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L^2 HARMONIC FORMS ON A COMPLETE STABLE HYPERSURFACES WITH CONSTANT MEAN CURVATURE*

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Abstract

We show that an n-dimensional $(2 \leq n \leq 5)$ complete noncompact strongly stable hypersurface M with constant mean curvature in an $(n+1)$-dimensional manifold $\overline{M}$ of nonnegative bi-Ricci curvature admits no nontrivial L^2 harmonic 1-forms.

1. Introduction

Let $\overline{M}$ be an $(n+1)$-dimensional orientable Riemannian manifold and let $x : M \to \overline{M}$ be an immersion with constant mean curvature H of an n-dimensional differentiable manifold M into $\overline{M}$. We recall that x is *strongly stable* if (see [1], [2], [6])

$$I(f) \equiv \int_M \{|\nabla f|^2 - (|A|^2 + \overline{\mathrm{Ric}}(n))f^2\}\, dM \geq 0 \tag{1.1}$$

for all $f : M \to R$ with compact support, where ∇f is the gradient of f and $|A|^2$ is the squared norm of the second fundamental form of x, and $\overline{\mathrm{Ric}}(n)$ is the Ricci curvature of $\overline{M}$ in the unit normal direction n. We recall x is *weakly stable* (c.f. p. 127 of [2]) if (1.1) is true for all f with compact support that satisfies

$$\int_M f\, dM = 0. \tag{1.2}$$

In [3], do Carmo and Peng proved that if M is a strongly stable complete minimal hypersurface of an $(n+1)$-dimensional Euclidean space R^{n+1} with finite absolute curvature, then M is a hyperplane. In [1] and [2], Barbosa, do Carmo and Eschenburg proved that round spheres are the only compact hypersurfaces with constant mean curvature in R^{n+1} that are weakly stable. Mori [8] and da Silveira [4] considered the complete and noncompact surfaces with constant mean

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curvature in R^3. Mori proved that if M is a strongly stable noncompact surface with constant mean curvature in R^3, then M is a plane. Da Silveira proved the same assertion under the assumption of weakly stable condition. But very little is known about the stability of complete and noncompact hypersurfaces M with constant mean curvature $H \neq 0$ for the higher dimension.

In [11], Tanno proved the following result

THEOREM 1 (see Theorem B of [11]). *Let M be a complete noncompact orientable minimal hypersurface in a Riemannian manifold of nonnegative bi-Ricci curvature. If M is stable, then there are no nontrivial L^2 harmonic 1-forms on M.*

This is a generalization of Palmer's result (when $\overline{M} = R^{n+1}$) and Miyaoka's result [7] (when $\overline{M}$ is of nonnegative sectional curvature).

When $H = 0$, we easily see that *strongly stable* reduces to *stable* of minimal hypersurface. In this paper, we generalize Theorem 1 to hypersurfaces with constant mean curvature, in fact, we obtain

THEOREM 2. *Let M be an n-dimensional $(2 \leq n \leq 5)$ complete and non-compact orientable hypersurface with constant mean curvature H in a Riemannian manifold of nonnegative bi-Ricci curvature. If M is strongly stable, then there are no nontrivial L^2 harmonic 1-forms on M.*

2. Preliminaries

We first recall the following definition

DEFINITION 1 ([10]). Let $\overline{M}$ be an $(n+1)$-dimensional Riemannian manifold, and u, v be orthonormal tangent vectors. We set

$$\text{b-Ric}(u, v) = \overline{\text{Ric}}(u) + \overline{\text{Ric}}(v) - \overline{K}(u, v),$$

and call it the bi-Ricci curvature in the directions u, v. Here $\overline{K}$ denotes the sectional curvature of the plane spanned by u, v.

From Definition 1, it is clear that the nonnegativity of the sectional curvature of $\overline{M}$ implies the nonnegativity of the bi-Ricci curvature of $\overline{M}$. If $n+1 = 2$ or $n+1 = 3$, then $\text{b-Ric}(u, v) = \overline{S}/2$, where $\overline{S}$ is the scalar curvature of $\overline{M}$.

