P. MIKRUT
KODAI MATH. J.
20 (1997), 107—119

BORDISM OF ORIENTED MANIFOLDS WITH LOCAL T^2 ACTION

PIOTR MIKRUT[1]

Abstract

The notion of pure T-structure of rank 2 with singularities is one of the notions introduced by Cheeger and Gromov in [CG1]. It is a generalization of both: a manifold with effective action of a torus T^2 and a manifold being the total space of a T^2 bundle with $\mathrm{Aff}(T^2)$ as a structure group. Using well known properties of the group $SL_2(Z)\cong Z_6 *_{Z_2} Z_4$ and a smooth substitute of a classifying map we show that a compact orientable manifold with local T^2 action with suitable assumptions on orbit types is equivariantly cobordant with CP^2 bundle over a manifold, where T^2 acts in a standard way on fibers. The result is an important step towards calculating bordism group of manifolds with mixed singular T-structures. In dimensions 4, 5 and 6 we calculate explicit generators.

0. Introduction

The notion of pure T-structure of rank 2 or local T^2 action on a manifold is a generalization of an effective action of a torus. The torus T^2 acts on open subsets of a covering of the manifold and these actions fit together in such a way that when two open sets intersect the actions differ by automorphism of T^2. If we assume that T^2 acts locally with fixed points then we say that the T-structure has singularities.

T-structure was introduced by Mikhael Gromov in the paper [G]. More general notions of an F-structure and nilpotent Killing structure appeared in [CG1], [CG2] and [CFG] in the context of collapsing Riemannian manifolds.

Bordism group of compact oriented n-manifolds with T-structure or singular T-structure can be defined. Bordism class of a given manifold is an invariant of the structure. It is an interesting question what this group really measures. The case of T^2 local action is relatively simple and thus suitable for testing.

AMS subject classification: 57R85, 57R75, 57R45.

Keywords and phrases: pure T-structure, bordism, $SL_2(Z)$.

[1] The author was partially supported by the Polish Committee of Scientific Research grant 4241/PB/IM/95.

Received June 7, 1996.

By means of a suitable classifying map for the T^2 fibration over the interior of the orbit space and some surgery constructions we find generators of the bordism group.

The method presented here works for a compact manifold with pure T^2 local action satisfying Assumption 0.4 and of dimension at least 4. Here we will make complete calculations in dimensions 4, 5 and 6.

The case of 4-manifolds was described in detail in [M].

I would like to thank Marek Lewkowicz and Tadeusz Januszkiewicz for introducing me to the subject of T-structures.

DEFINITION 0.1. We say that a smooth manifold M admits a *local T^2 action* if there is a covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of M by open sets such that for each $\alpha\in\Lambda$ there is a smooth and effective action of T^2 on U_α:

$$\theta_\alpha : T^2\times U_\alpha \longrightarrow U_\alpha$$

and if $U_{\alpha\beta}:=U_\alpha\cap U_\beta\neq\emptyset$ then there exists an automorphism

$$\xi_{\alpha\beta} : T^2 \longrightarrow T^2$$

such that the following diagram is commutative:

$$\begin{array}{ccc} T^2\times U_{\alpha\beta} & \xrightarrow{\ \theta_\alpha\ } & U_{\alpha\beta} \\ {\scriptstyle \xi_{\alpha\beta}\times \mathrm{id}_{U_{\alpha\beta}}}\downarrow & & {\scriptstyle \mathrm{id}_{U_{\alpha\beta}}}\downarrow \\ T^2\times U_{\alpha\beta} & \xrightarrow{\ \theta_\beta\ } & U_{\alpha\beta} \end{array}$$

Moreover if $U_\alpha\cap U_\beta\cap U_\gamma\neq\emptyset$ then $\xi_{\beta\gamma}\xi_{\alpha\beta}=\xi_{\alpha\gamma}$.

An *atlas* of the local action of tori is the collection of sets and maps

$$\langle \{U_\alpha\}_{\alpha\in\Lambda};\ \{\theta_\alpha\}_{\alpha\in\Lambda};\ \{\xi_{\alpha\beta}\}_{(\alpha,\beta)\in\Lambda_0}\rangle .$$

Here Λ_0 is the set of those pairs (α, β) for which $\xi_{\alpha\beta}$ is defined.

According to the terminology of [CG1] the local T^2 action on M is a *singular pure T-structure of rank* 2.

The set of all atlases of local actions of tori on a given manifold can be ordered by means of inclusion. Any atlas is included in the maximal atlas.

