bullet\log}_{X/Y}). \end{aligned}$$

Theorem B follows from 4.7. □

5. Log Hodge structures

The aim of this section is to prove Theorem C. A log Hodge structure in Theorem C is a log geometric interpretation of object called a limit mixed Hodge structure in [St1].

Let X be an fs log analytic space. For $x \in X$, let $\mathscr{Y}_x$ (resp. $\overline{\mathscr{Y}}_x$) be the set of all homomorphisms $\mathscr{M}^{gp}_{X,x} \to \boldsymbol{R}_{>0}$ (resp. $\mathscr{M}_{X,x} \to \boldsymbol{R}_{\geq 0}$) which are extensions of

$$\mathscr{O}^*_{X,x} \to \boldsymbol{R}_{>0}; \quad f \mapsto |f(x)|.$$

We introduce on $\mathscr{Y}_x$ (resp. $\overline{\mathscr{Y}}_x$) the topology in the following way. If $a_1, \dots, a_r$ are elements of $\mathscr{M}^{gp}_{X,x}$ (resp. $\mathscr{M}_{X,x}$) whose classes in $\mathscr{M}^{gp}_{X,x}/\mathscr{O}^*_{X,x}$ (resp. $\mathscr{M}_{X,x}/\mathscr{O}^*_{X,x}$) generate $\mathscr{M}^{gp}_{X,x}/\mathscr{O}^*_{X,x}$ as a group (resp. $\mathscr{M}_{X,x}/\mathscr{O}^*_{X,x}$ as a monoid), $\mathscr{Y}_x$ (resp. $\overline{\mathscr{Y}}_x$) has the topology as a subspace of $(\boldsymbol{R}_{>0})^r$ (resp. $(\boldsymbol{R}_{\geq 0})^r$) in which $\mathscr{Y}_x$ (resp. $\overline{\mathscr{Y}}_x$) is embedded by $\psi \to (\psi(a_i))_{1 \leq i \leq r}$.

We regard $\mathscr{Y}_x$ as a subspace of $\overline{\mathscr{Y}}_x$ by the fact that a homomorphism $\mathscr{M}^{gp}_{X,x} \to \boldsymbol{R}_{>0}$ is uniquely induced to one $\mathscr{M}_{X,x} \to \mathscr{M}^{gp}_{X,x} \to \boldsymbol{R}_{>0} \hookrightarrow \boldsymbol{R}_{\geq 0}$. Let ξ_x be the element of $\overline{\mathscr{Y}}_x$ that sends $\mathscr{M}^{gp}_{X,x} - \mathscr{O}^*_{X,x}$ to $0 \in \boldsymbol{R}_{\geq 0}$.

Lemma 5.1 (K. Kato). *Let $y \in X^{\log}$ and let $x = \tau(y) \in X$. Let $\mathscr{Y}_{x,y}$ be the set of homomorphisms $\phi : \mathscr{O}^{\log}_{X,y} \to \boldsymbol{C}$ having the following properties*:

(1) *ϕ is an extension of $\mathscr{O}_{X,x} \to \boldsymbol{C}$; $f \mapsto f(x)$.*

(2) *The composite $\mathscr{L}_y \hookrightarrow \mathscr{O}^{\log}_{X,y} \xrightarrow{\phi} \boldsymbol{C} \to \boldsymbol{C}/\boldsymbol{R} = \boldsymbol{R}\sqrt{-1}$ coincides with θ_y in 2.8. Then there exists a unique bijection*

$$\mathscr{Y}_x \to \mathscr{Y}_{x,y}; \quad \psi \mapsto \psi_y$$

satisfying

$$\psi(\exp(a)) = |\exp(\psi_y(a))|, \quad \textit{for } a \in \mathscr{L}_y.$$

Proof. Let $\eta = \mathrm{Re}(\psi_y)$. Then ψ_y is uniquely determined by η. Let $t_1, \dots, t_r$ be a family of elements of $\mathscr{L}_y$ whose image under exp is $\boldsymbol{Z}$-basis of $\mathscr{M}^{gp}_{X,x}/\mathscr{O}^*_{X,x}$. η (resp. ψ) is uniquely determined by the image of $t_1, \dots, t_r$ (resp. $\exp(t_1), \dots, \exp(t_r)$). Put $\eta(t_i) = \log(\psi(\exp(t_i)))$. Then we have the desired bijection. □

We assume that the fs log analytic space X satisfies the following condition:

5.2. Locally on X, there is an fs monoid P and an ideal Σ of P such that X is an open subspace of $(\mathrm{Spec}\ \boldsymbol{C}[P]/\Sigma)_{an}$ that is endowed with the log structure associated to $P \to \boldsymbol{C}[P]/\Sigma$.

