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ON THE FREQUENCY OF ZEROS OF A FUNDAMENTAL SOLUTION SET OF COMPLEX LINEAR DIFFERENTIAL EQUATIONS

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Abstract

We give a complete characterization of those equations of the form (1.1) which possess a fundamental solution set having few zeros. Some other results are also obtained.

1. Introduction and Results

Consider a linear differential equation of the form

$$f^{(n)}+p_{n-1}(z)f^{(n-1)}+\cdots+p_0(z)f=0, \tag{1.1}$$

where $n\geqq 2$ and $p_0(z), \dots, p_{n-1}(z)$ are polynomials with $p_0(z)\not\equiv 0$. It is well known that every solution f of equation (1.1) is either a polynomial or a transcendental entire function of positive rational order. Moreover, there are at most n distinct positive rational numbers which form the set of all the possible orders of transcendental solutions of equation (1.1). This list of rational numbers can be obtained either from the Newton-Puiseux diagram ([12], [13]), or from a simple arithmetic, which was developed in [7], with the degrees of the polynomial coefficients in (1.1). The interesting reader may refer to [7] for more specific information about the possible orders of transcendental solutions of an equation of the form (1.1).

In what follows, we denote the order of growth of an entire function f by $\rho(f)$, and the exponent of convergence of its zeros by $\lambda(f)$. We assume that the coefficients $p_0(z), \dots, p_{n-1}(z)$ in equation (1.1) are not all constants.

For equation (1.1), set $d_j=\deg p_j(z)$ and

$$\gamma=1+\max_{0\leq j\leq n-1}\frac{d_j}{n-j}. \tag{1.2}$$

Then it is known [8, p. 127] that any solution f of (1.1) satisfies

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$$\rho(f)\leqq\gamma. \tag{1.3}$$

In [7], it was shown that there always exists a solution of equation (1.1) that satisfies $\rho(f)=\gamma$, where γ is the constant in (1.2). In other words, the upper bound γ in (1.3) is always reached. Specifically, set

$$q=\min\left\{m\,\middle|\,\frac{d_m}{n-m}=\max_{0\leq j\leq n-1}\frac{d_j}{n-j}\right\}. \tag{1.4}$$

Then we have the following result, which is an easy consequence of Theorem 2 in [7].

THEOREM A. *If* $\{f_1, f_2, \dots, f_n\}$ *is a fundamental set of solutions of* (1.1), *then there are at least* $n-q$ *of them whose orders are* γ.

Note that, from (1.4), $0\leqq q\leqq n-1$. Hence, $n-q\geqq 1$. Therefore, it follows from Theorem A that, among any fundamental solution set $\{f_1, f_2, \dots, f_n\}$ of (1.1), there must exist a least one of them whose order is γ. Here γ is the constant in (1.2). There are examples in [7] which show that Theorem A is sharp.

In this paper we use the above Theorem A to investigate the zeros of a fundamental set of solutions of an equation of the form (1.1). Specifically, we characterize those equations of the form (1.1) which possess a fundamental solution set $\{f_1, f_2, \dots, f_n\}$ with the property that $\lambda(f_k)<\rho(f_k)$ for each $k=1, 2, \dots, n$. (In literature, such a property of f_k is called f_k has a Borel exceptional value at zero.)

Brüggemann [3] and Steinmetz [9] proved the following result independently.

THEOREM B. *Suppose that the coefficients of equation* (1.1) *are not all constants, and that the equation possesses a fundamental set of solutions* $\{f_1, f_2, \dots, f_n\}$ *with the property that* $\lambda(f_k)<\rho(f_k)$ *for each* $1\leqq k\leqq n$. *Then there exists a nonlinear polynomial* $Q(z)$ *such that*

$$f_k(z)=g_k(z)e^{Q(z)}, \qquad k=1, 2, \dots, n, \tag{1.5}$$

where $g_k(z)$ *is an entire function satisfying* $\rho(g_k)<\deg Q$.

Theorem B was originally a conjecture of Frank [5] and Wittich [11]. In [5] Frank proved Theorem B for the case when each f_k has only finitely many zeros. Both proofs of Theorem B in [3] and [9] are based on the theory of asymptotic integration.