DEFINITION 0.2. We say that two oriented n-manifolds M_1, M_2 with local T^2 action are cobordant if there exists an oriented $n+1$-manifold W admitting local T^2 action, such that $\partial W=M_1\cup -M_2$ and the local T^2 action on W restricted to ∂W gives the corresponding local actions of tori on M_1, M_2.

NOTATION 0.3. According to Definition 0.1 an orbit of a point is well defined. Let X denote the *orbit space* of M with the quotient topology. Let us denote the quotient map $M \to X$ by ϖ.

ASSUMPTION 0.4. In this paper we assume that the actions $\{\theta_\alpha\}_{\alpha \in A}$ have only three orbit types.

We assume that

— $\mathrm{int}(X)$ consists only of principal orbits.

— $\partial(X)$ consists only of orbits with isotropy subgroup isomorphic to T^1 or T^2.

NOTATION 0.5. Let us denote by

— $\mathrm{Pr} \subset M$ the stratum consisting of principal orbits.

— $\mathrm{Cr} \subset M$ the $n-2$ dimensional stratum of M with isotropy subgroup T^1 such that $\mathrm{Cr}/T^2 \subset \partial(X)$.

— $\mathrm{Fix} \subset M$ the $n-4$ dimensional stratum of the fixed point set.

Remark 0.6. From the slice theorem we obtain that there are tubular neighbourhoods of orbits, corresponding to $\partial(X)$, T^2 equivariantly diffeomorphic to

— $D^2 \times T^1 \times D^{n-3}$ for Cr or

— $D^2 \times D^2 \times D^{n-4}$ for Fix.

T^2 acts on the first two factors in a standard way.

Remark 0.7. According to [D] the orbit space X has a structure of a smooth manifold with corners. The cell structure on $\partial(X)$ is compatible with the cell structure given by orbit types.

Remark 0.8. Let us observe that the normal bundle to the stratum Fix has fiber $D^2 \times D^2$ and according to [D] and Definition 0.1 the structure group is the stabilizer of the pair $\{(1, 0); (0, 1)\} \subset Z^2$ in $\mathrm{Aff}(T^2)$, where $\mathrm{Aff}(T^2)$ acts on Z^2 according to the homomorphism $\mathrm{Aff}(T^2) \to GL_2(Z)$. Since the total space of the bundle is orientable we obtain additional restriction on monodromy.

DEFINITION 0.9. By a *collapse* of a torus T^k in the direction of a subtorus $T^l < T^k$ we mean replacing T^k by the quotient space $T^k/T^l \cong T^{k-l}$.

1. Classifying map for the T^2 fibration

DEFINITION 1.1. We will call a map from a manifold M to a surface H *stable* if it is generic in the space of C^∞ maps with corresponding topology. See [GG].

There are local coordinates $(u, x, z_1, \dots, z_{n-2})$ in M and some coordinates in H such that the map near singularity is of one of the forms:

$$(u, x, z) \longmapsto \left(u\,;\ \pm x^2 + \sum_{i=1}^{n-2} \pm z_i^2\right)$$

$$(u,\ x,\ z) \longmapsto \left(u\ ;\ \pm xu \pm x^3 + \sum_{i=1}^{n-2} \pm z_i^2\right).$$

The map is stable if for each open subset of M the restriction of the map to the subset is stable.

NOTATION 1.2. Let H^2 denote the upper half space. There is a natural action of $SL_2(Z)$ on H^2 by homographies and extension of this action to the action of $GL_2(Z)$. Let i, (respectively $e(1/6)$) denote a point in H^2, $H^2/SL_2(Z)$ or in $H^2/GL_2(Z)$ corresponding to i (and $\exp(2\pi i/6)$ respectively) in H^2. Let C denote the set of complex numbers. Let

$$J : H^2/SL_2(Z) \longrightarrow C$$

be a modular function for $SL_2(Z)$ having value ∞, 0, 1 at the cusp, i and $e(1/6)$ respectively.

DEFINITION 1.3. The space $H^2/SL_2(Z)$ is naturally identified with moduli space of conformal structures of flat metrics on T^2. By *a right-angled torus* we mean a conformal structure on T^2 represented by a pure imaginary complex number in H^2. Let us denote the set of right-angled tori in $H^2/SL_2(Z)$ by $\{iy : y \geqq 1\}$. We assume similar convention for other subsets of H^2.