Definition 5.3 (K. Kato). Let X be an fs log analytic space satisfying the condition 5.2. For $n \in \boldsymbol{Z}$, a log Hodge structure (log HS) $\mathscr{H}$ on X of weight n is a triplet $(\mathscr{H}_{\boldsymbol{Q}}, \mathscr{H}_{\mathscr{O}}, \iota_{\mathscr{H}})$ consisting of

• a sheaf of $\boldsymbol{Q}$-modules $\mathscr{H}_{\boldsymbol{Q}}$ on $X^{\log}$,

• a sheaf of $\mathcal{O}_X$-modules $\mathscr{H}_{\mathcal{O}}$ on X endowed with a descending filtration $(F^i\mathscr{H}_{\mathcal{O}})_{i\in \boldsymbol{Z}}$ and with an integrable connection

$$\nabla : \mathscr{H}_{\mathcal{O}} \to \omega_X^1 \otimes_{\mathcal{O}_X} \mathscr{H}_{\mathcal{O}},$$

• an isomorphism of $\mathcal{O}_X^{\log}$-modules

$$\iota_{\mathscr{H}} : \mathscr{H}_{\boldsymbol{Q}} \otimes_{\boldsymbol{Q}} \mathcal{O}_X^{\log} \cong \tau^* \mathscr{H}_{\mathcal{O}},$$

that satisfy the following conditions 5.4–5.9:

5.4. $\mathscr{H}_{\boldsymbol{Q}}$ is locally constant, and each stalk is free of finite rank as a $\boldsymbol{Q}$-module.

5.5. $\mathscr{H}_{\mathcal{O}}$ is locally free of finite rank as an $\mathcal{O}_X$-module.

5.6. $F^i\mathscr{H}_{\mathcal{O}} = \mathscr{H}_{\mathcal{O}}$ if $i \ll 0$, $F^i\mathscr{H}_{\mathcal{O}} = 0$ if $i \gg 0$.

5.7. Each $F^i\mathscr{H}_{\mathcal{O}}$ is an $\mathcal{O}_X$-submodule of $\mathscr{H}_{\mathcal{O}}$, and is locally an $\mathcal{O}_X$-direct summand of $\mathscr{H}_{\mathcal{O}}$.

5.8. $\nabla(F^i\mathscr{H}_{\mathcal{O}}) \subset \omega_X^1 \otimes_{\mathcal{O}_X} F^{i-1}\mathscr{H}_{\mathcal{O}}$ for each i.

5.9. Let $x \in X$. Then there exists an open neighborhood V of ξ_x in $\bar{\mathscr{Y}}_x$ such that for any $y \in \tau^{-1}(x)$ and $\psi \in \mathscr{Y}_x \cap V$, $\mathscr{H}_{\boldsymbol{Q},y}$ with the filtration $\boldsymbol{C} \otimes_{\mathcal{O}_{X,x}} F^i\mathscr{H}_{\mathcal{O},x}$ on $\boldsymbol{C} \otimes \mathscr{H}_{\boldsymbol{Q},y} = \boldsymbol{C} \otimes_{\mathcal{O}_{X,x}} \mathscr{H}_{\mathcal{O},x}$, the equality given by ψ_y, is a Hodge structure of weight n in the classical sense. Here $\mathcal{O}_{X,x} \to \boldsymbol{C}$ is $f \mapsto f(x)$.

Let $Y = \{z \in \boldsymbol{C} |\ |z| < 1\}$ be the unit disk with the log structure defined by the origin, Y^* the punctured disk. Via the mapping $U \to Y^*$; $u \mapsto \exp(2\pi i\, u)$, the upper half plane $U = \{u \in \boldsymbol{C} \mid \operatorname{Im}(u) > 0\}$ becomes the universal covering of Y^*. The fundamental group $\pi_1(Y^*) = \pi_1(Y^{\log})$ is generated by the translation $u \mapsto u + 1$. Consider the subsheaf $\boldsymbol{Q}[u] \subset \mathcal{O}_Y^{\log}$. Let σ be the monodromy of $\boldsymbol{Q}[u]$ around the origin. Then we have $\sigma : u \mapsto u - 1$ and $\log \sigma = -d/du$.