Remark 2.1. It is clear that P_2 nonnegativity of the sectional curvature of $\overline{M}$ in [11] is equivalent to the nonnegativity of the bi-Ricci curvature of $\overline{M}$ (in [10]).

Now let ω be an L^2 harmonic p-form on a complete orientable Riemannian manifold $M = (M, g)$. It is known that ω is closed and coclosed (see [5]). The Riemannian curvature tensor, the Ricci curvature tensor and the Riemannian connection are denoted by R^i_{jkl}, R_{jl} and ∇. The expression of $\Delta\omega$ is given by (c.f. [12])

$$\Delta\omega = \Delta\omega_{i_1\cdots i_p} = \nabla^r\nabla_r\omega_{i_1\cdots i_p} - \sum_{s=1}^{p} R_{i_s}{}^r\omega_{i_1\cdots r\cdots i_p} + \sum_{t<s}^{1\cdots p} R^{vu}{}_{i_t i_s}\omega_{i_1\cdots v\cdots u\cdots i_p}$$
$$= 0.$$

Putting $\|\omega\|^2 = \sum \omega_{i_1\cdots i_p}\omega^{i_1\cdots i_p}$ and $\|\nabla\omega\|^2 = \sum \nabla_r\omega_{i_1\cdots i_p}\nabla^r\omega^{i_1\cdots i_p}$, we obtain

$$\begin{aligned}(2.1)\quad \frac{1}{2}\Delta\|\omega\|^2 &= \|\nabla\omega\|^2 + \sum \omega_{i_1\cdots i_p}\nabla^r\nabla_r\omega^{i_1\cdots i_p} \\ &= \|\nabla\omega\|^2 + \sum R^r_{i_s}\omega_{i_1\cdots r\cdots i_p}\omega^{i_1\cdots i_p} - \sum_{t<s}^{1\cdots p} R^{vu}{}_{i_t i_s}\omega_{i_1\cdots v\cdots u\cdots i_p}\omega^{i_1\cdots i_p} \\ &= \|\nabla\omega\|^2 + p\sum R^r_{i_1}\omega_{r i_2\cdots i_p}\omega^{i_1\cdots i_p} - \sum_{t<s}^{1\cdots p} R^{vu}{}_{i_t i_s}\omega_{i_1\cdots v\cdots u\cdots i_p}\omega^{i_1\cdots i_p} \\ &= \|\nabla\omega\|^2 + p\sum R_{ij}\omega^i{}_{i_2\cdots i_p}\omega_{j i_2\cdots i_p} - \frac{p(p-1)}{2}\sum R_{kjih}\omega^{kj}{}_{i_3\cdots i_p}\omega^{ih i_3\cdots i_p}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(2.2)\qquad \frac{1}{2}\Delta\|\omega\|^2 &= \|\omega\|\Delta\|\omega\| + \|\nabla\|\omega\|\|^2 \\ &= \|\omega\|\Delta\|\omega\| + \|\nabla\omega\|^2 - F(\omega),\end{aligned}$$

where

$$(2.3)\qquad F(\omega) = \|\nabla\omega\|^2 - \|\nabla\|\omega\|\|^2,$$

and Kato's inequality implies

$$(2.4)\qquad F(\omega) \geq 0.$$

By (2.1) and (2.2), we get

$$(2.5)\quad \|\omega\|\Delta\|\omega\| = p\sum R_{ij}\omega^i{}_{i_2\cdots i_p}\omega^{j i_2\cdots i_p} - \frac{p(p-1)}{2}\sum R_{kjih}\omega^{kj}{}_{i_3\cdots i_p}\omega^{ih i_3\cdots i_p} + F(\omega).$$

3. Hypersurfaces with constant mean curvature H

Let M be an n-dimensional orientable hypersurface with constant mean curvature H in an $(n+1)$-dimensional Riemannian manifold $\overline{M}$. Let n be a unit normal vector field on M and let A be the shape operator with respect to n. We assume that M admits a nontrivial L^2 harmonic p-form ω. After Palmer [9] we use the following cut off function h. Let p be a point of M. By $B_r(p)$ we denote the geodesic r-ball centered at p (r-neighborhood of p in M). h is a smooth function such that $0 \leq h \leq 1$ and

(i) $h = 1$ on $B_{r/2}(p)$ and $h = 0$ outside $B_r(p)$,
(ii) $\|\nabla h\|^2 \leq c/r^2$, where c is a constant.