PROPOSITION 1.4. *Assume that M is a manifold admitting pure local T^2 action satisfying Assumptions* 0.4 *then there is a T^2 invariant metric on M with the following properties*:

—*There exists a T^2 invariant tubular neighbourhoods W, W' of the set Fix satisfying $W \subset \mathrm{cl}(W) \subset W'$ such that the metric restricted to a principal fibre in W is a right-angled flat torus.*

—*Let the map $\phi : \mathrm{int}(X) \to H^2/GL_2(Z)$ sends a principal orbit to the image of the conformal structure of its metric (a point in $H^2/SL_2(Z)$) in $H^2/GL_2(Z)$. ϕ has the property: For any $x \in \mathrm{int}(X) - \varpi(W')$ there is an open neighbourhood U of x such that the map $\phi|_U$ can be lifted to H^2 and the lift is stable.*

—$\phi^{-1}\{1/2 + iy : y \geqq \sqrt{3}/2\} \cap W = \emptyset$.

—*The submanifolds $\{1/2 + iy : y \geqq \sqrt{3}/2\}$, $\{\exp(yi) : \pi/3 + \varepsilon \geqq y \geqq \pi/3)$ for a small $\varepsilon > 0$ are transversal to the strata of singular values of the mapping i.e. is disjoined from points given by singularities of cusp type and is transversal to the curve given by singularity of fold type. Transversality is understood according to the smooth structure on H^2.*

Proof. Let us choose a small T^2 equivariant tubular neighbourhoods W and W' of Fix i.e. $D^2 \times D^2$ bundles over Fix with the standard T^2 local action along fibers such that $W \subset \mathrm{cl}(W) \subset W'$. We can choose a metric on W' such that each principal orbit inherits a flat metric with a right-angled conformal structure inherited from euclidean metric in $D^2 \times D^2 \subset R^2 \times R^2$, which is obviously T^2

invariant. This is possible since the structure group of the bundle is a subgroup of $O(4)$, the isometry group of $R^2\times R^2$ stabilizing (0, 0, 0, 0).

Let us extend the metric to any T^2 invariant metric on M. By a small deformation on $\text{int}(X)-\varpi(W)$ we can make the map ψ stable in $\text{int}(X)-\varpi(W')$ in the above sense. We can further assume that ψ satisfies the following regularity condition:

—For any $x\in\partial(X)-\varpi(W)$ the derivative of ψ in x restricted to a subspace normal to the boundary is non zero.

Let us delete from the upper half space H^2 open horoballs congruent modulo $SL_2(Z)$ action to $\{z:\text{Im}(z)>2\}$. The resulting space $\bar{H}^2$ is diffeomorphic to a compactification of H^2. There is a universal T^2 bundle over H^2 and corresponding bundle over $\bar{H}^2$. Let us assign to each boundary component of $\bar{H}^2$ a subgroup $T^1<T^2$, which is stabilized by the corresponding parabolic subgroup of $SL_2(Z)$. Let us collapse fibers of the universal T^2 bundle over $\partial(\bar{H}^2)$ in the direction of the subgroup T^1 assigned to a given component. The resulting space is a non compact 4-manifold B with T^2 action and orbit space $\bar{H}^2$. There is a global section of the orbit map and we can assume that there is an action of $SL_2(Z)$ on B covering the action on the orbit space. The manifold has a natural smooth structure. There is a T^2 invariant metric on the manifold B such that the conformal structure of the metric restricted to a principal orbit i.e. a point in $H^2/SL_2(Z)$ corresponds to the image of the orbit in the orbit space. See [M] for details of this construction. Such a metric can also be constructed by means of the metric on the universal covering of a manifold described in [CG1] example 1.7.

For any $x\in X-\varpi(W)$ there is an open neighbourhood U of x such that the map $\psi|_U$ has a lift to H^2, denoted $\tilde{\psi}|_U$, and the subset of M lying over U is a pull-back in the sense of [D] of the submanifold of B lying over the image of $\tilde{\psi}|_U$.

There is a small neighbourhood of the orbit lying over x T^2 equivariantly diffeomorphic to $T^2\times D^{n-2}$ in case of principal orbit and to $T^1\times D^2\times D^{n-3}$ in case of orbit with isotropy subgroup T^1. Similar neighbourhoods can be found near the image of the orbit in B. The diffeomorphisms can be chosen such that the map into B is of the form $\text{id}|_{T^2}\times\phi$ in the first case or $\text{id}|_{T^1\times T^2}\times\phi$ in the second for some function ϕ. We can define a T^2 invariant metric on the pull-back such that the map restricted to each set $T^2\times\{d\}$ $(d\in D^{n-2})$ or $T^1\times D^2\times\{d\}$ $(d\in D^{n-3})$ is an isometry.

