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AN n-DIMENSIONAL FLAT TORUS IN S^{2n-1} WHOSE EXTRINSIC DIAMETER IS EQUAL TO π

Dedicated to Professor Shukichi Tanno on his 60th birthday

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1. Introduction

Let S^3 be the 3-dimensional standard unit sphere in the complex Euclidean space $\boldsymbol{C}^2$. For each θ satisfying $0<\theta<\pi/2$, we consider a torus $M_\theta \subset S^3$ defined by

$$M_\theta = \{(z_1, z_2) \in \boldsymbol{C}^2 : |z_1| = \cos\theta, \ |z_2| = \sin\theta\}.$$

The torus M_θ is a flat Riemannian manifold equipped with the metric induced by the inclusion map $i_\theta : M_\theta \to S^3$. In [2] the author studied the question whether the flat torus $M_\theta \subset S^3$ is rigid or not, and he proved that every isometric deformation of $i_\theta : M_\theta \to S^3$ is trivial. Recently, concerning the question above, Enomoto, Weiner and the author proved the following rigidity theorem.

THEOREM 1.1 ([1]). *If $f : M_\theta \to S^3$ is an isometric embedding, then there exists an isometry A of S^3 such that $f = A \circ i_\theta$.*

There are two key ingredients in the proof of this theorem. One is the fact that every embedded flat torus in S^3 is invariant under the antipodal map of S^3 ([3]), and the other is the following:

THEOREM 1.2 ([1]). *Let $f : M_\theta \to S^3$ be an isometric immersion. If the diameter of the image $f(M_\theta)$ is equal to π, then there exists an isometry A of S^3 such that $f = A \circ i_\theta$.*

In this note we establish a higher dimensional generalization of Theorem 1.2. For $n \geqq 2$, let $\tau = (R_1, \dots, R_n)$ be an n-tuple of positive real numbers such that $\sum_{i=1}^n R_i^2 = 1$, and let M_τ be an n-dimensional torus in the $(2n-1)$-dimensional standard unit sphere $S^{2n-1} \subset \boldsymbol{C}^n$ defined by

$$M_\tau = \{(z_1, \dots, z_n) \in \boldsymbol{C}^n : |z_i| = R_i \text{ for } 1 \leqq i \leqq n\}.$$

Then M_τ is a flat Riemannian manifold equipped with the metric induced by

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the inclusion map $i_\tau: M_\tau \to S^{2n-1}$. For each isometric immersion $f: M_\tau \to S^{2n-1}$, we denote by Diam(f) the diameter of the image $f(M_\tau)$ in S^{2n-1}. Note that the inclusion map i_τ satisfies $\mathrm{Diam}(i_\tau)=\pi$. The following theorem, which will be proved in Section 2, is the main result of this note.

THEOREM 1.3. *Let $f: M_\tau \to S^{2n-1}$ be an isometric immersion. If* $\mathrm{Diam}(f)=\pi$, *then there exists an isometry A of S^{2n-1} such that $f=A\circ i_\tau$.*

Remark. Because of Theorem 1.3, it is interesting to ask the following question: *Does there exist an isometric immersion $f: M_\tau \to S^{2n-1}$ with* $\mathrm{Diam}(f)<\pi$? However, the author does not know the answer to the question even for $n=2$.

2. Proof of Theorem 1.3

We first prove the following algebraic lemma.

LEMMA 2.1. *Let v and v_{ij} $(1\leqq i, j\leqq n)$ be elements of a real vector space V. Suppose that $\sum_{i,j=1}^n x_i x_j v_{ij}=v$ for all $(x_1, \dots, x_n)\in \boldsymbol{R}^n$ with $|x_i|=1$ $(1\leqq i\leqq n)$. Then $v_{ij}+v_{ji}=0$ for all $i<j$, and $\sum_{i=1}^n v_{ii}=v$.*

