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EQUIVARIANT CATEGORY OF THE FREE PART OF A G-MANIFOLD AND OF THE SPHERE OF SPHERICAL HARMONICS

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Abstract

In this work we study the G-category of a G-manifold M by taking in consideration the fixed point set of a maximal torus of a compact Lie group G. The used method let us compute the G-category of sphere of every real irreducible, odd indexed representation V_l of the group $G=SO(3)$. An application to a nonlinear Dirichlet problem, one of several possible, is given. Simplifying a proof of estimate of the G-category of the free part of a sphere we also show that the complement of saturation of fixed point set of a maximal torus is an open invariant subset of larger G-category than the free part of action and give particular computation for the spherical harmonics.

0. Introduction

In study variational methods with symmetries it is very useful to apply invariant of mini-max type as genus of a G-space, G-category, or cohomological index of a G-space (see [Bar2] for a revue of recent results). In view of applications the most important is to know a value of such an invariant for the unit sphere of an orthogonal representation of a given compact Lie group G. It was first observed that if G is the torus $T=T^k$, or p-torus Z_p^k, p prime, then for every orthogonal representation V without fixed point of G on the sphere $S(V)$ a value of such an invariant for the sphere is equal to the complex dimension (or real dimension) of V (cf. [Fa], [C-P], [Ma1] and [Bar2] for other references). The situation changed drastically if the connected component G_0 of G is nonabelian (cf. [Bar1]). Using a method of classification of compact Lie groups with the Borsuk-Ulam property developed himself, T. Bartsch gave a condition on representation W_0 of $G=SO(3)$ (and an example) that for every other representation U of $SO(3)$, $U^G=0$, we have

$$\operatorname{cat}_G S(W_0 \oplus U) \leqq 2 \operatorname{cat}_G S(W_0),$$

and consequently does not depend monotonic on the dimension ([Bar2]). On the

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other hand real irreducible representations of $G=SO(3)$ are the spaces V_l, $l=1, 2, \dots$, of *spherical harmonics* of order l. The dimension of V_l is equal to $2l+1$ ([CLM]), thus tends to infinity if $l \to \infty$. It is worth of emphasize that the spaces V_l appear naturally as the defect spaces (kernels of the linearization) in nonlinear problems with $SO(3)$-symmetry, since they are the eigenspaces of Laplacian on the unit ball in $\boldsymbol{R}^3$.

The main purpose of this work (Theorem 2.3) is to show that for $G=SO(3)$ and any odd l

$$\operatorname{cat}_G S(V_l) \geqq \left[\frac{l}{2}\right]+1,$$

where $[x]$ denote the integer part of a real number x.

We begin with a generalization and improvement results of a recent paper of Balanov and Brodsky [Ba-Br] on the genus of free part of action on G-spheres. They extended the classical theorems of Krasnosielski [Kr] and A. Švartz for the finite cyclic group onto the case of arbitrary compact Lie. We show that instead of an invariant subset A of the G-sphere S such that the action on $S \backslash A$ is free we can take a G-subset A of G-manifold M such there exists a subgroup $K \subset G$ which acts freely on $M \backslash A$ (Theorem 1.1). Also we get rid of the condition on smoothness of A used in [Ba-Br] replacing it by a condition on the dimension of A as in the original Švartz formulation. An estimate of the G-category of $M \backslash A$ holds provided vanishing the cohomology groups $H^i(M; \boldsymbol{Z})$ in gradation greater than the codimension of A, which is satisfied if $M=S$ is the sphere (Proposition 1.2, Corollary 1.3). Our proof is much shorter. Next we restate an estimate of [Ma2] on the G-category of a G-space for which $X^T=\emptyset$, T a maximal torus of G. We remark that our formula gives sharper estimate than that one of [Bar-C] (Corollary 1.8, Remark 1.9). This leads to an observation that to pick up an open invariant subset $\mathcal{U}$ of sphere S of a large G-category it is more convenient to take the set $\mathcal{U}=S \backslash S^{(T)}$ instead of $\mathcal{U}=S_{(e)}$, as was taken in [Ba-Br]. Particular computations are done for the irreducible representations, i.e. *spherical harmonics*, of the group $G=SO(n)$ to compare with those of [Ba-Br] (Theorem 2.2). Using opportunity the author would like to thank to Z. Balanov for pointing out his attention on this problem and many interesting talks.