Thus, for an equation of the form (1.1) which admits a fundamental set of solutions $\{f_1, f_2, \dots, f_n\}$ satisfying $\lambda(f_k)<\rho(f_k)$ for each $k=1, 2, \dots, n$, Theorem B gives a characterization of all solutions in the fundamental set. But then, what can be said about the equation? Our first result addresses this question. We obtain the following result.

THEOREM 1. *Given an equation of the form* (1.1) *where the coefficients are not all constants. Let* $p_k(z)=A_k z^{d_k}+\cdots$, $k=0, 1, \ldots, n-1$, *where* A_k *is a nonzero constant and* d_k *is the degree of* $p_k(z)$. *For each* $k=0, 1, \ldots, n-1$, *set*

$$A_k^*=\begin{cases} A_k & \text{if } \dfrac{d_k}{n-k}=\max\limits_{0\leq j\leq n-1}\dfrac{d_j}{n-j}, \\ 0 & \text{otherwise.}\end{cases} \tag{1.6}$$

Define a polynomial $h(t)$ *as follows*:

$$h(t)=t^n+\sum_{k=0}^{n-1} A_k^* t^k . \tag{1.7}$$

Then the equation (1.1) *admits a fundamental solution set* $\{f_1, f_2, \ldots, f_n\}$ *with the property that* $\lambda(f_k)<\rho(f_k)$ *for all* $k=1, 2, \ldots, n$, *if and only if* $h(t)=(t+b)^n$ *for some constant* $b\neq 0$.

We remark that if $h(t)=(t+b)^n$ for some constant $b\neq 0$, then each coefficient in equation (1.1) takes a special form. Specifically, we have for $j=0, 1, \ldots, n-1$,

$$p_j(z)=\binom{n}{j} b^{n-j} z^{\alpha(n-j)}+\cdots , \tag{1.8}$$

where α is a positive integer, and $\binom{n}{j}$ is the binomial coefficient. In other words, an equation of the form (1.1) which possesses a fundamental solution set $\{f_1, f_2, \ldots, f_n\}$ satisfying $\lambda(f_k)<\rho(f_k)$ for each $k=1, 2, \ldots, n$, if and only if there exist a nonzero constant b and a positive integer α such that the coefficient $p_j(z)$ of the equation takes the form (1.8) for all $j=0, 1, \ldots, n-1$. Hence, Theorem 1 gives a complete characterization for those equations which admit an exceptional fundamental solution set in the sense of Borel.

As an application of Theorem 1, we look at a special type of equations of the form (1.1), where we assume the degrees of the coefficients in equation (1.1) satisfy

$$\frac{d_j}{n-j}\leqq\frac{d_0}{n}, \qquad j=1, 2, \ldots, n-1 . \tag{1.9}$$

Then it is known that in this case every solution $f\not\equiv 0$ of (1.1) has the order $\rho(f)=1+d_0/n$. (See [13, Chapter 5] or [7, Theorem 1]). Hence, from Theorem 1, if the polynomial $h(t)$ defined in (1.7) has at least two distinct zeros, then among any fundamental set of solutions $\{f_1, f_2, \ldots, f_n\}$, there is at least one of them, say f_k, that satisfies $\lambda(f_k)=\rho(f_k)=1+d_0/n$. In other words, we proved the following result.

COROLLARY 2. *Given an equation of the form* (1.1) *where the coefficients are* *not all constants and satisfy* (1.9). *Suppose that the polynomial* $h(t)$ *in* (1.7) *has at least two distinct zeros. Then, if* $\{f_1, f_2, \dots, f_n\}$ *is a fundamental set of solutions for* (1.1), $\lambda(f_k)=1+d_0/n$ *for at least one* k *in the set* $\{1, 2, \dots, n\}$.

We remark that Corollary 2 improves Satz 2 in [1], where the same conclusion was obtained under the stronger assumption that $h(t)$ has only simple zeros.