By means of partition of unity associated to the cover of M by saturated open sets including the above neighbourhoods we can patch together the metrics and obtain a T^2 invariant metric on M which on W agrees with the original metric and over $\text{int}(X)-\varpi(W)$ agrees with the deformed map ψ'. In fact, let ω_i for $i\in I$ (I set of indices) is a collection of metrics defined on open saturated neighbourhoods U_i of a point x. Let x lies in orbit F. Moreover assume that $\omega_i|_F=\omega$. Then $(\sum_i\phi_i\omega_i)|_F=\omega$, where $\{\phi_i\}$ is a set of functions non zero at x taken from the partition of unity.

2. Neighbourhood of $\psi^{-1}(e(1/6))$

FACT 2.1. *The stable map* ψ *described in Proposition* 1.4 *has the properties*:

(1) *For each* $x \in \psi^{-1}(e(1/6))$ *there is a local smooth coordinate* (z, y) (z *complex*) *near* x *and* z *in the neighbourhood* U *of* $J(\psi(x))$ *such that*

$$J(\psi(z, y)) = z^3 + 1 \quad or \quad J(\psi(z, y)) = \bar{z}^3 + 1 .$$

(2) $J(\psi)$ *is stable in* $\mathrm{int}(X) - \varpi(W')$ *outside a small neighbourhood of* $\psi^{-1}(e(1/6))$.

In particular we obtain that $\psi^{-1}(e(1/6))$ is a smooth codimension 2 submanifold of $\mathrm{int}(X)$. Moreover the normal D^2 bundle can be identified with $\psi^{-1}(U')$ for U'-sufficiently small disk neighbourhood of $e(1/6) \in H^2/GL_2(Z)$. The monodromy of the bundle is restricted by the fact that $\psi^{-1}(\{\exp(yi) : \pi/3 + \varepsilon \geqq y \geqq \pi/3\} \cap \partial U')$ define three local sections (if the diameter of U' is small when compared to ε in 1.4). Thus $\psi^{-1}(U')$ is diffeomorphic to

$$\tilde{\varphi}^{-1}(e(1/6)) \times_G D^2$$

where $\tilde{\psi}^{-1}(e(1/6))$ is the covering of $\psi^{-1}(e(1/6))$ corresponding to monodromy representation $\pi_1(\psi^{-1}(U')) \to PGL_2(Z) = GL_2(Z)/\pm I$ and G is the corresponding monodromy group. G is a subgroup of dyhedral group $D_6/\pm I \cong D_3$, i.e. the stabilizer of $e(1/6)$ in the group $PGL_2(Z)$ acting on H^2 in a standard way.

It is now easy to identify the subset of the manifold M having as the orbit space $\psi^{-1}(U')$ in case of existence of a zero section of the T^2 fibration. It is equivariantly diffeomorphic to

$$\tilde{\psi}^{-1}(e(1/6)) \times_{\tilde{G}} (D^2 \times T^2)$$

where $\tilde{G} < GL_2(Z)$ is the group that covers $G < PGL_2(Z)$. $\tilde{G}$ acts on the product $D^2 \times T^2$ by the corresponding monodromy representation on T^2 and on D^2 according to the homomorphism $\tilde{G} \to G \to D_6/\pm I < PGL_2(Z)$. $\tilde{G}$ acts on $\tilde{\psi}^{-1}(e(1/6))$ by covering transformations.

Remark 2.2. The manifold M satisfying the Assumption 0.4 can be constructed as a T^2 bundle over X with structure group $Aff(T^2)$ with fibers over $\partial(X)$ collapsed in the direction of isotropy subgroups. We can find a zero section of this bundle outside a codimension 2 complex in X, transversal to the strata in $\partial(X)$.

If M has dimension n then X has dimension $n-2$ and the obstruction chain to existing of a global section is supported in subcomplex of X of dimension $n-4$. Let us assume that the subcomplex is transversal to $\psi^{-1}(e(1/6))$ and that ε is sufficiently small.

In particular in Dimension 5 we can assume that there is a global section over $\psi^{-1}(U')$. In dimension at least 6 the obstruction chain intersects with $\tilde{\psi}^{-1}(e(1/6))$ transversaly in a complex of codimension 2.