We observe that for a compact Lie group G of rank 1 if the $S^T=S^0$ is the zero-dimensional sphere, and $S^{N(T)}=\emptyset$, $N(T)$ is the normalizer of T, then $\operatorname{cat}_G S \geqq (m+1)/(1+d(S))$, where m is the dimension of S and $d(S)$ is the maximal dimension of orbits on S (Theorem 1.10) (Note that in general $\operatorname{cat}_G(S \backslash S^{(T)})$ gives no impact on the G-category of the whole sphere.) This theorem let us estimate the G-category of $S(V_l)$, the sphere of irreducible representations (*i.e. spherical harmonics*) of the group $G=SO(3)$ if l is odd, because then the above assumption is satisfied (Theorem 2.3). Furthermore, as it was showed by Bartsch that the equivariant genus of every compact $SO(3)$-ANR is bounded from above by 52. Comparing it with the statement of Theorem 2.3 we show that for every odd

$l>103$ the equivariant category of $S(V_l)$ is strictly greater than its equivariant genus (Corollary 2.4).

Finally we apply our main theorem to establish a multiplicity of solutions of a nonlinear Dirichlet problem

$$\Delta u=\mu f(u), \quad \Delta_{|\partial D^3}=0$$

with a real parameter μ, which bifurcate from an eigenvalue of the Laplacian Δ on the unit ball $D^3(0, 1)$ in $\boldsymbol{R}^3$ (Proposition 2.8).

1. Estimation of the G-category

We shall use the standard notation of group transformation theory (cf. [Bre], [tD]). Let G be a compact Lie group and M a closed oriented manifold of dimension m. By (H) we denote an orbit type of the closed subgroup $H\subset G$. For a closed subgroup $H\subset G$ by $d(H)$ we denote the dimension of H. By a dimension of a subset $A\subset M$ of M we mean the cohomological dimension of A. We begin with the following theorem.

1.1 THEOREM. *Let a compact Lie group G act on a smooth closed oriented manifold M of dimension m and $A\subset M$ be a closed invariant subset of M such that $M\backslash A\subset M_{(H)}$ for an orbit type (H). Assume then there exists a subgroup $K\subset G$ such that K acts freely on G/H.*

Suppose that $\dim A\leqq k$, $k\leqq m-3$, *and* $H^{k+2}(M;\boldsymbol{Z})=\cdots=H^{m-1}(M;\boldsymbol{Z})=0$. *Then*

$$\mathrm{cat}_G(M\backslash A)\geqq\frac{m-k}{1+d(G)-d(H)}\,.$$

Proof. From the long exact sequence of the pair (M, A) with integer coefficients $\to H^{k-1}(M, A)\to H^{k-1}(M)\to H^{k-1}(A)\to H^k(M, A)\to H^k(M)\to H^k(A)\to H^{k+1}(M, A)\to H^{k+1}(M)\to H^{k+1}(A)\to H^{k+2}(M, A)\to H^{k+2}(M)\to H^{k+2}(A)\to\cdots$ it follows that $H^i(M, A)=0$ for $i\geqq k+2$ if $H^i(M)=0$ for $i\geqq k+2$. Consequently, by the Lefschetz-Poincaré duality, we have $H_q(M\backslash A)=H^{m-q}(M, A)=0$ for every $1\leqq q\leqq(m-k)-2$. This shows that $M\backslash A$ is $(m-k)-2$-acyclic over $\boldsymbol{Z}$. The last and the assumption on the existence of a subgroup $K\subset G$ as imposed above show that the assumptions of Theorem 3.2 of [Ma2] are fulfilled, because on $M\backslash A$ is only one orbit type (G/H). Consequently we have

$$\mathrm{cat}_G(M\backslash A)\geqq\left[\frac{(m-k)-1}{1+d(G)-d(H)}+1\right],$$

where $[r]$ denote the integer part of a real number r. The statement follows from the following inequality

$$(*)\qquad \left[\frac{l-1}{1+\alpha}\right]\geqq\frac{l}{1+\alpha}\,.$$

where l is nonnegative integer, and $\alpha \geqq 0$, is a real number. $\square$

Now we show that the result of Balanov and Brodsky [Ba-Br] can be deduced from Theorem 1.1. It is enough to use the following well known fact.

1.2 PROPOSITION. *Let $A=f(N)\subset M$ be a image of n-dimensional manifold N, $m>n$, throughout a C^1-map $f: N \rightarrow M$. Then* $\dim A \leqq n$.