Let $f = h\|\omega\|$ in (1.1), we have

$$I(h) = -\int_M h^2(\|\omega\|\Delta\|\omega\| + \|A\|\,\|\omega\|^2 + \overline{\mathrm{Ric}}(n)\|\omega\|^2) + \int_M \|\nabla h\|^2\|\omega\|^2. \tag{3.1}$$

By (2.5), we get

$$\begin{aligned} I(h) = -\int_M h^2\Big[p\sum R_{ij}\omega^i{}_{i_2\cdots i_p}\omega^{ji_2\cdots i_p} - \frac{p(p-1)}{2}\sum R_{kjih}\omega^{kj}{}_{i_3\cdots i_p}\omega^{ihi_3\cdots i_p} \\ + F(\omega) + \|A\|^2\|\omega\|^2 + \overline{\mathrm{Ric}}(n)\|\omega\|^2\Big] + \int_M \|\nabla h\|^2\|\omega\|^2. \end{aligned} \tag{3.2}$$

Now let $\{e_1, \ldots, e_n, e_{n+1} = n\}$ be a local orthonormal frame along M. Then we have the following Gauss equations

$$R_{ijkl} = A_{ik}A_{jl} - A_{il}A_{jk} + \bar{R}_{ijkl}, \tag{3.3}$$

$$\begin{aligned} R_{jl} &= nHA_{jl} - \sum_k A_{jk}A_{kl} + \sum_{k=1}^{n} \bar{R}_{kjkl} \\ &= nHA_{jl} - \sum_k A_{jk}A_{kl} + \bar{R}_{jl} - \bar{K}(e_{n+1}, e_j, e_{n+1}, e_l), \end{aligned} \tag{3.4}$$

where $H = (\operatorname{tr} A)/n$ is the mean curvature of M in $\bar{M}$.

Putting (3.3) and (3.4) into (3.2), we obtain

$$\begin{aligned} I(h) = &-\int_M h^2\Big[npH\sum A_{ij}\omega^i{}_{i_2\cdots i_p}\omega^{ji_2\cdots i_p} - p\sum A_{ik}A_{kj}\omega^i{}_{i_2\cdots i_p}\omega^{ji_2\cdots i_p} \\ &- \frac{p(p-1)}{2}\sum A_{ki}A_{jh}\omega^{kj}{}_{i_3\cdots i_p}\omega^{ihi_3\cdots i_p} \\ &+ \frac{p(p-1)}{2}\sum A_{kh}A_{ij}\omega^{kj}{}_{i_3\cdots i_p}\omega^{ihi_3\cdots i_p} \\ &+ p\sum \bar{R}_{kikj}\omega^i{}_{i_2\cdots i_p}\omega^{ji_2\cdots i_p} - \frac{p(p-1)}{2}\sum \bar{R}_{kjih}\omega^{kj}{}_{i_3\cdots i_p}\omega^{ihi_3\cdots i_p} \\ &+ F(\omega) + \|A\|^2\|\omega\|^2 + \overline{\mathrm{Ric}}(n)\|\omega\|^2\Big] + \int_M \|\nabla h\|^2\|\omega\|^2 \\ = &-\int_M h^2\Big[npH\sum A_{ij}\omega^i{}_{i_2\cdots i_p}\omega^{ji_2\cdots i_p} - p\sum A_{ik}A_{kj}\omega^i{}_{i_2\cdots i_p}\omega^{ji_2\cdots i_p} \\ &- p(p-1)\sum A_{ki}A_{jh}\omega^{kj}{}_{i_3\cdots i_p}\omega^{ihi_3\cdots i_p} + F(\omega) \\ &+ \|A\|^2\|\omega\|^2 + Q(\omega)\Big] + \int_M \|\nabla h\|^2\|\omega\|^2, \end{aligned} \tag{3.5}$$