Next, we investigate the quantity $\max_{1\leq j\leq n}\{\lambda(f_j)\}$. Here $\{f_1, f_2, \dots, f_n\}$ is any fundamental solution set for an equation of the form (1.1). From Theorem A we observe that $\max_{1\leq j\leq n}\{\rho(f_j)\}=\gamma$, where γ is the constant in (1.2). Thus, it follows that $\max_{1\leq j\leq n}\{\lambda(f_j)\}\leqq\gamma$. This relationship raises a natural question, namely what if $\max_{1\leq j\leq n}\{\lambda(f_j)\}<\gamma$? The following result gives a complete answer to this question.

THEOREM 3. *Given an equation of the form* (1.1) *where the coefficients are not all constants. Let* γ *be the constant in* (1.2) *and* $h(t)$ *the polynomial in* (1.7). *Then equation* (1.1) *admits a fundamental solution set* $\{f_1, f_2, \dots, f_n\}$ *satisfying* $\max_{1\leq j\leq n}\{\lambda(f_j)\}<\gamma$, *if and only if* $h(t)=(t+b)^n$ *for some contant* $b\neq 0$.

From inspection of Theorem 1 and Theorem 3, we see that if $\{f_1, f_2, \dots, f_n\}$ is a fundamental set of solutions of (1.1), then $\max_{1\leq j\leq n}\{\lambda(f_j)\}<\gamma$ if and only if $\lambda(f_k)<\rho(f_k)$ for all $1\leqq k\leqq n$.

Notice that if $h(t)=(t+b)^n$ for some constant $b\neq 0$, then the coefficient $p_j(z)$ in (1.1) takes the special form (1.8). From (1.8) we see that, in this case, the degree d_j of the coefficient $p_j(z)$ satisfies $d_j/(n-j)=d_0/n$ for all $j=1, 2, \dots, n-1$. Thus, from Theorem 3, we obtain immediately the following result.

COROLLARY 4. *Suppose that* $d_0/n\neq\max_{0\leq j\leq n-1}d_j/(n-j)$. *Then any fundamental solution set* $\{f_1, f_2, \dots, f_n\}$ *of* (1.1) *satisfies*

$$\max_{1\leq j\leq n}\{\lambda(f_j)\}=\gamma=1+\max_{0\leq j\leq n-1}\frac{d_j}{n-j}.$$

Finally, we examine the product function $E=f_1f_2\cdots f_n$, where $\{f_1, f_2, \dots, f_n\}$ is a fundamental set of solutions for an equation of the form (1.1). It is easily seen that $\lambda(E)=\max_{1\leq j\leq n}\{\lambda(f_j)\}$. Hence, Corollary 2, Theorem 3 and Corollary 4 also hold for E. However, for the order of E, we have in general only that $\rho(E)\leqq\max_{1\leq j\leq n}\{\rho(f_j)\}$. Note that, from Theorem A, $\max_{1\leq j\leq n}\{\rho(f_j)\}=\gamma$ where γ is the constant in (1.2). Hence, it follows that $\rho(E)\leqq\gamma$. Our final result in this paper shows that the situation $\rho(E)<\gamma$ will not occur unless all the coefficients of the equation are constants. We prove the following result.

THEOREM 5. *Given an equation of the form* (1.1) *where the coefficients are not all constants. Let* $\{f_1, f_2, \dots, f_n\}$ *be a fundamental set of solutions of* (1.1), *and let* $E=f_1f_2\cdots f_n$. *Then*

$$\rho(E)=\gamma=1+\max_{0\leq j\leq n-1}\frac{d_j}{n-j}.$$

Bank and Laine [2] studied intensively the product function $E=f_1f_2$, where f_1 and f_2 are two linearly independent solutions of

$$f''+A(z)f=0, \tag{1.10}$$

where $A(z)$ is an entire function. When $A(z)$ in (1.10) is a polynomial, they proved, among other things, that $\lambda(E)=\rho(E)=1+(1/2)\deg A$. The proof of this result is based on an identity satisfied by the function E, see [2]. Such an identity, however, does not seem to exist for the general higher order equation like (1.1). Apparently, Theorem 3 and Theorem 5 generalize this result of Bank and Laine. More than this, we provide a unified proof for all cases.