Proof. Since $n<m$, every point $x \in N$ is critical i.e. $N=C$ the set of critical points. It is not difficult to show that locally (in local coordinates $\mathcal{U}$) the $(n+i)$-dimensional, $i \geqq 1$, Lebesque measure $\mu(f(\mathcal{U} \cap C))=0$ ([Mi] 3., or [St] Chapter II 3). This shows that $h_{n+i}(f(\mathcal{U}))=0$ for the $(n+i)$-dimensional Hausdorff measure. Since N is covered by countable number of such local coordinates, $h_{n+i}(f(N))=0$. This means that Hausdorff dimension of $f(N)$ is less or equal to n. Consequently the covering dimension of $A=f(N)$ is less or equal to n (cf. [Hu-W] Theorem VII 3) which gives the same inequality for the cohomological dimension of A ([Hu-W] Theorem VIII 4). $\square$

Applying the above Proposition we get the following version of main theorem of [Ba-Br].

1.3 COROLLARY. *Let G be a compact Lie group of dimension $d(G)$ acting on the sphere S^m. Let A be a closed G-invariant subset of S^m such that the G-space $S^m \backslash A$ is free. Suppose, further that A is an image of an n-dimensional smooth compact manifold under a smooth map with $n<m$ (if A is empty then it is thought as the image of the (-1)-dimensional manifold under the empty map). Then*

$$\operatorname{cat}_G(S^m \backslash A) \geqq \frac{m-n}{1+d(G)}. \qquad \square$$

1.4 *Remark.* Note that Balanov and Brodsky showed the above inequality for the genus of a G-space which is less or equal to the G-category. To get that one should modify the proof of Theorem 3.2 of [Ma2] which we applied in our argument. On the other hand we get rid of the assumption on smoothness of action used by them (we assume only that M is a manifold to have the Alexander-Lefschetz-Spanier duality). If the action is smooth it is natural to take the principal orbit type as (H).

We have just proved that the set $S_{(H)}$ consisting of principal orbit type (H) has G-category which growth depends on the dimension of $S_{(H)}$. In general it is difficult to compute $\dim M_{(H)}=\dim M^{(H)}$ even for a manifold being the sphere S. Now we show that taking out from a manifold M the saturated fixed point set $M_{(T)}$ of the maximal torus $T \subset G$ one get an open invariant subset $M \backslash M^{(T)}$ of G-category not less that a constant depending on $\dim M$, $\dim G$, $\dim T$, provided that M has the cohomology property as in Theorem 1.1. A proof of this fact is based on an lower estimate of G-category of a G-space without fixed point of maximal torus given the author ([Ma2] and similar to that one given by Bartsch and Clapp [Bar-C].

For a subgroup $H\subset G$ by $\mathrm{rk}(H)$ we denote the rank of H i.e. the dimension of a maximal subtorus $T\subset H$ of H. By $\delta(G)$ we denote the number $d(G)-\mathrm{rk}(G)$. For a given G-set X, by

$$d(X)=\max\{d(G)-d(G_x)\}=d(G)-\min\{d(G_x)\}.$$

Note that if G acts smoothly on a G-manifold M then $d(M)=d(G)-d(H)$, where (H) is the principal orbit type on M.

1.5 PROPOSITION. *Let G be a compact Lie group acting smoothly on a manifold M of dimension m and $T\subset G$ be the maximal torus of G. Then*

$$\dim M^{(T)}\leqq\dim M^T+\delta(G).$$

Proof. Note that $\dim M^{(T)}=\dim M_{(T)}$. (cf. [Bre] IV, Lemma 3.5). Since $M_{(T)}$ is a fiber bundle with fiber G/T base $M_{(T)}/G$ and structure group $N(T)/T$ ([Bre] IV, Theorem 3.3), we have $\dim M^{(T)}=\dim M_{(T)}/G+d(G)-d(T)$. On the other hand $M_{(T)}/G$ is homeomorphic to $M^T_{(T)}/N(T)$ (cf. [Bre] II, Corollary 5.10), because there is only one orbit type on $M_{(T)}$. Since the Weyl group of maximal torus $W(T)=N(T)/T$ is finite, we have $\dim(M_{(T)})^T/NT)=\dim(M_{(T)})^T/W(T)=\dim(M^{(T)})^T=\dim M^T$. Combining the above we get

$$\dim M^{(T)}=\dim M^T+d(G)-d(T),$$

which completes the proof. □

Note that $d(T)=\mathrm{rk}\,G$ by the definition. For a given action of G on M we set $t:=\dim M^T$ for the maximal torus $T\subset G$.