3. CP^2 and CP^2 bundles

Let us represent the standard action of T^2 on CP^2 as:

$$(\tau_1, \tau_2)(z_1, z_2, z_3)=(\tau_1 z_1, \tau_2 z_2, z_3)$$

where (z_1, z_2, z_3) are homogeneous coordinates of CP^2. Let us set: $t_1=z_1/z_3$, $t_2=z_2/z_3$ if $z_3\neq 0$. The action of T^2 can be written by the formula:

$$(\tau_1, \tau_2)(t_1, t_2)=(\tau_1 t_1, \tau_2 t_2)$$

Let us put $r^2=|t_1|^2+|t_2|^2$, $r_1^2=|t_1|^2$, $r_2^2=|t_2|^2$, $t_1=r_1e^{i\phi_1}$, $t_2=r_2e^{i\phi_2}$ for $0\leqq\phi_1, \phi_2 \leqq 2\pi$ and $r_1=r\cos\psi$, $r_2=r\sin\psi$ for $0\leqq\psi\leqq\pi/2$.

LEMMA 3.1. *The folowing maps*

$$(z_1, z_2, z_3)\longmapsto(\bar{z}_2, \bar{z}_3, \bar{z}_1)$$

$$(z_1, z_2, z_3)\longmapsto(z_2, z_1, z_3)$$

generate a group of automorphisms of the standard T^2 action on CP^2 corresponding to the stabilizer of $e(1/6)$ in $GL_2(Z)$.

Proof. The map

$$(z_1, z_2, z_3)\longmapsto(\bar{z}_2, \bar{z}_3, \bar{z}_1)$$

when restricted to the torus $\{(z_1, z_2, z_3): |z_1|=|z_2|=|z_3|\}$ corresponds to the automorphism of the torus:

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

if we take ϕ_1, ϕ_2 as coordinates of T^2.

Similarly the map

$$(z_1, z_2, z_3)\longmapsto(z_2, z_1, z_3)$$

corresponds to the automorphism of the torus:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Both maps are the isometries of Fubini-Study metric on CP^2.

Since the action of T^2 on CP^2 is global, i.e. the monodromy representation is trivial, we can replace the classifying map ψ in 1.4 by its lift to H^2, denoted $\tilde{\psi}$. The above automorphisms commute with $\tilde{\psi}$ given by the Fubini-Study metric.

3.2. *Metrics on CP^2 bundles*: Let us define a map

$$(r, \psi, \phi_1, \phi_2)\longmapsto(r, \psi, \phi_1 h_1(r, \psi), \phi_2 h_2(r, \psi))$$

where h_1, h_2 are smooth maps with respect to coordinates (r, ϕ). In case h_1, h_2 are constant we obtain an isometry of CP^2 with Fubini-Study metric. Let T^2 denote the group of isometries of Fubini-Study metric represented by pairs h_1, h_2 of constant maps. The group generated by T^2 and the group of automorphism described in 3.1 is isomorphic to a subgroup G' of $Aff(T^2)$.

Any bundle with fibre CP^2 and structure group G' can be given a metric which when restricted to a fibre gives the Fubini-Study metric.

4. Surgery operation in the neighbourhood of $\phi^{-1}\{1/2+iy : y \geqq \sqrt{3}/2\}$

The submanifold of M with orbit space $\phi^{-1}(U')$ is the total space of T^2 fibration with structure group $G' < Aff(T^2)$. We can assume that the transition functions are constant with respect to coordinates of D^2, where D^2 is the fibre of the normal bundle to $\phi^{-1}(e(1/6))$. (see 2.1). The classifying map determines metric on the vertical subbundle of the tangent bundle of the T^2 fibration over $\phi^{-1}(U')$.

There is an isotopy of the metric, not changing the classifying map, such that the resulting metric over $\phi^{-1}(U')$ is isometric to the metric on the submanifold of CP^2 bundle over $\phi^{-1}(e(1/6))$ associated to the T^2 bundle.

Here metrics on the fibers of CP^2 bundles are Fubini-Study metrics.

We can perform a surgery operation:

$$(M-V)\cup N-(-V')$$

where V is a subset of M lying over $\phi^{-1}(U')$, V' is the isometric manifold in the CP^2 bundle and the gluing map is the isometry on the boundary. The surgery corresponds to adjacent connected sum and thus is realized by bordism. The obtained metric is smooth when restricted to fibers. The metric can be smoothed in such a way that the map ϕ will change only in the neighbourhood of the gluing area. We can assume that the changed map is sufficiently close in C^0 topology to the map before the smoothing.

