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KODAI MATH. J.
20 (1997), 135—142

SUBMANIFOLDS WHOSE QUADRIC REPRESENTATIONS SATISFY $\Delta\tilde{x}=B\tilde{x}+C$

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Abstract

Let $x: M^n \rightarrow E^m$ be an isometric immersion of an n-dimensional Riemannian manifold into the m-dimensional Euclidean space. Then the map $\tilde{x}=xx^t$ (where t denotes transpose) is called the quadric representation of M^n. In this paper, we give some results on submanifolds in the Euclidean space E^m which satisfy $\Delta\tilde{x}=B\tilde{x}+C$, where B and C are two constant matrices.

1. Introduction

Let $x: M^n \rightarrow E^m$ be an isometric immersion of an n-dimensional Riemannian manifold into the m-dimensional Euclidean space, and $SM(m)$ be the $m \times m$ real symmetric matrices space (this space becomes the standard $(1/2)m(m+1)$-dimensional Euclidean space when equipped with the metric $g(P, Q)=(1/2)\mathrm{tr}(PQ)$[4]). We regard x as a column matrix in E^m and denote by x^t the transpose of x. Let $\tilde{x}=xx^t$. Then we obtain a smooth map $\tilde{x}: M^n \rightarrow SM(m)$. Since the coordinates of $\tilde{x}$ depend on the coordinates of x in a quadric manner, we call $\tilde{x}$ the quadric representation of M^n ([5]). $\tilde{x}$ is an important map, because it has many interesting relations with the geometric properties of the submanifold. In fact, for the hypersphere centered at the origin embedded in the Euclidean space in the standard way, the quadric representation is just the second standard embedding of the sphere. In [5], I. Dimitric established some general results about the quadric representation, in particular those relative to the condition of $\tilde{x}$ being of finite type. In [8], the author gave some classification results for hypersurfaces in E^m which satisfy $\Delta\tilde{x}=B\tilde{x}+C$ with B and C are two constant matrices. In this paper, we will study submanifolds in E^m which satisfy the same condition $\Delta\tilde{x}=B\tilde{x}+C$. We prove that an n-dimensional submanifold with parallel mean curvature vector which satisfy $\Delta\tilde{x}=B\tilde{x}+C$ must be the n-dimensional Euclidean space or contained in a quadric hypersurface, but there is no submanifold in E^m satisfying $\Delta\tilde{x}=B\tilde{x}$. We also prove that the only submanifold in E^m satisfying $\Delta\tilde{x}=C$ is a lower dimensional Euclidean space.

Keywords: submanifold, quadric representation, mean curvature vector. 53C40.
Received November 26, 1996; revised April 7, 1997.

2. Preliminaries

Let us fix the notations first. Let $x: M^n \to E^m$ be an isometric immersion of an n-dimensional Riemannian manifold into the m-dimensional Euclidean space. We denote by H the mean curvature vector of M^n in E^m. Let $e_1, \dots, e_n, e_{n+1}, \dots, e_m$ be local orthonormal vector fields along M^n, such that $e_1, \dots, e_n$ are tangent to M^n, $e_{n+1}, \dots, e_m$ are normal to M^n, and e_{n+1} is parallel to H. Then $H=\alpha e_{n+1}$, where α is the mean curvature of M^n in E^m. Let $\langle\,,\,\rangle$ and $\bar{\nabla}$ be the Euclidean metric and the connection of E^m, and denote by ∇, h, D, A_r, $|A_r|$ respectively, the connection of M^n, the second fundamental form of M^n in E^m, the normal connection of M^n in E^m, the Weigarten endomorphism relative to the normal direction e_r, and the length of A_r, $r=n+1, \dots, m$.

In this setting, the indices i, j, k always range from 1 to n, r, s, t from $n+1$ to m and β, γ from $n+2$ to m. At any point $x \in M^n$, for any column vector V in E^m, we denote by $V_T=\sum_i\langle V, e_i\rangle e_i$, $V_N=\sum_r\langle V, e_r\rangle e_r$, and $V_{\bar{N}}=\sum_\beta\langle V, e_\beta\rangle e_\beta$.