LEMMA 5.10. *Let V be a $\boldsymbol{Q}$-vector space, $N : V \to V$ nilpotent homomorphism and $\boldsymbol{Q}[u]$ a polynomial ring in one variable over $\boldsymbol{Q}$. We define the endomorphism Δ of $V \otimes_{\boldsymbol{Q}} \boldsymbol{Q}[u]$ to be $N \otimes 1 - 1 \otimes d/du$. Then*

$$\ker \Delta = W := \left\{ \sum_{m=0}^{\infty} \frac{N^m(x)}{m!} \otimes u^m;\ x \in V \right\}.$$

Proof. Let f be an element of $V \otimes \boldsymbol{Q}[u]$. We can write $f = \sum_{i=0}^{m} x_i \otimes u^i$, $x_i \in V$. Then we have

$$\Delta(f) = \sum_{i=0}^{m-1} \{Nx_i - (i+1)x_{i+1}\} \otimes u^i + Nx_m \otimes u^m.$$

Hence, $f \in \ker \Delta$ implies $x_i = N^i(x_0)/i!$ for $(i \geq 1)$, therefore $f \in W$. It is clear that $W \subset \ker \Delta$. □

LEMMA 5.11. *Let X be the analytic space* Spec $\boldsymbol{C}_{\mathrm{an}}$ *endowed with the log structure associated to $\boldsymbol{N} \to \boldsymbol{C}$; $n \mapsto 0^n$ and $\mathscr{F}$ a locally constant sheaf of $\boldsymbol{Q}$-vector spaces on the topological space $X^{\log}$. Let t be a section of the sheaf of monoids on X associated to its log structure such that t is an image of $1 \in \boldsymbol{N}$, and consider the subsheaf $\boldsymbol{Q}[u] \subset \mathcal{O}_X^{\log}$ where $u = (2\pi\sqrt{-1})^{-1} \log t$. Let N be the logarithm of the monodromy of $\mathscr{F}$. Assume N is nilpotent. Then the restriction map of the sheaf $\mathscr{F} \otimes_{\boldsymbol{Q}} \boldsymbol{Q}[u]$*

$$\Gamma(X^{\log}, \mathscr{F} \otimes \boldsymbol{Q}[u]) \to \mathscr{F}_\alpha \otimes \boldsymbol{Q}[u], \quad (\alpha \in X^{\log})$$

factors through the submodule

$$\exp(uN)\mathscr{F}_\alpha := \left\{ \sum_{n=0}^{\infty} \frac{N^n(x)}{n!} \otimes u^n \ ; \ x \in \mathscr{F}_\alpha \right\} \subset \mathscr{F}_\alpha \otimes \boldsymbol{Q}[u],$$

and moreover, $\Gamma(X^{\log}, \mathscr{F} \otimes Q[u]) \to \exp(uN)\mathscr{F}_\alpha$ is an isomorphism.

Proof. Let Δ be $N \otimes 1 - 1 \otimes d/du$. Since Δ is the logarithm of the monodromy of $\mathscr{F} \otimes \boldsymbol{Q}[u]$, we have $\Gamma(X^{\log}, \mathscr{F} \otimes \boldsymbol{Q}[u]) \xrightarrow{\sim} \ker \Delta \subset \mathscr{F} \otimes \boldsymbol{Q}[u]$. From 5.10, we have the desired isomorphism $\Gamma(X^{\log}, \mathscr{F} \otimes \boldsymbol{Q}[u]) \xrightarrow{\sim} \exp(uN)\mathscr{F}_\alpha$. □

5.12. Let T be a topological space and $\mathscr{F}$ a sheaf on T. For a subset S of T, we omit $\Gamma(S, \mathscr{F}|_S)$ as $\Gamma(S, \mathscr{F})$.

PROPOSITION 5.13 (F. Kato). *Let Y be a unit disk with the log structure defined by the origin and $f : X \to Y$ a proper morphism of fs log analytic spaces that satisfies* 3.2. *Let D be $f^{-1}(0)$ and $\tilde{X}^*$ the fibre product of X and the universal covering of Y^* over Y^*. Let $\tau : Y^{\log} \to Y$ be the canonical map. For $\alpha \in \tau^{-1}(0)$, we have*

(i) $p : H^m((f^{\log})^{-1}(\alpha), \boldsymbol{C}) \xrightarrow{\sim} H^m(\tilde{X}^*, \boldsymbol{C})$ (*resp.* $H^m((f^{\log})^{-1}(\alpha), \boldsymbol{Q}) \xrightarrow{\sim} H^m(\tilde{X}^*, \boldsymbol{Q})$).