1.6 THEOREM. *Suppose that a compact Lie group G acts smoothly on a smooth closed oriented manifold M of dimension m with the principal orbit type (H). Assume that $t+\delta(G)\leqq m-3$ and $H^{t+\delta(G)+2}(M;\boldsymbol{Z})=\cdots=H^{m-1}(M;\boldsymbol{Z})=0$. Then*

$$\mathrm{cat}_G(M\backslash M^{(T)})\geqq\frac{m-t-\delta(G)}{1+d(G)-d(H)}.$$

Proof. From Proposition 1.5 it follows that $\dim M^{(T)}\leqq t+\delta(G)$. The assumption on the cohomology of M. It is sufficient to show that $M\backslash M^{(T)}$ is $m-t-\delta(G)-2$-acyclic. The statement follows from Proposition 3.7 of [Ma2] and inequality (*), since there is only a finite number of distinct orbit types on M. (The last and $(M\backslash M^{(T)})^T=\emptyset$ yield the existence a subgroup $\boldsymbol{Z}_p\subset T\subset G$ such that the action of $\boldsymbol{Z}_p$ on $M\backslash M^{(T)}$ is free.) □

1.7 COROLLARY. *Let G be a compact Lie group of dimension $d(G)$ acting smoothly on the sphere S^m with the principal orbit type (H). Let $t=\dim S^T$ for a maximal torus $T\subset G$, and $t+\delta(G)\leqq m-3$. Then*

$$\mathrm{cat}_G S^m\backslash(S^m)^{(T)}\geqq\frac{m-t-\delta(G)}{1+d(G)+d(H)}.$$

In particular, if there exists a point $x\in S^m$ such that $d(G_x)=0$ (or $(H)=\boldsymbol{e}$ i.e. the free part is not empty) then

$$\mathrm{cat}(S\backslash S^{(T)})\geqq\frac{m-t-\delta(G)}{1+d(G)}.$$ □

1.8 COROLLARY. *Suppose that S^m is as in Corollary 1.7 and $\dim(S^m)^T=0$. Then*

$$\mathrm{cat}_G(S^m\backslash S^{(T)})\geqq\frac{m-\delta(G)}{1+d(G)-d(H)},$$

where (H) is the principal orbit type. □

1.9 *Remark.* It is worth of pointing out the estimate of corollary is better that similar one of [C-P] and [Bar-C] (see [Bar2] Corollary 2.21). Indeed for $G=SO(3)$ let V_l be the l-th irreducible representation of G, $l>2$. Then $\dim V_l^T=1$, and the principal orbit type is equal to $\boldsymbol{e}$ (see [Bar2], [CLM]). Corollary 2.21 of [Bar2] gives the following estimate

$$\mathrm{cat}_G(S\backslash S^{(T)})\geqq\frac{\dim((V_l^T)\perp)}{2(1+d(G)-d(T))}=\frac{2l}{2\cdot 3}=l/3.$$

On the other hand, in this case Corollary 1.8 gives

$$\mathrm{cat}_G(S\backslash S^{(T)})\geqq l/2.$$

1.10 THEOREM. *Let S be a manifold being cohomology sphere over $\boldsymbol{Z}$ of dimension m on which acts smoothly a compact Lie group G of rank 1. Suppose that $\dim S^T=0$ for a maximal torus $T\subset G$ and $S^{N(T)}=\emptyset$ for the normalizer $N(T)$ of T in G. Then*

$$\mathrm{cat}_G S\geqq\frac{m+1}{1+d(S)}.$$

Proof. Note that there are only three compact connected Lie groups of rank 1, namely S^1, S^3 and $SO(3)$ (cf. [Bre]).

From the Smith theory (or Borel localization theorem) (cf. [Bre] or [H]) it follows that m is even and $S^T\cong S^0$ as a submanifold consists of two points s_1 and s_2. Since $S^{N(T)}=\emptyset$, $W(T)=N(T)/T$ acts transitively on S^0, and consequently there is only one orbit of the action $N(T)$ equal to $s_1\cup s_2$ and isomorphic to

$N(T)/T=W(T)$ as a $N(T)$-set. This shows that $W(T)$ is equal to $\boldsymbol{Z}_2$, and $N(T)$ is an extension of the torus T by $\boldsymbol{Z}_2$.