We define a map $*$ from $E^m \times E^m$ into $SM(m)$ by $V*W=VW^t+WV^t$, for column vectors V and W in E^m. Then $V*W=W*V$. Let $\tilde{\nabla}$ denotes the Euclidean connection of $SM(m)$, then we have ([5]):

$$\tilde{\nabla}_V(W_1*W_2)=(\bar{\nabla}_V W_1)*W_2+W_1*(\bar{\nabla}_V*W_2), \tag{2.1}$$

$$g(V_1*V_2, W_1*W_2)=\langle V_1, W_1\rangle\langle V_2, W_2\rangle+\langle V_1, W_2\rangle\langle V_2, W_1\rangle, \tag{2.2}$$

and

$$\Delta(V*W)=(\Delta V)*W+V*(\Delta W)-2\sum_i(\bar{\nabla}_{e_i}V)*(\bar{\nabla}_{e_i}W), \tag{2.3}$$

where V, W, W_1, W_2, V_1 and V_2 are all vectors in E^m, and Δ is the Laplacian operator of M^n.

Using (2.3), by a lengthy but direct computation, we have

$$\Delta\tilde{x}=-n\alpha e_{n+1}*x-\sum_i e_i*e_i, \tag{2.4}$$

and

$$\begin{aligned}\Delta^2\tilde{x}=&4ne_{n+1}*\mathrm{grad}\alpha\\ &+(n^2\alpha^2+2|A_{n+1}|^2)e_{n+1}*e_{n+1}\\ &+2n(D_{\mathbf{grad}\,\alpha}e_{n+1})*x-n\alpha(\Delta^D e_{n+1})*x\\ &+4\textstyle\sum_\beta(\mathrm{tr}A_{n+1}A_\beta)e_{n+1}*e_\beta-n^2\alpha(\mathrm{grad}\,\alpha)*x\\ &-2n(A_{n+1}\,\mathrm{grad}\,\alpha)*x-2n\alpha(\textstyle\sum_i A_{D_{e_i}e_{n+1}}e_i)*x\\ &+4n\alpha\textstyle\sum_i(D_{e_i}e_{n+1})*e_i+2\textstyle\sum_{\beta,\gamma}(\mathrm{tr}A_\beta A_\gamma)e_\beta*e_\gamma\end{aligned} \tag{2.5}$$

$$-n\alpha\sum_\beta(\mathrm{tr}A_\beta A_{n+1})e_\beta*x-2\sum_{r,i}(A_re_i)*(A_re_i)$$
$$-n(\Delta\alpha+\alpha|A_{n+1}|^2)e_{n+1}*x-2n\alpha\sum_i(A_{n+1}e_i)*e_i .$$

Without noting, in this paper, we always denote by X, Y and Z the tangent vector of M^n, by ξ and η the normal vector of M^n in E^m, and V and W the column vector in E^m.

3. Submanifolds satisfying $\Delta\tilde{x}=B\tilde{x}+C$

THEOREM 3.1. *Let $x: M^n\to E^m$ be an isometric immersion with the parallel mean curvature vector. If its quadric representation satisfies $\Delta\tilde{x}=B\tilde{x}+C$, then M^n must be (a piece of) the n-dimensional Euclidean space or contained in a quadric hypersurface.*

Proof. If M^n is a minimal submanifold of E^m, that is $\alpha=0$, then (2.4) and (2.5) become $\Delta\tilde{x}=-\sum_i e_i*e_i$ and $\Delta^2\tilde{x}=2\sum_{r,s}(\mathrm{tr}A_rA_s)e_r*e_s-2\sum_{r,i}(A_re_i)*(A_re_i)$. Since $\Delta\tilde{x}=B\tilde{x}+C$, then $\Delta^2\tilde{x}=B(\Delta\tilde{x})$. Applying $g(\tilde{\ }, e_r*e_r)$ to this relation and summing on r, we have $\sum_r|A_r|^2=0$. Then M^n is a totally geodesic submanifold of E^m, that is to say that M^n is (a piece of) the n-dimensional Euclidean space.

Moreover, we can easily prove that when $M^n=E^n$,

$$\Delta\tilde{x}=-\begin{pmatrix} 2 & & & \\ & \ddots & & \\ & & 2 & \\ & & & 0 \end{pmatrix}.$$

Now we suppose $\alpha\neq0$. Since M^n has the parallel curvature vector in E^m, then $De_{n+1}=0$ and α is a constant. Differentiating $\Delta\tilde{x}=B\tilde{x}+C$ along X, an arbitrary tangent vector of M^n, we have

$$\begin{aligned}(3.1)\qquad & n\alpha e_{n+1}*X+2\sum_r(A_rX)*e_r\\ &+(BX)x^t+(Bx)X^t-n\alpha(A_{n+1}X)*x=0.\end{aligned}$$

Finding the e_r*e_s component of (3.1) we have

$$(3.2)\qquad \langle BX, e_r\rangle\langle x, e_s\rangle+\langle BX, e_s\rangle\langle x, e_r\rangle=0.$$

In (3.2), let $r=s$ and sum on s, we know

$$(3.3)\qquad \langle BX, x_N\rangle=0.$$

From (3.2) we also have

(3.4) $$\langle BX, e_r\rangle x_N+\langle x, e_r\rangle(BX)_N=0.$$

Combining (3.3) and (3.4), we have $(BX)_N=0$ or $x_N=0$.