Proof. We prove the lemma by induction on n. For $n=1$, the assertion of the lemma is trivial. Choose $(x_1, \dots, x_{n-1})\in \boldsymbol{R}^{n-1}$ such that $|x_i|=1$ for all $1\leqq i\leqq n-1$. Then $\sum_{i,j=1}^n x_i x_j v_{ij}=v$ for $x_n=\pm 1$. This shows

$$\sum_{i,j<n} x_i x_j v_{ij} \pm \sum_{i<n} x_i(v_{in}+v_{ni})=v-v_{nn}\,.$$

Hence

$$\sum_{i,j<n} x_i x_j v_{ij}=v-v_{nn}\,, \tag{2.1}$$

$$\sum_{i<n} x_i(v_{in}+v_{ni})=0\,. \tag{2.2}$$

By (2.1) and the induction hypothesis it follows that $v_{ij}+v_{ji}=0$ for all $i<j<n$ and that $\sum_{i=1}^{n-1} v_{ii}=v-v_{nn}$. On the other hand (2.2) implies that $v_{in}+v_{ni}=0$ for $i<n$. Hence we obtain the assertion of the lemma. □

For each $u=(u_1, \dots, u_n)\in \boldsymbol{R}^n$, we consider a transformation $T_u: M_\tau \to M_\tau$ given by

$$T_u(p)=(z_1\exp(\sqrt{-1}u_1/R_1), \dots, z_n\exp(\sqrt{-1}u_n/R_n)),$$

where $p=(z_1, \dots, z_n)\in M_\tau$. Note that

$$T_{u+v}=T_u\circ T_v \qquad \text{for all} \quad u, v\in \boldsymbol{R}^n\,. \tag{2.3}$$

Now we denote by Ω the set of all $\omega=(\omega_1, \dots, \omega_n)\in \boldsymbol{R}^n$ such that $|\omega_i|=R_i$ for all $1\leqq i\leqq n$. Then

LEMMA 2.2. *Let* $f: M_\tau\to S^{2n-1}$ *be an isometric immersion, and let* $p\in M_\tau$. *If there exists a point* $q\in M_\tau$ *such that* $f(q)=-f(p)$, *then for each* $\omega\in\Omega$ *the curve* $\gamma(t)=f(T_{t\omega}(p))$ *is a unit speed geodesic in* S^{2n-1}.

Proof. Let $d(\ ,\ )$ denote the distance function on M_τ induced by the Riemannian metric on M_τ. Then it follows that

$$d(x, y)\leqq\pi \quad \text{for all} \quad x, y\in M_\tau, \tag{2.4}$$

where the equality holds if and only if $i_\tau(y)=-i_\tau(x)$. Since $|\omega_i|=R_i$, the curve $\gamma(t)$ is a unit speed curve in S^{2n-1} satisfying $\gamma(t+2\pi)=\gamma(t)$. So it is sufficient to show that $\gamma(\pi)=-\gamma(0)$. Since the immersion f is isometric, the assumption $f(q)=-f(p)$ implies that $d(p, q)\geqq\pi$. Hence it follows from (2.4) that $i_\tau(q)=-i_\tau(p)$. Therefore $i_\tau(q)=-i_\tau(p)=i_\tau(T_{\pi\omega}(p))$, and so $q=T_{\pi\omega}(p)$. Hence $\gamma(\pi)=f(T_{\pi\omega}(p))=f(q)=-f(p)=-\gamma(0)$. □

LEMMA 2.3. *Let* $f: M_\tau\to S^{2n-1}$ *be an isometric immersion, and let* σ *be the second fundamental form of the immersion. If* $\mathrm{Diam}(f)=\pi$, *then* $\sigma(X_\omega, X_\omega)=0$ *for all* $\omega\in\Omega$, *where* X_ω *denotes the vector field induced by the one parameter group of transformation* $T_{t\omega}(t\in\boldsymbol{R})$.