(ii) $\Gamma(\tau^{-1}(0), R^m f_*^{\log} \boldsymbol{C} \otimes \mathcal{O}_Y^{\log}) \xrightarrow{\sim} H^m((f^{\log})^{-1}(\alpha), \boldsymbol{C})$ (*resp.* $\Gamma(\tau^{-1}(0), R^m f_*^{\log} \boldsymbol{Q} \otimes \boldsymbol{Q}[u]) \xrightarrow{\sim} H^m((f^{\log})^{-1}(\alpha), \boldsymbol{Q})$).

(iii) *Let ι be a morphism as in Theorem B. Taking $\Gamma(\tau^{-1}(0), \quad)$ on ι, we got an isomorphism $q : H^m(D, \omega_D^\bullet) \xrightarrow{\sim} H^m((f^{\log})^{-1}(\alpha), \boldsymbol{C})$. Then the composite map $p \circ q$ is the same isomorphism as* [*St*1, (2.16)].

Proof. See [Usu] and [FKa, pp. 21–22]. □

5.14. let $Y := \{z \in \boldsymbol{C} \mid |z| < 1\}$ be the unit disk, and $f : X \to Y$ a projective surjective morphism of complex manifolds. We assume that f is smooth over the punctured disk $Y^* = Y - \{0\}$ and that $X_0 = f^{-1}(0)$ is a reduced divisor with normal crossings. Let $P \in X_0$. We assume that there exists a coordinate neighborhood U of P with coordinates $(z_0, \dots, z_n)$ and an integer r with $1 \le r \le n$ such that $P = (0, \dots, 0)$ and $f|U(z_1, \dots, z_n) = z_1 \cdots z_r = z$. Let $\mathscr{M}_Y$ (resp. $\mathscr{M}_X$) be a sheaf of holomorphic functions on Y (resp. X) which are invertible outside the origin (resp. X_0).

THEOREM 5.15 (Usui). *Let $f : X \to Y$ a morphism of fs log analytic spaces that satisfies* 5.14. *Then $f^{\log} : X^{\log} \to Y^{\log}$ is a locally topologically trivial family over the base. Moreover $R^m f_*^{\log} \boldsymbol{Q}$ is a locally constant sheaf.* (*This is a special case of* [*Usu, Theorem* 3.4].)

THEOREM C. *Let $f : X \to Y$ be as in* 5.14. *Let $\mathscr{H}_{\boldsymbol{Q}} = R^m f_*^{\log} \boldsymbol{Q}$, $\mathscr{H}_{\mathscr{O}} = R^m f_* \omega^\bullet_{X/Y}$ endowed with a filtration $\mathscr{F}^\iota := R^m f_* \omega^{\bullet \ge \iota}_{X/Y}$ and ι the isomorphism as in Theorem B. Then the triplet $(\mathscr{H}_{\boldsymbol{Q}}, \mathscr{H}_{\mathscr{O}}, \iota)$ is a log Hodge structure on Y.*

Proof. To show Theorem C, we will verify the conditions from 5.4 to 5.9. It is well known that the pair $(\mathscr{H}_{\mathscr{O}}, \mathscr{F}^\bullet)$ satisfies from 5.5 to 5.8. 5.4 is direct from 5.15. Let $y \in Y$ be a smooth point, then it is well known that 5.9 is satisfied for y from the theory of variation of Hodge structure. We verify 5.9 for the origin y of Y as follows. Let $w \in \tau^{-1}(y) \subset Y^{\log}$ and u' an element of $\mathscr{L}_w$ whose image under exp is the $\boldsymbol{Z}$-basis of $\mathscr{M}^{gp}_{Y,y}/\mathscr{O}^*_{Y,y}$, i.e., $\exp(u') = \exp(2\pi i u) = z$. Let $\psi_w : \mathscr{O}^{\log}_{Y,w} \to \boldsymbol{C}$ be an element of $\mathscr{Y}_{y,w}$ such that $\psi_w(u') = a$, $\psi : \mathscr{M}^{gp}_{Y,y} \to \boldsymbol{R}_{>0}$ the corresponding element of $\mathscr{Y}_y$. We have the following commutative diagram