Moreover, since $T=S^1\subset G$ acts on $S\backslash S^T$ without fixed point and there is only a finite number of orbit types of action T on S, we can find a subgroup $K\cong \boldsymbol{Z}_p\subset S^1=T$, p-prime, such that $S^K=S^T\cong S^0$, i.e. the action of K is free outside S^0.

Let $g\in N(T)\backslash T$ be an element of $N(T)$ such that $gs_1=s_2$. By the choice $g^2\in T$. By the definition the ordering $[g]\mapsto g^{-1}(\ )g$ is the homomorphism from $W(T)$ into $\mathrm{Aut}(S^1)\cong GL(1, \boldsymbol{Z})\cong \{1, -1\}$. Consequently or $g^{-1}(t)g=t$, either $g^{-1}(t)g=-t$ for every $t\in T$.

In the first case g commutes with T and the extension $T\subset N(T)\to W(T)$ is trivial (the product). If $g^{-1}(\ )g=-\mathrm{Id}$, or equivalently $g^{-1}tg=t^{-1}$ for every $t\in T$, then the extension $T\subset N(T)\to W(T)$ is nontrivial. Substituting $t=g^2$, we get $g^4=1$, consequently or $g^2=1$ either $g^2=-1\in T$. In second case $\tilde{g}=g(-1)$ is an element of $N(T)\backslash T$ of order 2. For simplicity we denote it also by g. The above shows that the chosen element g acts in the same way on $K=\boldsymbol{Z}_p$, consequently a group H generated by a generator $h\in H$ and g is isomorphic or to the cyclic group $\boldsymbol{Z}_{2p}$ either to the dihedral group D_p, with the cyclic normal subgroup K. We shall establish the theorem if we show that

$$(**)\qquad \mathrm{cat}_H(S)=m+1$$

under our assumption. Indeed, it is easy to check that

$$\mathrm{cat}_G(X)\geqq \mathrm{cat}_H(X)\cdot\max\{\mathrm{cat}_H(G/G_x)\}\geqq \mathrm{cat}_H(X)(1+\max\dim(G/G_x)),$$

(cf. [Bar2] or [Ma2]), which gives desired estimate in view of equality (**).

Since $\dim S=m$ and H is finite we have $\mathrm{cat}_H S\leqq m+1$ ([Bar2], [Ma1]). We shall show the opposite inequality.

Let $\{U_i\}_1^k$ be an H-invariant minimal cover of S that the closure of any element of which is equivariantly deformed in S to an orbit H/H_i. Suppose that for $1\leqq i\leqq l$, $K_i=K$ and $H_i\neq K$ for $l+1\leqq i\leqq k$. In the first case the end map r_1^i, of H-equivariant deformation r_t^i, is an H-equivariant map of $\mathrm{cl}\,\mathcal{U}_i$ onto one orbit $s_1\cup s_2$, with $s_2=gs_1$. Note also that if $s_1\in \mathcal{U}_i$ then r_1^i is an equivariant deformation of $\mathrm{cl}\,\mathcal{U}_i$ onto s_1, since $r_0^i(s_1)=s_1$, $r_t^i(S^K)\subset S^K$ and $r^i(s_1\times I)$ is an connected set. Analogously for s_2. Remark also that from it follows that $l\geqq 1$.

We need the following lemma.

1.11 LEMMA. *Let S and H be as above. Set $\mathcal{U}=\bigcup_1^l \mathcal{U}_i$.*

Then there exists an H-equivariant map $r:\mathrm{cl}\,\mathcal{U}\to S^K=\{s_1, s_2\}$ such that $r(s_i)=s_i$, $i=1, 2$.

Proof. For a given $A\subset S$ let $[A]$ denote its image in the orbit space S/H. It is clear that it is enough to show the thesis for the counterimage, throughout the projection on orbit space, of each connected component of $[\mathrm{cl}\,\mathcal{U}]$. Next let $\mathcal{U}_0$ be the cover of connected component of $[\mathrm{cl}\,\mathcal{U}]$ containing $[S^0]$. We shall have established the lemma if we prove the existence of stated map for $\mathrm{cl}\,\mathcal{U}_0$, because for other components of $[\mathrm{cl}\,\mathcal{U}]$ a proof is analogous—even easier with respect to lack of the last requirement then. Decomposing each $\mathcal{U}_i$ following connected components of $[\mathrm{cl}\,\mathcal{U}_i]$ if necessary we can assume that $[\mathrm{cl}\,\mathcal{U}_i]$ is connected for every $1 \leqq i \leqq n$.