We also remark that a similar result to Theorem 5 was indicated in [9] and [10].

Finally, we remark that all the results obtained in this paper are no longer true without the assumption that the coefficients in (1.1) are not all constants. Let $c_1, c_2, \dots, c_n$ be the n distinct roots of $c^n=1$, and let $f_j(z)=\exp(c_jz)$ for each $j=1, 2, \dots, n$. Then $\{f_1, f_2, \dots, f_n\}$ forms a fundamental set of solutions of the following equation

$$f^{(n)}-f=0. \tag{1.11}$$

Furthermore, for this particular fundamental set, the function $E=f_1f_2\cdots f_n=1$. It can be seen that all the results obtained are not valid for this fundamental set of equation (1.11). Note that (1.11) is an equation of the form (1.1) but with constant coefficients.

In § 2 we give a lemma which we will use in the proofs of Theorem 1 and Theorem 3. We prove Theorem 1 in § 3. We prove the necessity part of Theorem 1 by using Theorem A and Theorem B, and then we use Theorem 1 to prove Theorem 3 in § 4. Theorem 5 follows from Theorem 3, and the proof will be given in § 5. In § 6 we give a remark on an improvement of Theorem 1.

2. An auxiliary result

We use the following lemma in the proofs of Theorem 1 and Theorem 3.

LEMMA. *Let $f\not\equiv 0$ be a solution of an equation of the form* (1.1), *and $Q\not\equiv 0$ be an entire function. Set*

$$u=fe^{-Q}. \tag{2.1}$$

Then u satisfies the following equation

$$u^{(n)}+b_{n-1}(z)u^{(n-1)}+\cdots+b_0(z)u=0, \tag{2.2}$$

where the coefficients $b_0(z), \dots, b_{n-1}(z)$ may be expressed in the form

$$b_{n-1}=p_{n-1}+nQ',$$
(2.3)
$$b_j=p_j+\sum_{k=1}^{n-j}\left[\binom{j+k}{k}(Q')^k+H_{k-1}(Q')\right]p_{j+k}$$

for $j=0, 1, \dots, n-2$. *Here* $H_{k-1}(Q')$ *stands for a differential polynomial of total degree* $\leqq k-1$ *in* Q' *and its derivatives, with constant coefficients, and* $\binom{j+k}{k}$ *stands for the binomial coefficients. Moreover, we set* $p_n(z)\equiv 1$ *in* (2.3).

Furthermore, suppose that the constant γ *in* (1.2) *is an integer which is greater than one, and that the function* $Q(z)$ *in* (2.1) *is a polynomial of degree* γ *having the form*

$$Q(z)=\frac{1}{\gamma}cz^{\gamma}+\cdots, \qquad c\neq 0. \tag{2.4}$$

Let $h(t)$ *be the polynomial defined in* (1.7). *Then the coefficients of equation* (2.2) *are polynomials taking the form*

$$b_j(z)=\frac{h^{(j)}(c)}{j!}z^{(n-j)(\gamma-1)}+\cdots, \qquad j=0, 1, \dots, n-1. \tag{2.5}$$

Proof. From (2.1) we have $f=ue^Q$. Then, by induction, it can be proved that for each $p=1, 2, \dots, n$,

$$f^{(p)}=\left(u^{(p)}+pQ'u^{(p-1)}+\sum_{k=2}^{p}\left[\binom{p}{k}(Q')^k+H_{k-1}(Q')\right]u^{(p-k)}\right)e^Q. \tag{2.6}$$

Substituting (2.6) into equation (1.1) we obtain (2.2) and (2.3).

To get (2.5), we make use of (2.3). By (1.6), we may rewrite $p_k(z)$ in the following form

$$p_k(z)=A_k^* z^{(n-k)(\gamma-1)}+\cdots, \tag{2.7}$$

for all $k=0, 1, \dots, n-1$. Then, by inspection ot the leading term of $b_j(z)$ in (2.3), we obtain (2.5) from (2.4), (2.7), and the definition of $h(t)$ in (1.7). This proves the lemma.