where

$$(3.6)\qquad Q(\omega) = p\sum \bar{R}_{kikj}\omega^{i}{}_{i_2\cdots i_p}\omega^{ji_2\cdots i_p} - \frac{p(p-1)}{2}\sum \bar{R}_{kjih}\omega^{kj}{}_{i_3\cdots i_p}\omega^{ihi_3\cdots i_p} + \overline{\mathrm{Ric}}(n)\|\omega\|^2.$$

4. L^2 harmonic 1-forms

Let M be a complete orientable hypersurface in $\bar{M}$. We assume that M admits a non-trivial L^2 harmonic 1-form ω and let ω^* denote the vector field dual to ω with respect to the Riemannian metric. Choosing $p = 1$, in (3.5), we have

$$(4.1)\qquad I(h) = -\int_M h^2[D(\omega^*) + F(\omega) + Q(\omega)] + \int_M \|\nabla h\|^2\|\omega\|^2,$$

where

$$(4.2)\qquad D(\omega^*) = nHA(\omega^*,\omega^*) - \langle A\omega^*, A\omega^*\rangle + \|A\|^2\|\omega\|^2,$$

$$(4.3)\qquad Q(\omega) = \sum_k \bar{R}(e_k,\omega^*,e_k,\omega^*) + \overline{\mathrm{Ric}}(e_{n+1},e_{n+1})\|\omega^*\|^2 = \overline{\mathrm{Ric}}(\omega^*,\omega^*) + \overline{\mathrm{Ric}}(e_{n+1},e_{n+1})\|\omega^*\|^2 - \bar{K}(e_{n+1},\omega^*,e_{n+1},\omega^*),$$

where $e_1,\ldots,e_n$ are local orthonormal basis and $e_{n+1} = n$. Let $Ae_i = \lambda_i e_i$, i.e., $A(e_i,e_j) = \lambda_i\delta_{ij}$, $\omega^* = \sum a_i e_i$, then

$$nH = \lambda_1 + \cdots + \lambda_n,\quad A(\omega^*,\omega^*) = \sum a_i a_j \lambda_i \delta_{ij} = \sum \lambda_i a_i^2,$$
$$\langle A\omega^*, A\omega^*\rangle = \sum a_i^2\lambda_i^2.$$

We first prove the following lemma

LEMMA 4.1. *For any tangent vector field $v = \sum_i b_i e_i$ on M, we have*

$$(4.4)\qquad D(v) = nHA(v,v) - \langle Av, Av\rangle + \|A\|^2\|v\|^2 = (\lambda_1+\cdots+\lambda_n)\sum_i b_i^2\lambda_i - \sum_i b_i^2\lambda_i^2 + (\lambda_1^2+\cdots+\lambda_n^2)(b_1^2+\cdots+b_n^2) \geq 0,\quad \text{when}\quad 2\leq n\leq 5.$$

Proof. For $1\leq i\leq n$, we let

$$F_i = (\lambda_1+\cdots+\lambda_n)b_i^2\lambda_i - b_i^2\lambda_i^2 + (\lambda_1^2+\cdots+\lambda_n^2)b_i^2 = [(\lambda_1+\cdots+\lambda_n)\lambda_i - \lambda_i^2 + (\lambda_1^2+\cdots+\lambda_n^2)]b_i^2.$$

When $n=2$, $F_i = 1/2[(\lambda_1+\lambda_2)^2 + \lambda_1^2 + \lambda_2^2]b_i^2 \geq 0$.

When $n=3$, $F_1 = 1/2[(\lambda_1+\lambda_2)^2 + (\lambda_1+\lambda_3)^2 + \lambda_2^2 + \lambda_3^2]b_1^2 \geq 0$, similarly, $F_i\geq 0$, $i = 2,3$.