$$
\begin{array}{ccccc}
H^m(D, \omega^\bullet_D) & \xrightarrow{\sim} & H^m(\tilde{X}^*, \boldsymbol{C}) & \hookleftarrow & H^m(\tilde{X}^*, \boldsymbol{Q}) \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
\Gamma(\tau^{-1}(0), \mathscr{H}_{\mathscr{O}} \otimes \mathscr{O}^{\log}_Y) & \xrightarrow{\sim} & \Gamma(\tau^{-1}(0), \mathscr{H}_{\boldsymbol{Q}} \otimes \mathscr{O}^{\log}_Y) & \hookleftarrow & \Gamma(\tau^{-1}(0), \mathscr{H}_{\boldsymbol{Q}} \otimes \boldsymbol{Q}[u]) \\
{\scriptstyle res}\downarrow & & {\scriptstyle res}\downarrow & & {\scriptstyle res}\downarrow \\
H^m(D, \omega^\bullet_D) \otimes \mathscr{O}^{\log}_{Y,w} & \xrightarrow{\sim} & H^m(f^{\log -1}(w), \boldsymbol{Q}) \otimes \mathscr{O}^{\log}_{Y,w} & \hookleftarrow & H^m(f^{\log -1}(w), \boldsymbol{Q}) \otimes \boldsymbol{Q}[u] \\
{\scriptstyle \psi_w}\downarrow & & {\scriptstyle \psi_w}\downarrow & & \\
H^m(D, \omega^\bullet_D) & \underset{\iota_a}{\xrightarrow{\sim}} & H^m(f^{\log -1}(w), \boldsymbol{Q}) \otimes \boldsymbol{C}. & &
\end{array}
$$

Here *res* is a restriction map. By 5.11, the image of $H^m(\tilde{X}^*, \boldsymbol{Q})$ in $H^m(f^{\log -1}(w), \boldsymbol{Q}) \otimes \mathscr{O}^{\log}_{Y,w}$ at the above diagram is $\exp(uN)\mathscr{H}_{\boldsymbol{Q},\alpha}$. We have $\psi_a \circ res$ is the identity map. Hence an image of $H^m(\tilde{X}^*, \boldsymbol{Q})$ in the left hand side of ι_a at the diagram is canonical. Consider $H^m(\tilde{X}^*, \boldsymbol{Q})$ as a submodule of

$H^m(D, \omega_D^\bullet)$ in this way. Since the above diagram is commutative, the image of $H^m(\tilde{X}^*, \boldsymbol{Q})$ by ι_a is $\exp((a/2\pi i)N)\mathscr{H}_{\boldsymbol{Q},w}$. Let $-$ (resp. $\bar{-}$) be the complex conjugation mapping associated to the $\boldsymbol{Q}$-structure $H^m(\tilde{X}^*, \boldsymbol{Q})$ (resp. $\mathscr{H}_{\boldsymbol{Q},w}$). Then we have $\bar{-} = \exp(-(a/2\pi i)N) \circ - \circ \exp((a/2\pi i)N)$. Hence

$$\begin{aligned}\mathscr{F}^\bullet \oplus \overline{\overline{\mathscr{F}}}{}^\bullet &= \mathscr{F}^\bullet \oplus \exp\Big(-\frac{a}{2\pi i}N\Big)\overline{\exp\Big(\frac{a}{2\pi i}N\mathscr{F}^\bullet\Big)} \\ &\cong \exp\Big(\frac{a}{2\pi i}N\Big)\mathscr{F}^\bullet \oplus \overline{\exp\Big(\frac{a}{2\pi i}N\Big)\mathscr{F}^\bullet}.\end{aligned}$$

By nilpotent orbit theorem [Sch, (4.9)], $(H^m(\tilde{X}^*, \boldsymbol{Q}), H^m(D, \mathscr{W}_D^\bullet), \exp((a/2\pi i)N)\mathscr{F}^\bullet)$ is a Hodge structure if $\mathrm{Im}(a/2\pi i) \gg 0$. This is equivalent to say that $(\mathscr{H}_{\boldsymbol{Q},w} \otimes \boldsymbol{C}, \mathscr{H}_{\boldsymbol{Q},w}, \mathscr{F}^\bullet)$ is a Hodge structure if $\psi(z) \ll 0$. □

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