Proof. Let M_τ^* be the set of all $p\in M_\tau$ such that $f(p)=-f(q)$ for some $q\in M_\tau$. Since $X_\omega(p)=(d/dt)T_{t\omega}(p)|_{t=0}$, it follows from Lemma 2.2 that $\sigma(X_\omega(p), X_\omega(p))=0$ for $p\in M_\tau^*$. So it is sufficient to show that $M_\tau=M_\tau^*$. Since $\mathrm{Diam}(f)=\pi$, the set M_τ^* is not empty. Let $p_0\in M_\tau^*$, and let $\{\alpha_1, \dots, \alpha_n\}$ be a basis of $\boldsymbol{R}^n$ satisfying $\alpha_i\in\Omega$. Now we take a point $q\in M_\tau$. Then there exist real numbers $x_1, \dots, x_n$ such that $q=T_{x_1\alpha_1+\cdots+x_n\alpha_n}(p_0)$. We consider a sequence of points $p_1, \dots, p_n\in M_\tau$ defined by the relation $p_i=T_{x_i\alpha_i}(p_{i-1})$. Since $p_0\in M_\tau^*$ and $\alpha_1\in\Omega$, it follows from Lemma 2.2 that $\gamma(t)=f(T_{t\alpha_1}(p_0))$ is a unit speed geodesic in S^{2n-1}. This shows $p_1\in M_\tau^*$. Similarly we see that $p_2, \dots, p_n$ are contained in M_τ^*. On the other hand (2.3) implies that $q=T_{x_2\alpha_2+\cdots+x_n\alpha_n}(p_1)=\cdots=T_{x_n\alpha_n}(p_{n-1})=p_n$. Hence $q\in M_\tau^*$, and so $M_\tau=M_\tau^*$. □

Now we denote by $\{e_1, \dots, e_n\}$ the standard basis of $\boldsymbol{R}^n$, and define an orthonormal frame field $\{E_1, \dots, E_n\}$ on M_τ by

$$E_i(p)=\frac{d}{dt}T_{te_i}(p)\Big|_{t=0},$$

where $p\in M_\tau$. Then

LEMMA 2.4. *Let* $f: M_\tau\to S^{2n-1}$ *be an isometric immersion, and let* σ *be the second fundamental form of the immersion. If* $\mathrm{Diam}(f)=\pi$, *then*

(1) $\sigma(E_i, E_j)=0$ *for* $i\neq j$,

(2) $\sum_{i=1}^n R_i^2\sigma(E_i, E_i)=0$,

(3) $h(\sigma(E_i, E_i), \sigma(E_j, E_j))=-1$ *for* $i\neq j$,
(4) $h(\sigma(E_i, E_i), \sigma(E_i, E_i))=R_i^{-2}-1$,
(5) $D(\sigma(E_i, E_i))=0$,
where h *and* D *denote the induced metric and the induced connection on the normal bundle of the immersion* f, *respectively.*

Proof. Let $(x_1, \dots, x_n)\in \boldsymbol{R}^n$ such that $|x_i|=1$ for all $1\leqq i\leqq n$, and let $\omega=(x_1R_1, \dots, x_nR_n)$. Since $\omega\in\Omega$, it follows from Lemma 2.3 that $\sigma(X_\omega, X_\omega)=0$. On the other hand it is easy to see that $X_\omega=x_1R_1E_1+\cdots+x_nR_nE_n$. So we obtain

$$\sum_{i,j=1}^{n} x_i x_j v_{ij}=0,$$

where $v_{ij}=R_iR_j\sigma(E_i, E_j)$. Since $v_{ij}=v_{ji}$, it follows from Lemma 2.1 that $v_{ij}=0$ for $i\neq j$, and $v_{11}+\cdots+v_{nn}=0$. This shows the assertions (1) and (2). By the equations of Gauss we have

$$1-\delta_{ij}=h(\sigma(E_i, E_j), \sigma(E_i, E_j))-h(\sigma(E_i, E_i), \sigma(E_j, E_j)).$$

So the assertion (3) follows from (1). Combining (2) and (3), we obtain the assertion (4). Since the vector fields $E_1, \dots, E_n$ are parallel with respect to the Riemannian metric on M_τ, it follows from the equations of Codazzi that $D_{E_j}(\sigma(E_i, E_i))=D_{E_i}(\sigma(E_i, E_j))$. Hence (1) yields

$$D_{E_j}(\sigma(E_i, E_i))=0 \quad \text{for } i\neq j. \tag{2.5}$$

On the other hand, differentiating (2), we obtain

$$\sum_{i=1}^{n} R_i^2 D_{E_j}(\sigma(E_i, E_i))=0. \tag{2.6}$$

Combining (2.5) and (2.6), we see that $D_{E_j}(\sigma(E_i, E_i))=0$ for all $1\leqq i, j\leqq n$. This implies the assertion (5). □

LEMMA 2.5. *Let* f *and* $\tilde{f}$ *be isometric immersions of the flat torus* M_τ *into the unit sphere* S^{2n-1}. *If* $\mathrm{Diam}(f)=\mathrm{Diam}(\tilde{f})=\pi$, *then there exists an isometry* A *of* S^{2n-1} *such that* $\tilde{f}=A\circ f$.