We have obtained a manifold in which the set $\phi^{-1}(e(1/6))$ is empty. Moreover we can assume that the classifying map ϕ is regular in the sense of 1.4 except the condition concerning the set Fix. We have obtained some components of Fix with the fixed metric in the neighbourhood (Fubini-Study in the fibers) for which the classifying map is different from that in W. The condition in the neighbourhood of Fix that the image of ϕ intersects trivially with $\phi^{-1}\{1/2+iy : y \geqq \sqrt{3}/2\}$ is satisfied.

A connected component of $\phi^{-1}\{1/2+iy : y \geqq \sqrt{3}/2\}$ is mapped by ϕ into the line $\{1/2+iy : y > \sqrt{3}/2\}$ since $\phi^{-1}(e(1/6)) = \emptyset$. The line is contractible to the cusp corresponding to $y = \infty$ (e.g. in one point compactification of $H^2/SL_2(Z)$). Thus the monodromy matrices corresponding to loops in the component $\phi^{-1}\{1/2+iy : y \geqq \sqrt{3}/2\}$ preserve the subgroup T^1 corresponding to the cusp.

Here the subgroup T^1 is properly defined if we take some local trivialization on $\phi^{-1}\{1/2+iy : y\geqq\sqrt{3}/2\}$,

Let us define a surgical operation in the following way. Let us cut the manifold along the submanifold $\phi^{-1}\{1/2+iy : y\geqq\sqrt{3}/2\}$. Let us collapse orbits over the resulting boundary in the direction of assigned subgroup T^1, depending on connected component. The subgroup T^1 is a new isotropy subgroup. The boundary of each connected component of $\phi^{-1}\{1/2+iy : y\geqq\sqrt{3}/2\}$ corresponds to the same T^1 isotropy subgroup.

The resulting manifold satisfies assumptions 0.4 and from the behaviour of the classifying map ϕ on $X-\phi^{-1}\{1/2+iy : y\geqq\sqrt{3}/2\}$ we deduce the property:

(4.1) Each connected component of M has the linear part of monodromy group conjugated to a subgroup of the group generated by the matrices:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

5. Surgery in case all principal orbits have right-angled flat metric

Let M satisfy 0.4 and 4.1. We can put a T^2 invariant metric on M such that each principal orbit inherits a right-angled conformal structure and a neighbourhood of Fix i.e. $D^2\times D^2$ bundle over Fix, with the standard T^2 local action along fibers, has metric which on fibers is inherited from euclidean metric in $R^2\times R^2$. The metric on fiber is then represented by an imaginary number $ix : x\geqq 1$. In fact we have a R_+-bundle associated with the T^2-bundle over the principal part of X with monodromy Z_2 generated by $x\mapsto 1/x$. It corresponds to the group permuting the isotropy subgroups represented by integer vectors (1, 0); (0, 1)$\in Z^2$. The conformal structure is represented by a section of the R_+-bundle.

We may assume that the intersection Y of the section giving the conformal structure with the section $1\in R_+$ is transversal. Thus Y is a smooth $n-3$-submanifold of int(X). Its closure in X has boundary in Fix. The intersection with the sum of sections $1\pm\varepsilon$, for $\varepsilon>0$ sufficiently small, outside some neighbourhood of Fix can be identified with the normal S_0 bundle. The normal S^0 bundle can be compactified to give a part of the boundary of a $n-2$-submanifold of X- a tubular neighbourhood of Y. We may assume that the boundary of the $n-2$-manifold (denoted by S) is transversal to $\partial(X)$. We can assign each connected component of S with a subgroup $T^1<T^2$. The 1-dimensional subgroup assigned to the component corresponds to one of two cusps determined by the line Re(z)=0 of right-angled tori in H^2—the one closer to the section $(1\pm\varepsilon)i$ determining S with respect to local trivializations.

We perform surgery operation along S in an analogous way as we did in section 4 along $\phi^{-1}\{1/2+iy : y\geqq\sqrt{3}/2\}$. After the surgery we obtain a manifold consisting of two disjoined parts.

(1) The first one has $\mathrm{Fix}=\emptyset$ and its monodromy group is generated by the subgroup of the monodromy group stabilizing each subgroup from the pair $T^1\times\{1\}$; $\{1\}\times T^1$ and the nontrivial isotropy subgroup corresponds to one of the subgroups $T^1\times\{1\}$ or $\{1\}\times T^1$. It coincides with the subgroup assigned to the component of S, which is a part of the boundary of the orbit space after the surgery. The local action of the T^1 is semi-free thus there is a standard T^1 equivariant filling (see [U]), which is also T^2 equivariant. The filling locally corresponds to replacing a principal orbit T^2 by $T^1\times D^2$ and $T^1\times D^2$ bundle over Cr/T^2 by the associated $T^1\times D^3$ bundle.