3. Proof of Theorem 1

Proof of Necessity. Suppose that the equation (1.1) possesses a fundamental solution set $\{f_1, f_2, \dots, f_n\}$ which satisfies $\lambda(f_k)<\rho(f_k)$ for all $k=1, 2, \dots, n$. Then we show that $h(t)=(t+b)^n$ for some constant $b\neq 0$.

From Theorem B, there exists a nonlinear polynomial $Q(z)$ such that

$$f_k(z)=g_k(z)e^{Q(z)}, \qquad k=1, 2, \dots, n, \tag{3.1}$$

where $g_k(z)$ is an entire function satisfying

$$\rho(g_k)<\deg Q(z), \qquad k=1, 2, \dots, n. \tag{3.2}$$

Hence, it follows from (3.1) and (3.2) that $\rho(f_k)=\deg Q$ for all $1\leqq k\leqq n$. Combining this fact with Theorem A imply that $\deg Q=\gamma$, where γ is the constant in (1.2). Thus, γ is an integer which is greater than one.

Write

$$Q(z)=\frac{1}{\gamma}cz^{\gamma}+\cdots, \tag{3.3}$$

where $c\neq 0$ is a constant, and set

$$f=ge^{Q}. \tag{3.4}$$

By substituting (3.4) into equation (1.1), we obtain from the lemma in § 2 that g solves the following equation

$$g^{(n)}+b_{n-1}(z)g^{(n-1)}+\cdots+b_0(z)g=0, \tag{3.5}$$

where the coefficients $b_0(z), \dots, b_{n-1}(z)$ are polynomials having the form

$$b_j(z)=\frac{h^{(j)}(c)}{j!}z^{(n-j)(\gamma-1)}+\cdots, \qquad j=0, 1, \dots, n-1. \tag{3.6}$$

Here $h(t)$ is the polynomial defined in (1.7) and γ is the contant in (1.2).

Since $\{f_1 f_2, \dots, f_n\}$ is a fundamental solution set for (1.1), it follows from (3.1) and (3.4) that $\{g_1, g_2, \dots, g_n\}$ is a fundamental solution set for equation (3.5). Moreover, from (3.2) and (3.3), we see that

$$\rho(g_k)<\gamma \qquad \text{for all} \quad k=1, 2, \dots, n. \tag{3.7}$$

Therefore, by Theorem A, we conclude from (3.6) that

$$h^{(j)}(c)=0 \qquad \text{for all} \quad j=0, 1, \dots, n-1. \tag{3.8}$$

Since otherwise, from (3.6), we would obtain for equation (3.5) that

$$1+\max_{0\leq j\leq n-1}\frac{\deg b_j}{n-j}=\gamma,$$

and then, from Theorem A, there would be at least one of the solutions in the set $\{g_1, g_2, \dots, g_n\}$ which has order γ. But this contradicts (3.7).

Note that, by (1.7), $h(t)$ is a polynomial of degree n. Therefore, from (3.8), we obtain

$$h(t)=(t-c)^n,$$

where c is the constant in (3.3). By taking $b=-c$, and observing that c is a nonzero constant, we obtain our result. This proves the necessity part of Theorem 1.

Proof of Sufficiency. Let $h(t)$ be the polynomial defined in (1.7), and suppose that $h(t)=(t+b)^n$ for some constant $b\neq 0$. Then, by the definition of $h(t)$ in

(1.7), we see that the leading term of each coefficient in (1.1) takes a special form, namely,

$$p_j(z)=\binom{n}{j}b^{n-j}z^{\alpha(n-j)}+\cdots, \qquad j=0, 1, \dots, n-1, \tag{3.9}$$

where α is a positive integer; see also (1.8). Hence, the relation (1.9) is satisfied with equality, and consequently, every solution $f\not\equiv 0$ of (1.1) has the order

$$\rho(f)=1+\alpha. \tag{3.10}$$

Set

$$f=u\exp\left(-\frac{b}{1+\alpha}z^{1+\alpha}\right). \tag{3.11}$$

Since f is a solution of (1.1), a substitution of (3.11) into (1.1) implies that u satisfies the equation

$$u^{(n)}+b_{n-1}(z)u^{(n-1)}+\cdots+b_0(z)u=0. \tag{3.12}$$

Here the coefficients $b_0(z), \dots, b_{n-1}(z)$ are polynomials taking the form

$$b_j(z)=B_jz^{\alpha(n-j)-1}+\cdots, \qquad j=0, 1, \dots, n-1, \tag{3.13}$$

where B_j is a constant, which may or may not be zero. Note that we obtain the leading term of $b_j(z)$ in (3.13) by applying a similar argument as to (2.5) in § 2.