When $n=4$, $F_1 = [(\lambda_1/2+\lambda_2)^2 + (\lambda_1/2+\lambda_3)^2 + (\lambda_1/2+\lambda_4)^2 + \lambda_1^2/4]b_1^2 \geq 0$, similarly, $F_i\geq 0$, $i=2,3,4$.

When $n=5$, $F_1=[(\lambda_1/2+\lambda_2)^2+(\lambda_1/2+\lambda_3)^2+(\lambda_1/2+\lambda_4)^2+(\lambda_1/2+\lambda_5)^2]b_1^2 \geq 0$, similarly, $F_i \geq 0$, $i \geq 2$. Thus, the left hand side of (4.4) $=\sum_i F_i \geq 0$. □

Remark 4.1. Note that, if $n=6$, for example, $\lambda_1=-1, \lambda_2=\cdots=\lambda_6=1/2$, $b_1 \neq 0, b_2=\cdots=b_6=0$. In this case, $F_1=-b_1^2/4<0, F_2=\cdots=F_6=0$, thus the left hand side of (4.4) is negative. We see that the condition $n \leq 5$ in Lemma 4.1 is essential.

5. The proof of Theorem 2

Let $\overline{M}$ be an $(n+1)$-dimensional Riemannian manifold of nonnegative bi-Ricci curvature. Then by (4.3)

$$
\begin{aligned}
(5.1)\qquad Q(\omega) &= \overline{\mathrm{Ric}}(\omega^*,\omega^*)+\overline{\mathrm{Ric}}(e_{n+1},e_{n+1})\|\omega^*\|^2-\overline{K}(e_{n+1},\omega^*,e_{n+1},\omega^*)\\
&= [\overline{\mathrm{Ric}}(e,e)+\overline{\mathrm{Ric}}(e_{n+1},e_{n+1})-\overline{K}(e_{n+1},e,e_{n+1},e)]\|\omega^*\|^2\\
&\geq 0,
\end{aligned}
$$

where $e=\omega^*/\|\omega^*\|$ is the unit tangent vector field on M. Now we assume that M is an n-dimensional noncompact complete strongly stable hypersurface with constant mean curvature H in $\overline{M}$, and that there is a nontrivial L^2 harmonic 1-form ω on M. So we have by (4.1), (1.1) and the definition of function h

$$
\begin{aligned}
(5.2)\qquad 0 \leq I(h) &= -\int_M h^2[D(\omega^*)+F(\omega)+Q(\omega)]+\int_M \|\nabla h\|^2\|\omega\|^2\\
&\leq -\int_{B_{r/2}(p)}[D(\omega^*)+F(\omega)+Q(\omega)]+\frac{c}{r^2}\int_M \|\omega^*\|^2.
\end{aligned}
$$

Letting $r \to \infty$, in view of (2.4), Lemma 4.1 and (5.1), we have $Q(\omega)=F(\omega)=D(\omega^*)=0$. The equality $F(\omega)=0$ implies $2\|\omega\|^2\nabla_i\omega_j=(\nabla_i\|\omega\|^2)\omega_j$. So $\delta\omega=0$ implies $\omega^i\nabla_i\|\omega\|^2=0$. Furthermore, $d\omega=0$ implies $\|\omega\|$ is constant and ω^* is parallel. Thus $\mathrm{Ric}(\omega^*,\omega^*)=0$, and we have by (3.4)

$$(5.3)\qquad nHA(\omega^*,\omega^*)-\langle A\omega^*,A\omega^*\rangle+\overline{\mathrm{Ric}}(\omega^*,\omega^*)-\overline{K}(e_{n+1},\omega^*,e_{n+1},\omega^*)=0.$$

By (4.2), $D(\omega^*)=0$ reduces to

$$(5.4)\qquad nHA(\omega^*,\omega^*)-\langle A\omega^*,A\omega^*\rangle+\|A\|^2\|\omega\|^2=0.$$