Proof. Let B (resp. $\tilde{B}$) denote the normal bundle of the immersion f (resp. $\tilde{f}$), and let D (resp. $\tilde{D}$) be the induced connection on the normal bundle B (resp. $\tilde{B}$). The second fundamental form of f (resp. $\tilde{f}$) is denoted by σ (resp. $\tilde{\sigma}$), and the induced metric on the bundle B (resp. $\tilde{B}$) is denoted by h (resp. $\tilde{h}$). We denote by $\Gamma(B)$ and $\Gamma(TM_\tau)$ the sets of the smooth cross sections of the normal bundle B and the tangent bundle TM_τ, respectively. Then, by the fundamental theorem for submanifolds [4, Chapter 7], the assertion of Lemma 2.5 follows from the existence of a bundle isomorphism $\Phi: B\to\tilde{B}$ such that

$$h(\xi, \eta)=\tilde{h}(\Phi\xi, \Phi\eta) \quad \text{for all} \quad \xi, \eta\in\Gamma(B), \tag{2.7}$$

$$\Phi(\sigma(X, Y))=\tilde{\sigma}(X, Y) \quad \text{for all} \quad X, Y\in\Gamma(TM_\tau), \tag{2.8}$$

$$\Phi(D_X\xi)=\tilde{D}_X(\Phi\xi) \quad \text{for all} \quad X\in\Gamma(TM_\tau) \text{ and all } \xi\in\Gamma(B). \tag{2.9}$$

To establish the existence of such a bundle isomorphism, we set $\xi_i=\sigma(E_i, E_i)$ and $\tilde{\xi}_i=\tilde{\sigma}(E_i, E_i)$. Then it follows from Lemma 2.4 (2)-(5) that

$$\sum_{i=1}^{n} R_i^2\xi_i=0, \quad \sum_{i=1}^{n} R_i^2\tilde{\xi}_i=0, \tag{2.10}$$

$$h(\xi_i, \xi_j)=\tilde{h}(\tilde{\xi}_i, \tilde{\xi}_j)=R_i^{-1}R_j^{-1}\delta_{ij}-1, \tag{2.11}$$

$$D_X\xi_i=0, \quad \tilde{D}_X\tilde{\xi}_i=0. \tag{2.12}$$

For each $p\in M_\tau$, we denote by B_p (resp. $\tilde{B}_p$) the fiber of B (resp. $\tilde{B}$) over the piont p. Then (2.11) implies that $\{\xi_1(p), \dots, \xi_{n-1}(p)\}$ and $\{\tilde{\xi}_1(p), \dots, \tilde{\xi}_{n-1}(p))\}$ are basis of B_p and $\tilde{B}_p$, respectively. So there exists a bundle isomorphism Φ: $B\to\tilde{B}$ such that $\Phi(\xi_i)=\tilde{\xi}_i$ for $1\leqq i\leqq n-1$. Since (2.10) yields $\Phi(\xi_n)=\tilde{\xi}_n$, it follows from Lemma 2.4 (1) that

$$\Phi(\sigma(E_i, E_j))=\tilde{\sigma}(E_i, E_j). \tag{2.13}$$

By (2.11)-(2.13) we see that the bundle isomorphism Φ satisfies (2.7)-(2.9). □

Now the assertion of Theorem 1.3 follows from Lemma 2.5, since the inclusion map i_τ: $M_\tau\to S^{2n-1}$ satisfies $\mathrm{Diam}(i_\tau)=\pi$.

References

[1] K. Enomoto, Y. Kitagawa and J. L. Weiner, A rigidity theorem for the Clifford tori in S^3, Proc. Amer. Math. Soc., **124** (1996), 265-268.
[2] Y. Kitagawa, Rigidity of the Clifford tori in S^3, Math. Z., **198** (1988), 591-599.
[3] Y. Kitagawa, Embedded flat tori in the unit 3-sphere, J. Math. Soc. Japan, **47** (1995), 275-296.
[4] M. Spivak, A Comprehensive Introduction to Differential Geometry, Vol. 4, Publish or Perish, Berkeley, 1977.

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