CASE 1. $x_N=0$. In this case $x=x_T$ and for any tangent vector Y of M^n, $0=Y\langle e_{n+1}, x\rangle=-\langle A_{n+1}x, Y\rangle$. Thus $A_{n+1}x=0$. Finding the $e_{n+1}*Y$ component of (3.1) we have

(3.5) $$\langle BX, e_{n+1}\rangle\langle x, Y\rangle+2n\alpha\langle X, Y\rangle$$
$$+\langle Bx, e_{n+1}\rangle\langle X, Y\rangle+4\langle A_{n+1}X, Y\rangle=0.$$

In (3.5), let $X=Y=x$, we know

(3.6) $$\langle Bx, e_{n+1}\rangle=-n\alpha.$$

In (3.5), let $X=Y=e_i$ and sum on i, we have

(3.7) $$(n+1)\langle Bx, e_{n+1}\rangle+2n(n+2)\alpha=0.$$

Combining (3.6) with (3.7) we obtain $\alpha=0$, this is a contradiction with the assumption $\alpha\neq 0$.

CASE 2. $x_N\neq 0$, but $\langle x, e_{n+1}\rangle=0$. Obviously $(BX)_N=0$. Finding the $e_{n+1}*Y$ component of (3.1) we have

(3.8) $$\langle Bx, e_{n+1}\rangle\langle X, Y\rangle+2n\alpha\langle X, Y\rangle+4\langle A_{n+1}X, Y\rangle=0.$$

In (3.8), let $X=Y=e_i$, we have

(3.9) $$\langle Bx, e_{n+1}\rangle=-2(n+2)\alpha.$$

Combining (3.8) with (3.9), we obtain $A_{n+1}X=\alpha X$. Moreover, differentiating $\langle e_{n+1}, x\rangle$ along X, we have $A_{n+1}x_T=0$. Then we have $\alpha x_T=0$. If $x_T=0$, for any tangent vector X, we know that $X\langle x, x\rangle=0$ and $\langle x, x\rangle$ is a constant. This means that M^n is contained in a hypersphere, which is certainly a quadric hypersurface. If $x_T\neq 0$ we have $\alpha=0$. This is a contradiction with the assumption that $\alpha\neq 0$.

CASE 3. $\langle e_{n+1}, x\rangle\neq 0$. Obviously $x_N\neq 0$ and $(BX)_N=0$. Then Finding the $e_{n+1}*Y$ component of (3.1) we have

(3.10) $$-2n\alpha\langle A_{n+1}X, Y\rangle\langle e_{n+1}, x\rangle$$
$$+2n\alpha\langle X, Y\rangle+\langle Bx, e_{n+1}\rangle\langle X, Y\rangle$$
$$+\langle BX, Y\rangle\langle x, e_{n+1}\rangle+4\langle A_{n+1}X, Y\rangle=0.$$

From the above relation we know

(3.11) $$\langle BX, Y\rangle=\langle X, BY\rangle.$$

Substituting $Y=x_T$ in (3.10), we have

$$
\begin{aligned}
(3.12)\qquad & -2n\alpha\langle A_{n+1}X,\ x\rangle\langle e_{n+1},\ x\rangle \\
& +2n\alpha\langle X,\ x\rangle+\langle Bx,\ e_{n+1}\rangle\langle X,\ x\rangle \\
& +\langle BX,\ x_T\rangle\langle x,\ e_{n+1}\rangle+4\langle A_{n+1}X,\ x\rangle=0.
\end{aligned}
$$

Finding the $Y*Z$ component of (3.1) we obtain

$$
\begin{aligned}
(3.13)\qquad & \langle BX,\ Y\rangle\langle x,\ Z\rangle+\langle BX,\ Z\rangle\langle x,\ Y\rangle \\
& +\langle Bx,\ Z\rangle\langle X,\ Y\rangle-2n\alpha\langle A_{n+1}X,\ Y\rangle\langle x,\ Z\rangle \\
& +\langle Bx,\ Y\rangle\langle X,\ Z\rangle-2n\alpha\langle A_{n+1}X,\ Z\rangle\langle x,\ Y\rangle=0.
\end{aligned}
$$