By (4.3), $Q(\omega)=0$ becomes

$$(5.5)\qquad \overline{\mathrm{Ric}}(\omega^*,\omega^*)+\overline{\mathrm{Ric}}(e_{n+1},e_{n+1})\|\omega^*\|^2-\overline{K}(e_{n+1},\omega^*,e_{n+1},\omega^*)=0.$$

Combining (5.3), (5.4) with (5.5), we have

$$(5.6)\qquad \|A\|^2\|\omega\|^2+\overline{\mathrm{Ric}}(e_{n+1},e_{n+1})\|\omega\|^2=0.$$

Let u be an arbitrary unit tangent vector field to M. From the nonnegativity of the bi-Ricci curvature of $\bar{M}$, for an orthonormal pair $\{u, e_{n+1}\}$, we have

$$\overline{\mathrm{Ric}}(u,u) + \overline{\mathrm{Ric}}(e_{n+1}, e_{n+1}) - \bar{K}(u, e_{n+1}, u, e_{n+1}) \geq 0. \tag{5.7}$$

By Gauss equation (3.4), we get from (5.6) and (5.7)

$$\begin{aligned} \mathrm{Ric}(u,u) &= \overline{\mathrm{Ric}}(u,u) - \bar{K}(u, e_{n+1}, u, e_{n+1}) + nHA(u,u) - \langle Au, Au\rangle \\ &\geq -\overline{\mathrm{Ric}}(e_{n+1}, e_{n+1}) + nHA(u,u) - \langle Au, Au\rangle \\ &= \|A\|^2 + nHA(u,u) - \langle Au, Au\rangle. \end{aligned} \tag{5.8}$$

By use of Lemma 4.1, we can conclude that

$$\mathrm{Ric}(u,u) \geq 0. \tag{5.9}$$

Thus the Ricci curvature of M is nonnegative. Because M is complete and noncompact, the volume of M is infinite ([13]). This contradicts that ω is an L^2 harmonic 1-form and $\|\omega\|$ is constant. □

Remark 5.1. By Dodziuk's result [5] the existence of a nontrivial L^2 harmonic 1 form follows from a topological condition that there exists a cycle of codimension one in M which does not disconnect M (c.f. Palmer [9] or Tanno [11]).

6. L^2 harmonic 2-forms

In this section, we will prove the following result

THEOREM 6.1. *Let M be an n-dimensional $(2 \leq n \leq 4)$ complete noncompact orientable hypersurface with constant mean curvature H in an $(n+1)$-dimensional Euclidean space R^{n+1}. If M is strongly stable and M admits a nontrivial L^2 harmonic 2-form ω, then ω is parallel on M.*

We first prove the following Lemma

LEMMA 6.1. *Let A, B be $n \times n$ real matrices such that*
(i) *A is symmetric*
(ii) *B is skew-symmetric.*
If $2 \leq n \leq 4$, then

$$\|A\|^2\|B\|^2 + 2\mathrm{tr}(AB)^2 + 2\mathrm{tr}(A^2B^2) - 2\mathrm{tr}\,A \cdot \mathrm{tr}(AB^2) \geq 0.$$

Proof of Lemma 6.1. First we diagonalize A to the form $(a_i\delta_{ij})$ by an orthonormal transformation. Let $B = (b_{ij})$, then we have the following

$$\|A\|^2\|B\|^2 = \left(\sum_i a_i^2\right)\left(\sum_{i\neq j} b_{ij}^2\right), \quad \mathrm{tr}(AB)^2 = -\sum_{i\neq j} a_i a_j b_{ij}^2,$$

$$\mathrm{tr}(A^2B^2) = -\sum_{i\neq j} a_i^2 b_{ij}^2, \quad -2\mathrm{tr}(A)\cdot\mathrm{tr}(AB^2) = 2\sum_i a_i \sum_{j\neq k} a_j b_{jk}^2.$$