We show the existence of r on $\mathrm{cl}\,\mathcal{U}_0$ by an induction over i. For an H-equivariant map $r^i : \mathrm{cl}\,\mathcal{U}_i \to S^0$ by $\mathcal{U}_i^+$ or $\mathcal{U}_i^-$, we denote the set $(r^i)^{-1}(s_1)$, or correspondingly $(r^i)^{-1}(s_2)$. Of course $\mathrm{cl}\,\mathcal{U}_i^+ \cap \mathrm{cl}\,\mathcal{U}_i^- = \emptyset$. Suppose that $n=1$. By our assumption $s_1 \in \mathcal{U}_1$, and if $r^1(s_1)=s_2$ then taking $\tilde{r}^1 = gr^1$ we get an H-equivariant map from $\mathrm{cl}\,\mathcal{U}_1$ onto S^0 satisfying the second requirement. Indeed for an element $gk^q \in H$, k a generator of K, we have $\tilde{r}^1(gk^q x) := gr^1(gk^q x) = g^2 k^q r^1(x) = r^1(x)$, since r^1 is H-equivariant and K acts trivially on S^0. On the other hand $gk^q \tilde{r}^1(x) := gk^q gr^1(x) = r(x)$, since $gk^q g = k^q$ or $gk^q g = k^{-q}$.

Assume the required map exists for a cover $\{\mathcal{U}_i\}_1^n$ of invariant set as above. This means that for every $1 \leqq i$, $j \leqq n$ if $\mathrm{cl}\,\mathcal{U}_i \cap \mathrm{cl}\,\mathcal{U}_j = \emptyset$ then $\mathrm{cl}\,U_i^+ \cap \mathrm{cl}\,\mathcal{U}_j^+ = \emptyset$ and $s_1 \in \mathcal{U}_1$. By our assumption $\mathcal{W} := \bigcup_1^n \mathcal{U}_i$ consists of two components $\mathcal{W}^+ = \bigcup_1^n \mathcal{U}_i^+$ and $\mathcal{W}^- = \bigcup_1^n \mathcal{U}_i^-$, since $[\mathcal{W}]$ is connected. Take the set $\mathcal{U}_{n+1}$ the $(n+1)$-th element of the cover.

If $\mathrm{cl}\,\mathcal{W}^+ \cap \mathrm{cl}\,\mathcal{U}_{n+1}^+ \neq \emptyset$ then the required map is well defined. If $\mathrm{cl}\,\mathcal{W}^+ \cap \mathrm{cl}\,\mathcal{U}_{n+1}^+ = \emptyset$ then we have to take an H-equivariant map $\tilde{r}^{n+1} = gr^{n+1}$ instead of r^{n+1} as in the first step of induction. This shows that for $\{\mathcal{U}_I\}_1^n$ and $\mathcal{U}_{n+1}$, $\tilde{r}^{n+1}$ the required map is well defined. Note that if $\{s_0, s_1\} \in \mathrm{cl}\,\mathcal{U}_i$ then $r^i(s_j) = s_j$, as it has been already pointed out. This proves the lemma by induction. □

Observe that Lemma 1.11 shows that $k-l \geqq 1$ if $m \geqq 1$. Indeed, otherwise $S^m = \bigcup_1^l \mathcal{U}_i$ and Lemma 1.11 states that there exists an equivariant map from S^m onto $S^T = S^0$, which leads to a contradiction if $m \geqq 1$.

We complete the proof of Theorem 1.10 by the Borsuk-Ulam theorem argument. Form an invariant cover of S taking $\mathcal{U}_0 = \bigcup_1^l \mathcal{U}_i$ and $\mathcal{U}_i$, $l+1 \leqq i \leqq k$. Next take $V_0 = \boldsymbol{R}^1$ the real, one-dimensional nontrivial representation of H defined by the projection $H \mapsto H/K \cong \boldsymbol{Z}_2$ and the isomorphism $\boldsymbol{Z}_2 = O(1)$. Observe that the map $r : \mathrm{cl}\,\mathcal{U} \to S^0 = \{s_1, s_2\}$ defines an H-equivariant map from $\mathrm{cl}\,\mathcal{U}$ onto $O(1) = O(V)$ given by $s_1 \mapsto 1$, $s_2 \mapsto -1$, $\{1, -1\} \subset \boldsymbol{R}^1 = V_0$.