Hence, from (1.3) and (3.13), we obtain that the order of any solution of (3.12) satisfies

$$\rho(u)\leqq 1+\max_{0\leq j\leq n-1}\frac{\deg b_j}{n-j}\leqq 1+\max_{0\leq j\leq n-1}\frac{\alpha(n-j)-1}{n-j}<1+\alpha. \tag{3.14}$$

Now, let $\{u_1, u_2, \dots, u_n\}$ be a fundamental solution set for (3.12). Then, by choosing

$$f_k=u_k\exp\left(-\frac{b}{1+\alpha}z^{1+\alpha}\right), \qquad k=1, 2, \dots, n, \tag{3.15}$$

we obtain from (3.11) that $\{f_1, f_2, \dots, f_n\}$ forms a fundamental set of solutions for (1.1). For this particular solution set, we see from (3.15) and (3.14) that $\lambda(f_k)\leqq\rho(u_k)<\rho(f_k)$ holds for all $k=1, 2, \dots, n$. This shows the existence of a fundamental solution set for (1.1) that has the desired property. Hence, the sufficiency part is proved.

4. Proof of Theorem 3

The sufficiency is covered by the corresponding part in Theorem 1. We only prove the necessity.

We suppose that the equation (1.1) admits a fundamental solution set $\{f_1,$

$f_2, \dots, f_n\}$ satisfying $\max_{1\le j\le n}\{\lambda(f_j)\}<\gamma$, where γ is the constant in (1.2). Since γ is the order of at least one solution in this fundamental set (by Theorem A), γ is an integer.

Again, let $h(t)$ be the polynomial defined in (1.7). Then we will show that $h(t)=(t+b)^n$ for some constant $b\neq 0$.

We choose a number $c_0\neq 0$ such that $h(c_0)\neq 0$, and set

$$f=u\exp\left(\frac{1}{\gamma}c_0z^\gamma\right). \tag{4.1}$$

Then, by substituting (4.1) into equation (1.1), we obtain from the lemma in § 2 that u satisfies the equation

$$u^{(n)}+b_{n-1}(z)u^{(n-1)}+\cdots+b_0(z)u=0, \tag{4.2}$$

where $b_0(z), \dots, b_{n-1}(z)$ are polynomials having the form

$$b_j(z)=\frac{h^{(j)}(c_0)}{j!}z^{(n-j)(\gamma-1)}+\cdots \tag{4.3}$$

for all $j=0, 1, \dots, n-1$.

Set

$$u_k=f_k\exp\left(-\frac{1}{\gamma}c_0z^\gamma\right), \qquad k=1, 2, \dots, n. \tag{4.4}$$

Since $\{f_1, f_2, \dots, f_n\}$ is a fundamental set, so is $\{u_1, u_2, \dots, u_n\}$. Specifically, from (4.1) and (4.4), $\{u_1, u_2, \dots, u_n\}$ is a fundamental set of solutions of equation (4.2).

Recall that the constant c_0 is so chosen that $h(c_0)\neq 0$. Hence, from (4.3) we see that $\deg b_0=n(\gamma-1)$ and $(\deg b_j)/(n-j)\leqq(\deg b_0)/n$, $j=1, 2, \dots, n-1$. Thus, it follows that every solution $u\not\equiv 0$ of (4.2) has the order $1+\deg b_0/n=\gamma$. (See the paragraph before Corollary 2 in § 1). In particular, we have