(2) The second one is a manifold which is a sum of:

—The S^3 bundle over Y associated with the T^2 bundle over Y given by principal orbits. Generators of linear part of monodromy act on S^3 by

$$(z_1, z_2)\longmapsto(z_2, \bar{z}_1)\qquad (z_1, z_2)\longmapsto(\bar{z}_1, z_2)$$

where $S^3=\{(z_1, z_2): |z_1|^2+|z_2|^2=1\}$ and T^2 can be identified with $\{(z_1, z_2)\in S^3: |z_1|=|z_2|\}$. Translational part of the structure group (i.e. T^2) acts on S^3 in a standard way:

$$(t_1, t_2)(z_1, z_2)\longrightarrow(t_1z_1, t_2z_2)$$

—$D^2\times D^2$ bundle over Fix which is a tubular neighbourhood of Fix in M. Here we take $D^4\cong D^2\times D^2$ without corners such that it has smooth boundary S^3.

The manifolds are glued T^2 equivariantly along the boundaries diffeomorphic to S^3 bundle over $\partial(Y)\cong\mathrm{Fix}$.

The first manifold is a part of the boundary of associated D^4 bundle over Y, where D^4 is a smooth cone over S^3. The local T^2 action can obviously be extended to the D^4 bundle.

Let us glue the D^4 bundle over Y with D^4 bundle over Fix along the S^3 bundle over $\partial(Y)\cong\mathrm{Fix}$. The sum of the D^4 bundle over $\partial(Y)$ with the D^4 bundle over Fix is T^2 equivariantly diffeomorphic with S^4 bundle over $\partial(Y)$, where the T^2 action on S^4 is a suspension of the T^2 action on S^3. There is T^2 equivariant filling by D^5 bundle over $\partial(Y)$ such that the S^3 bundle over $\partial(Y)$ is a corner. Here D^5 is a smooth cone over S^4 with corner S^3.

Thus the second manifold is null cobordant.

6. Generators of the bordism group

From above constructions we have obtained that a manifold M satisfying 0.4 is cobordant with a CP^2 bundle over a manifold Z with linear part of the monodromy being a subgroup of the stabilizer of $e(1/6)$ in $GL_2(Z)$.

The base Z of a CP^2 bundle is orientable. It follows from the fact that if the monodromy matrix along a loop in Z is nonorientable then the normal disk bundle changes orientation and the tangent bundle to X also changes orientation. We choose orientation on Z such that together with the fixed orientation on the fiber CP^2 it gives chosen orientation on the total space of the bundle. We thus obtain an element of $\Omega_{n-4}(BG')$, where G' (as in Section 3) is a subgroup of $\mathrm{Aff}(T^2)$ with image in $GL_2(Z)$ equal to the stabilizer of $e(1/6)$. We thus obtain

THEOREM 6.1. *A compact orientable manifold of dimension at least* 4 *with local* T^2 *action satisfying Assumption* 0.4 *is equivariantly cobordant with the total space of a* CP^2 *bundle over a manifold* Z.

In particular the problem of calculating generators of the bordism group of compact orientable manifolds of dimension n with local T^2 action satisfying Assumption 0.4 reduces to calculation of the group $\Omega_{n-4}(BG')$.

6.2. DIMENSION 4. Let 0.4 be satisfied. We obtain $\Omega_0(BG')=Z$ and the corresponding generator is CP^2 with standard T^2 action. Here we use the convention that $\{-1\}\times CP^2$ represents $-CP^2$, where $-1\in\Omega_0(BG')$ is the generator. T^2 action on CP^2 is determined up to automorphism of T^2 but according to Definition 0.1 all such actions give the same local action. Thus we obtain that the bordism group of pure rank 2 local action of tori in Dimension 4 is isomorphic to Z. The same result, using slightly different method was obtained in [M].

6.3. DIMENSION 5. If 0.4 is satisfied we obtain $\Omega_1(BG')=H_1(G')=G'/[G', G']=Z_2\oplus Z_2$. There are two corresponding generators. Both are CP^2 bundles over S^1 with monodromies (see 3.1):

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

6.4. DIMENSION 6. If 0.4 is satisfied then in Dimension 6 we have obtained $\Omega_2(BG')$. Let us take N_1: the total space of a T^2 bundle over an orientable surface X with structure group G' representing an element of $\Omega_2(BG')$. The obstruction to a global section can be described by a pair of integers (a, b). The integer vector is in the image of the matrix operating on integer vectors:

$$B:=\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

It is easy to see if we observe that $B(-1, 1)=(1, 0)$ and $B(-1, 0)=(0, 1)$. Thus $B(-a-b, a)=(a, b)$.