Let $Y=Z=e_i$ in the above equation and sum on i, we have

$$(3.14)\qquad \langle BX,\ x_T\rangle+\langle Bx,\ X\rangle-2n\alpha\langle A_{n+1}X,\ x_T\rangle=0.$$

Combining (3.12) with (3.14) we have

$$(3.15)\qquad (\langle Bx,\ e_{n+1}\rangle+2n\alpha)x_T-\langle x,\ e_{n+1}\rangle(Bx)_T+4A_{n+1}x_T=0.$$

In (3.13), let $X=Y=e_i$ and sum on i, we know

$$
\begin{aligned}
(3.16)\qquad & (\sum_i\langle Be_i,\ e_i\rangle-2n^2\alpha^2)x_T \\
& +Bx_T+(n+1)(Bx)_T-2n\alpha A_{n+1}x_T=0.
\end{aligned}
$$

But by using (3.11), we can from (3.14) obtain

$$(3.17)\qquad Bx_T+(Bx)_T=2n\alpha A_{n+1}x_T.$$

Combining (3.17) with (3.15) and (3.16), we know

$$
\begin{aligned}
(3.18)\qquad & 4nA_{n+1}x_T \\
& =-\{n\langle Bx,\ e_{n+1}\rangle+(\textstyle\sum_i\langle Be_i,\ e_i\rangle-2n^2\alpha^2)\langle e_{n+1},\ x\rangle+2n^2\alpha\}x_T.
\end{aligned}
$$

In (3.10), let $X=Y=e_i$ and sum on i, we obtain

$$
\begin{aligned}
(3.19)\qquad & n\langle Bx,\ e_{n+1}\rangle+2n^2\alpha+4n\alpha \\
& +(\textstyle\sum_i\langle Be_i,\ e_i\rangle-2n^2\alpha^2)\langle e_{n+1},\ x\rangle=0.
\end{aligned}
$$

Combining (3.18) with (3.19) we know

$$(3.20)\qquad A_{n+1}x_T=\alpha x_T.$$

Then

$$
\begin{aligned}
(3.21)\qquad & X\langle Bx,\ x\rangle=\langle BX,\ x\rangle+\langle Bx,\ X\rangle \\
& =2n\alpha\langle A_{n+1}X,\ x\rangle=2n\alpha^2\langle X,\ x\rangle=n\alpha^2X\langle x,\ x\rangle,
\end{aligned}
$$

where the second equation used (3.14) and the third equation used (3.20). Thus $X\langle(B-n\alpha^2I)x, x\rangle=0$ and

$$\langle(B-n\alpha^2I)x, x\rangle=C_0,$$

where I is the identity matrix in $SM(m)$ and C_0 is a constant.

If $B=n\alpha^2I$, substituting $B=n\alpha^2I$ in (3.19) we obtain $2n(n+2)\alpha=0$. Thus $\alpha=0$, this is a contradiction. If $B\neq n\alpha^2I$, then the above equation tell us that M^n is contained in a quadric hypersurface. ∎

THEOREM 3.2. *There does not exist a submanifold in E^m whose quadric representation satisfies $\Delta\tilde{x}=B\tilde{x}$.*

Proof. Obviously a submanifold in E^m satisfying $\Delta\tilde{x}=B\tilde{x}$ can not be minimal. Then $\alpha\neq0$ and

$$B\tilde{x}+\sum_i e_i*e_i+n\alpha e_{n+1}*x=0. \tag{3.22}$$

Finding the e_j*e_j component of the above equation and sum on j, we have

$$\langle Bx, x_T\rangle+2n=0. \tag{3.23}$$

Applying $g(\tilde{~}, e_{n+1}*Y)$ to (3.22), we have

$$2n\alpha\langle Y, x\rangle+\langle Bx, e_{n+1}\rangle\langle x, Y\rangle+\langle Bx, Y\rangle\langle e_{n+1}, x\rangle=0, \tag{3.24}$$

and then

$$(2n\alpha+\langle Bx, e_{n+1}\rangle)x_T+\langle e_{n+1}, x\rangle(Bx)_T=0. \tag{3.25}$$

But applying $g(\tilde{~}, e_{n+1}*e_{n+1})$ to (3.22), we have

$$(2n\alpha+\langle Bx, e_{n+1}\rangle)\langle e_{n+1}, x\rangle=0. \tag{3.26}$$

Then $(2n\alpha+\langle Bx, e_{n+1}\rangle)=0$ or $\langle e_{n+1}, x\rangle=0$.

CASE 1. $\langle e_{n+1}, x\rangle=0$ but $2n\alpha+\langle Bx, e_{n+1}\rangle\neq0$. Then from (3.25) we know that $x_T=0$, this is a contradiction with (3.23).