$$\rho(u_k)=\gamma, \qquad k=1, 2, \dots, n. \tag{4.5}$$

On the other hand, from (4.4), we have $\lambda(u_k)=\lambda(f_k)$ for each $k=1, 2, \dots, n$. Hence, by hypothesis, we obtain that $\max\{\lambda(u_j)\}=\max\{\lambda(f_j)\}<\gamma$. Combining this fact with (4.5) we conclude that the set $\{u_1, u_2, \dots, u_n\}$, which is a fundamental solution set for (4.2), has the property that $\lambda(u_k)<\rho(u_k)$ for all $k=1, 2, \dots, n$. Therefore, we may apply Theorem 1 to equation (4.2). Doing this implies that there exists a nonzero constant b_0 such that

$$t^n+\sum_{j=0}^{n-1}\frac{h^{(j)}(c_0)}{j!}t^j=(t+b_0)^n \tag{4.6}$$

holds for any complex number t. Here the left-hand side of (4.6) is the polynomial in (1.7) with respect to equation (4.2).

By comparing likewise terms, we obtain from (4.6) that

$$\frac{h^{(j)}(c_0)}{j!}=\binom{n}{j}b_0^{n-j},$$

i.e.,

$$h^{(j)}(c_0)=\frac{n!}{(n-j)!}b_0^{n-j}, \qquad j=0, 1, \dots, n-1. \tag{4.7}$$

Recall the definition of $h(t)$ in (1.7), we obtain from (4.7) that

$$\frac{n!}{(n-j)!}c_0^{n-j}+\frac{(n-1)!}{(n-j-1)!}c_0^{n-j-1}A_{n-1}^*+\cdots+j!A_j^*=\frac{n!}{(n-j)!}b_0^{n-j} \tag{4.8}$$

for all $j=0, 1, \dots, n-1$.

Next, we will use the recurrence relation (4.8) to show that

$$A_j^*=\binom{n}{j}(b_0-c_0)^{n-j}, \qquad j=0, 1, \dots, n-1. \tag{4.9}$$

Observe that, once (4.9) has been established, then necessity part of Theorem 3 will follow immediately, because from (4.9) and (1.7), we obtain $h(t)=(t+b_0-c_0)^n$. Obviously, $b_0\neq c_0$, since otherwise, by (4.9), all A_j^* would be zero, which is impossible by the definition of A_j^* (see (1.6)). Hence, by taking $b=b_0-c_0$, we obtain the necessity part of Theorem 3.

To prove (4.9), we use induction on j. Consider $j=n-1$. Then we obtain from (4.8) that

$$nc_0+A_{n-1}^*=nb_0.$$

Hence, it follows that $A_{n-1}^*=n(b_0-c_0)$, which shows that (4.9) holds when $j=n-1$.

For the induction step, we now assume that for some $0\leqq m\leqq n-2$, (4.9) holds for all $j=m+1, m+2, \dots, n-1$, i.e.,

$$A_j^*=\binom{n}{j}(b_0-c_0)^{n-j}, \qquad j=m+1, m+2, \dots, n-1. \tag{4.10}$$

Then we show that (4.9) holds for $j=m$.

Substituting (4.10) into (4.8) with $j=m$, we obtain

$$\frac{n!}{(n-m)!}c_0^{n-m}+\frac{(n-1)!}{(n-m-1)!}c_0^{n-m-1}\binom{n}{n-1}(b_0-c_0)+\cdots$$

$$\cdots+\frac{(m+1)!}{1!}c_0\binom{n}{m+1}(b_0-c_0)^{n-m-1}+m!A_m^*=\frac{n!}{(n-m)!}b_0^{n-m},$$

$$\frac{n!}{(n-m)!}\left[c_0^{n-m}+\binom{n-m}{1}c_0^{n-m-1}(b_0-c_0)+\cdots+\binom{n-m}{n-m-1}c_0(b_0-c_0)^{n-m-1}\right]$$

$$+m!A_m^*=\frac{n!}{(n-m)!}b_0^{n-m},$$

$$\frac{n!}{(n-m)!}(c_0+(b_0-c_0))^{n-m}-\frac{n!}{(n-m)!}(b_0-c_0)^{n-m}+m!A_m^*=\frac{n!}{(n-m)!}b_0^{n-m},$$

from which it follows that

$$A_m^*=\frac{n!}{(n-m)!\cdot m!}(b_0-c_0)^{n-m}. \tag{4.11}$$

However, (4.11) is (4.9) with $j=m$. Hence, this proves the induction step, and therefore completes the proof of the necessity of Theorem 3.