Let N_2 be the total space of T^2 bundle over $Y=S^1\times D^2\cup[0, 1]\times D^2$, where $\{0\}\times D^2\subset[0, 1]\times D^2$ is glued with a disk $D^2\subset\partial(S^1\times D^2)$. Let us assume that the linear part of monodromy along the S^1 is equal to:

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

We assume that the obstruction chain to existing of a global section (1-chain with coefficients $Z\oplus Z$ twisted by monodromy) is supported in a sum of circle $S^1\times\{0\}\subset Y$ for $0\in D^2$ and an interval joining the circle with $\{1\}\times\{0\}\subset\{1\}\times D^2$ in the boundary of Y. Along the circle we put the vector $(-a-b, a)$ and along the interval the vector (a, b) as local coefficients.

Let us construct the manifold $N_3=N_1\times[0, 1]\cup N_2$, where N_2 is glued to $N_1\times\{1\}$ by identifying T^2 bundle over $\{1\}\times D^2\subset Y$ with a part of the T^2 bundle $N_1\times\{1\}$ lying over a disk D^2 containing the point supporting obstruction to global section.

N_3 represents a bordism of the T^2 bundle N_1 with a T^2 bundle with global section over a surface $X\# T^2$. Thus we need only to calculate $\Omega_2(B(D_6))=H_2(D_6)$, where $D_6<GL_2(Z)$ as a stabilizer of $(1+\sqrt{3})/2\in H^2$ is isomorphic to dyhedral group having 12 elements.

Let us assume that an orientable surface X represents element of $H_2(D_6)$. Let us take a flat R^1 bundle over X determined by the homomorphism:

$$\pi_1(X)\longrightarrow D_6\longrightarrow Z_2$$

the determinant of the monodromy matrix. A generic section intersects zero section over a curve in X, which we denote by w_1. Without changing homology class of X we can assume that w_1 is connected. If w_1 separates X then the monodromy along w_1 is trivial and X is homologous to the cycle with w_1 trivial. The cycle then represents an element of $H_2(Z_6)=0$. If w_1 does not separates X then $X=X_1\# T^2$ and the torus supports w_1. X is homologous with a disjoined sum of X_1 and T^2. X_1 represents element of $H_2(Z_6)=0$. If a generator of $\pi_1(T^2)$ has monodromy I then the cycle T^2 is homologous to zero. Otherwise it is homologous to T^2 with monodromy matrices along generators of $\pi_1(T^2)$:

$$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\quad\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

The corresponding generator of 6-dimensional bordism group of manifolds with local T^2 action is the CP^2 bundle over T^2 with monodromy matrices along generators of $\pi_1(T^2)$ as above.

REFERENCES

[CG1] J. CHEEGER AND M. GROMOV, Collapsing Riemannian manifolds while keeping their curvature bounded I, J. Differential Geom., **23** (1986), 309-346.

[CG2] J. CHEEGER AND M. GROMOV, Collapsing Riemannian manifolds while keeping their curvature bounded II, J. Differential Geom., **32** (1990), 269-298.

[CFG] J. CHEEGER, K. FUKAYA AND M. GROMOV, Nilpotent structures and invariant metrics on collapsed manifolds, J. Amer. Math. Soc., **5** (1992), 327-372.

[D] M.W. DAVIS, Smooth G-manifolds as collections of fibre bundles, Pacific J. Math., **77** (1978), 315-363.

[GG] M. GOLUBITSKY AND V. GUILLEMIN, Stable Mappings and Their Singularities, Grad. Texts in Math., **14**, Springer-Verlag, New York, 1973.

[G] M. GROMOV, Volume and bounded cohomology, Inst. Hautes Études Sci. Publ. Math., **56** (1983), 5-99.

[M] P. MIKRUT, Local T^2 actions on compact 4-manifolds, Thesis, Polish Academy of Sciences, Warsaw, 1995.

[U] F. UCHIDA, Cobordism groups of semi-free S^1- and S^3-actions, Osaka J. Math., 7 (1970), 345-351.

MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW
POLAND
e-mail: mikrut@math.uni.wroc.pl