Moreover, for any $l+1 \leqq i \leqq k$ the mapping r^i maps $\mathrm{cl}\,\mathcal{U}_i$ onto an orbit Hx_i of $x_i \in S \backslash S^T$ with the isotropy group H_{x_i}, which is equal either to the trivial group $\boldsymbol{e}$ or conjugate to the two-elements cyclic group generated by g. In the first case the whole H-equivariant deformation $r_t^i : \mathrm{cl}\,\mathcal{U}_i \times I \to S_e \subset S$ lies in the H-free part of S. By the argument of Proposition 2.2 of [Ma1], there exists a K-equivariant deformation of $\tilde{r}_t^i : \mathrm{cl}\,\mathcal{U}_i \times I \to S_e$ onto an K-orbit $K_x \cong K$. In the second case we have an H-equivariant, thus K-equivariant, deformation which end is a K-equivariant map $r_1^i : \mathrm{cl}\,\mathcal{U}_I \to H/\{g\} = K$. Summing up, for every $l+1 \leqq i \leqq k$ we have a K-equivariant map $r^i : \mathrm{cl}\,U_I \to K$.

Put $\mathcal{W}_0 := \bigcup_{l+1}^k \mathcal{U}_i$. Finally, using the maps $\{r, r^{l+1}, \dots, r^{l+k}\}$ and a K-invariant partition of unity refined into $\{\mathcal{U}_{l+1}, \dots, \mathcal{U}_k\}$ we can construct a

K-equivariant map $h: \mathrm{cl}\,\mathcal{W}_0 \to S(V)$, where V is a unitary, free representation of $K = \boldsymbol{Z}_p$ of dimension $[(k-l+1)/2]$ and such that f is homotopic (not-equivariantly!) to the map onto a point if $k-l = \mathrm{cat}_K(\mathrm{cl}\,\mathcal{W}_0)$ and is an odd number (Lemmas 2.7 and 2.8 of [Ma1]). We can assume that $\mathrm{cat}_K(\mathrm{cl}\,\mathcal{W}_0) = k-l$, otherwise, adding $\mathcal{U}_0$ to any K-categorial invariant cover of $\mathrm{cl}\,\mathcal{W}_0$, we get K-categorial invariant cover of S consisting of less than k members.

Using next a K-invariant partition of unity ϕ, ψ refined to $\mathcal{U}_0$ and $\mathcal{W}_0$, and K-equivariant mappings r and h we can construct a K-equivariant map

$$f: S \longrightarrow S(V_0 \oplus V) = S(V_0) * S(V),$$

given by the formula $f(x) := [\phi(x) r(x), \psi(x) h(x)]$.

By its construction, the map $f^K: S^K = S^0 \to S^0 = S(V_0) = S(V_0 \oplus V)^K$ is the identity map, up to an identification, thus $\deg f^K \neq 0 \mod (p)$.

From the Borsuk-Ulam theorem for the group $G = \boldsymbol{Z}_p$ (cf. [Bar1], [Bar2], [C-P], [Ma1] Theorem 2.9, or [Ma2] Theorem 1.7 for an orthogonal action on S) it follows that $m-1 \leqq k-l-1$, which gives $k-l \geqq m$. Since $l \geqq 1$ we finally get the estimate $\mathrm{cat}_H(S) \geqq m+1$, which proves Theorem 1.10. □

1.12 *Remark.* It is worth of pointing out that the referred version ([Ma1] Thm. 2.9) of the Borsuk-Ulam theorem for $G = T$, $\boldsymbol{Z}_p^q$ is stated as follows. Let $f: X \to Y$, $X = Y = S^m$, be an equivariant map, $\deg f \neq 0$ ($\neq 0$ in $\boldsymbol{Z}_p$), then $X^G = S^r = Y^G$ and $\deg f \neq 0$ ($\neq 0$ in $\boldsymbol{Z}_p$). Here we need just opposite formulation, but a proof by the Borel localization theorem ([H]) also holds in this case (cf. [Ma1]). For a special case when $S = S(V)$ is the unit sphere of orthogonal representation V of $\boldsymbol{Z}_p$ the corresponding version of the Borsuk-Ulam theorem is just Theorem 1.7 c) of [Ma2].