CASE 2. $2n\alpha+\langle Bx, e_{n+1}\rangle=0$ but $\langle e_{n+1}, x\rangle\neq0$. Then from (3.25) we know $(Bx)_T=0$. This is also a contradiction with (3.23).

CASE 3. $2n\alpha+\langle Bx, e_{n+1}\rangle=0$ and $\langle e_{n+1}, x\rangle=0$. Finding the $e_\beta*e_\gamma$ component of (3.22) we have

$$\langle Bx, e_\beta\rangle\langle x, e_\gamma\rangle+\langle Bx, e_\gamma\rangle\langle x, e_\beta\rangle=0, \tag{3.27}$$

and

$$\langle Bx, e_\beta\rangle x_{\bar{N}}+\langle x, e_\beta\rangle(Bx)_{\bar{N}}=0. \tag{3.28}$$

In (3.27) let $\beta=\gamma$ and sum on γ, then we have

(3.29) $$\langle x_{\bar{N}},\ (Bx)_{\bar{N}}\rangle=0.$$

Combining (3.29) and (3.28), we know $x_{\bar{N}}=0$ or $(Bx)_{\bar{N}}=0$. Finding the $e_\beta * Y$ component of (3.22) we have

(3.30) $$\langle Bx,\ e_\beta\rangle\langle x,\ Y\rangle+\langle x,\ e_\beta\rangle\langle Bx,\ Y\rangle=0.$$

If $x_{\bar{N}}=0$, then $x_N=0$ and (3.23) become $\langle Bx,\ x\rangle+2n=0$. Differentiating this relation along tangent vector x, we have $\langle Bx,\ x\rangle=0$. This is a contradiction with $\langle Bx,\ x\rangle+2n=0$.

If $x_{\bar{N}}\neq 0$, we must have $(Bx)_{\bar{N}}=0$. Then from (3.30) we know that $(Bx)_T=0$, This is a contradiction with (3.23).

Thus in any case, the condition $\Delta\tilde{x}=B\tilde{x}$ can not hold. ■

THEOREM 3.3. *The only n-dimensional submanifold in E^m satisfying $\Delta\tilde{x}=C$ is the n-dimensional subspace of E^m.*

Proof. From the proof of Theorem 3.1, we also know that the only minimal n-dimensional submanifold in E^m satisfying $\Delta\tilde{x}=C$ is the n-dimensional subspace of E^m. Let M^n be a submanifold of E^m satisfying $\Delta\tilde{x}=C$, to complete this proof, the only thing we need to prove is that $\alpha=0$.

Suppose $\alpha\neq 0$. Differentiating $\Delta\tilde{x}=C$ along X, an arbitrary tangent vector of M^n, we have

(3.31) $$nX(\alpha)e_{n+1}*x+n\alpha e_{n+1}*X$$
$$+n\alpha(\bar{\nabla}_X e_{n+1})*x+2\textstyle\sum_i(\bar{\nabla}_X e_i)*e_i=0.$$

Finding the $e_{n+1}*e_{n+1}$ component of (3.31), we have

(3.32) $$X(\alpha)\langle e_{n+1},\ x\rangle=0.$$

If $\langle e_{n+1},\ x\rangle\neq 0$, then α is constant. Finding the $e_{n+1}*Y$ component of (3.31), we have

(3.33) $$(n\alpha\langle e_{n+1},\ x\rangle-2)\langle A_{n+1}X,\ Y\rangle=n\alpha\langle X,\ Y\rangle.$$

Since $\alpha\neq 0$, from (3.33) we know that $n\alpha\langle e_{n+1},\ x\rangle-2\neq 0$, and then

(3.34) $$A_{n+1}X=\frac{n\alpha}{n\alpha\langle e_{n+1},\ x\rangle-2}X.$$

Finding the e_i*e_i component of (3.31) and summing on i, we obtain

(3.35) $$2n\alpha\langle A_{n+1}X,\ x\rangle=0.$$

Combining (3.34) with (3.35), we have $\langle X,\ x\rangle=0$, that is $x_T=0$. Moreover, computing the e_k*e_k component of $\Delta^2\tilde{x}=0$ and summing on k, we have $\alpha=0$, this is a contradiction with the supposition $\alpha\neq 0$.

Suppose that $\langle e_{n+1}, x\rangle=0$. Finding the $e_{n+1}*e_{n+1}$ component of $\Delta^2\tilde{x}=0$, we can have $n^2\alpha^2+2|A_{n+1}|^2=0$, then $\alpha=0$. This is also a contradiction. ∎

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