5. Proof of Theorem 5

Let $\{f_1, f_2, \dots, f_n\}$ be a fundamental set of solutions of (1.1), and $E=f_1f_2\cdots f_n$. Recall that $\lambda(E)=\max_{1\leq j\leq n}\{\lambda(f_j)\}$ and that $\lambda(E)\leqq\rho(E)\leqq\gamma$, where γ is the constant in (1.2). Hence, if $\max_{1\leq j\leq n}\{\lambda(f_j)\}=\gamma$, then it follows immediately that $\rho(E)=\gamma$.

Thus, we need only to consider the case when $\max_{1\leq j\leq n}\{\lambda(f_j)\}<\gamma$. In this case, from the remark following Theorem 3, we see that the fundamental set $\{f_1, f_2, \dots, f_n\}$ satisfies $\lambda(f_k)<\rho(f_k)$ for all $1\leqq k\leqq n$. Hence, by Theorem B, there exists a nonlinear polynomial $Q(z)$ such that

$$f_k(z)=g_k(z)e^{Q(z)}, \qquad k=1, 2, \dots, n, \tag{5.1}$$

where $g_k(z)$ is an entire function satisfying

$$\rho(g_k)<\deg Q(z), \qquad k=1, 2, \dots, n. \tag{5.2}$$

Moreover, it follows from Theorem A that

$$\deg Q=\gamma. \tag{5.3}$$

Therefore, from (5.1) we obtain

$$E=f_1f_2\cdots f_n=(g_1g_2\cdots g_n)e^{nQ}. \tag{5.4}$$

Then, by (5.2), (5.3), and (5.4), if follows immediately that $\rho(E)=\gamma$. This proves Theorem 5.

6. A remark

We remark that Theorem 1 in § 1 can be improved in a certain sence. Specifically, let $\lambda_{NR}(f)$ denote the exponent of convergence of *nonreal* zeros of f, then we can characterize those equations of the form (1.1) which possess a fundamental solution set having few *nonreal* zereos. Corresponing to Theorem 1, we have the following result.

THEOREM 6. *Assume the hypotheses of Theorem* 1. *Then the equation* (1.1) *admits a fundamental solution set* $\{f_1, f_2, \ldots, f_n\}$ *with the property that*

$$\lambda_{NR}(f_k)<\rho(f_k), \qquad k=1, 2, \ldots, n, \tag{6.1}$$

if and only if $h(t)=(t+b)^n$ *for some constant* $b\neq 0$.

For the proof of Theorem 6, we may apply the same argument as in § 3. Since $\lambda_{NR}(f)<\lambda(f)$ holds trivially for any entire function f, the sufficiency part of Theorem 6 follows immediately from that of Theorem 1. To prove the necessity part of Theorem 6, we use a result of Brüggemann, see [4, Theorem 5]. As see from § 3, the proof of the necessity of Theorem 1 depends heavily on Theorem B in § 1. However, Brüggemann [4] showed that the conclusion of Theorem B still holds under the weaker assumption that the equation (1.1) possesses a fundamental set of solutions $\{f_1, f_2, \ldots, f_n\}$ satisfying (6.1). Hence, by using this result of Brüggemann instead, and applying the same argument as in § 3, we obtain the necessity part of Theorem 6.

The following result follows easily from Theorem 6 and the fact discussed in the paragraph before Corollary 2 in § 1.

COROLLARY 7. *Let* $P(z)$ *be a polynomial of degree* $d\geqq 1$, *and let* $\{f_1, f_2, \ldots, f_n\}$ *be a fundamental set of solutions of*

$$f^{(n)}+P(z)f=0, \qquad n\geqq 2.$$

Then at least one of $f_1, f_2, \ldots, f_n$ *has the property that its sequence of nonreal zeros has exponent of convergence equal to* $(n+d)/n$.

We remark that Corollary 7 generalizes Theorem 1 in [6], where the same result was obtained for the case $n=2$.

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