Therefore, we obtain

$$\begin{aligned}&\|A\|^2\|B\|^2 + 2\mathrm{tr}(AB)^2 + 2\mathrm{tr}(A^2B^2) - 2\mathrm{tr}A\cdot\mathrm{tr}(AB^2)\\&\quad = 2b_{12}^2[a_3^2 + a_4^2 + \cdots + a_n^2 - 2a_1a_2 + (a_1+\cdots+a_n)(a_1+a_2)]\\&\qquad + 2b_{13}^2[\cdots] + \cdots + 2b_{n-1n}^2[\cdots].\end{aligned}$$

When $n = 2$, $(a_1+a_2)^2 - 2a_1a_2 \geq 0$. When $n = 3$,

$$\begin{aligned}&a_3^2 - 2a_1a_2 + (a_1+a_2+a_3)(a_1+a_2)\\&\quad = (a_3/2 + a_1)^2 + (a_3/2+a_2)^2 + a_3^2/2 \geq 0.\end{aligned}$$

When $n = 4$,

$$\begin{aligned}&a_3^2 + a_4^2 - 2a_1a_2 + (a_1+a_2+a_3+a_4)(a_1+a_2)\\&\quad = \frac{1}{2}(a_1+a_3)^2 + \frac{1}{2}(a_1+a_4)^2 + \frac{1}{2}(a_2+a_3)^2 + \frac{1}{2}(a_2+a_4)^2 \geq 0. \qquad \square\end{aligned}$$

Remark 6.1. When $\mathrm{tr}\,A = 0$, Lemma 6.1 reduces to Lemma 1 of Tanno [11]. Just as in Tanno [11], the condition $n \leq 4$ in Lemma 6.1 is essential.

Proof of Theorem 6.1. We assume that a complete orientable hypersurface M with constant mean curvature H in R^{n+1} is strongly stable and M admits a nontrivial L^2 harmonic 2-form ω. Let $p = 2$ in (3.5), we have

$$\begin{aligned}(6.1)\quad I(h) &= -\int_M h^2\Big[2nH\sum A_{ij}\omega_k^i\omega^{jk} - 2\sum A_{ik}A_{kj}\omega_s^i\omega^{js} - 2\sum A_{ki}A_{jh}\omega^{kj}\omega^{ih}\\&\qquad + F(\omega) + \|A\|^2\|\omega\|^2\Big] + \int_M \|\nabla h\|^2\|\omega\|^2\\&= -\int_M h^2[D_1(\omega) + F(\omega)] + \int_M \|\nabla h\|^2\|\omega\|^2,\end{aligned}$$

where

$$(6.2)\qquad D_1(\omega) = -2\mathrm{tr}(A)\mathrm{tr}(AB^2) + 2\mathrm{tr}(A^2B^2) + 2\mathrm{tr}(AB)^2 + \|A\|^2\|B\|^2,$$

where $A = (A_{ij})$ and $B = (\omega_{ij})$.

Lemma 6.1 implies that $D_1(\omega) \geq 0$ holds on M. Then (6.1) and the definition of function h imply the following

$$(6.3)\qquad 0 \leq I(h) \leq -\int_{B_{r/2}(p)} [D_1(\omega) + F(\omega)] + (c/r^2)\int_M \|\omega\|^2.$$

Letting $r \to \infty$, $F(\omega) = D_1(\omega) = 0$. The equality $F(\omega) = 0$ implies (c.f. [11])

$$2\|\omega\|^2 \nabla_k \omega_{ij} = (\nabla_k \|\omega\|^2)\omega_{ij}. \tag{6.4}$$

We consider (6.4) on an open set where $\omega \neq 0$. $\delta\omega = 0$ implies that $\omega^{kj}\nabla_k \|\omega\|^2 = 0$ holds. Furthermore, $d\omega = 0$ is equivalent to

$$\nabla_k \omega_{ij} + \nabla_i \omega_{jk} + \nabla_j \omega_{ki} = 0.$$

By (6.4) and the last equality multiplied by ω^{ij}, we get $\nabla_k \|\omega\|^2 = 0$, and hence $\|\omega\|$ is constant. By (6.4), we conclude that ω is parallel. □

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