2. $SO(n)$-category of spherical harmonics

Now we will study equivariant category of sphere, or its invariant subsets, of an irreducible representation of the group $G = SO(n)$. For any odd $n = 2m+1$ we apply results of Section 1 (Cor. 1.7) to show that the complement of the fixed point set of the maximal torus $T^m \subset SO(n)$ has a "large" category. In general it is difficult to compute $\dim M_{(H)} = \dim M^{(H)}$ even for a manifold being the sphere S. It is worth of pointing out that Balanov and Brodsky applied their version of Theorem 1.1 to derive an estimate for G-category of the free part of sphere of complex spherical harmonics $\mathcal{H}(n, l)$ i.e. the l-th irreducible representations of the group $SO(n)$, n-odd. To show it they did complicated computations in $\mathcal{H}(n, l)$ and their formula contains also a term equal to the maximum of dimensions of fixed point sets of cyclic subgroups [Ba-Br]. We use an observation that a larger set $S \backslash S^{(T)}$ has the G-category greater than the set $S_{(H)}$ for every representation of any compact Lie group G and easier to derive by use of Corollary 1.7. Moreover for the particular case of $G = SO(n)$ and $S = S(\mathcal{H}(n, l))$ the G-category of $S \backslash S^{(T)}$ is estimated by a function for which seems be easier to prove the monotonicity with respect to the both variables n and l. We shall work with the real representations, since the topological dimension corresponds to an algebraic then. However the calculations of Theorem 2.1 and Proposition 2.2 hold if we substitute corresponding terms multiplied by 2. Opposite the proof of Theorem 2.3 works in the real case only.

Let $n=2m+1$ be an odd number, $n\geqq 3$, and l be a natural number. Denote by $\mathcal{P}(n, l)$ the linear space of all real homogenous polynomials of degree l in n variables and by $\mathcal{H}(n, l)$ the corresponding space of spherical harmonics ($\mathcal{H}(n, l) \subset \mathcal{P}(n, l)$). Let $\xi, x_1, y_1, \dots, x_m, y_m$ be an orthogonal basis in $\boldsymbol{R}^n$, $n=2m+1$. It is well known ([BtD]) that a polynomial $f\in\mathcal{P}(n, l)$ of the form

$$f=\sum_{k=0}^{l}\frac{\xi^k}{k!}f_k(x_1, y_1, \dots, x_m, y_m)$$

belongs to $\mathcal{H}(n, l)$ iff for every k, $0\leqq k\leqq l-2$ we have $f_{k+2}=-\Delta f_k$, where Δ is the Laplace operator and $f_k\in\mathcal{P}(n, l-k)$. The group $G=SO(n)$ acts on $\mathcal{P}(n, l)$, thus also on $\mathcal{H}(n, l)$ by the formula $(gf)(u)=f(g^{-1}u)$, $g\in SO(n)$.

We start with $n=3$. Using the chose basis we can identify the maximal torus $T\subset SO(3)$ with the set of matrices of the form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix},$$

where $\phi\in[0, 2\pi]$. Note that $f(x_1, y_1)$ is an $S^1=T\subset SO(3)$ invariant iff it is of the form $f(x_1, y_1)=\omega(x_1^2+y_1^2)$, where $\omega(r_1^2)$, $r_i=x_i^2+y_i^2$ is homogeneous polynomial of even degree.

This means that for $l=2k+1$ odd if $f\in\mathcal{H}(3, l)^T$ then $f_0=0$ and $f_1(x_1, y_1)=c(x_1^2+y_1^2)^k$. For $l=2k$ we have the opposite $f_0=c(x_1^2+y_1^2)^k$ and $f_1=0$. In other words

$$\dim\mathcal{H}(3, l)^T=1.$$

It is well known that representations $\{\mathcal{H}(n, l)\}$, $l=0, 1, \dots$ form the complete set of all irreducible real representations of $SO(n)$.

Since $\delta(SO(3))=2$, we get the following application of Corollary 1.7.

2.1 PROPOSITION. *Let $V=\bigoplus_{i=1}^r\mathcal{H}(3, l_i)$, $l_i>0$ and $l_i>1$, for at least one i, be an orthogonal representation of $G=SO(3)$. Then there exists an invariant, open subset $\mathcal{U}\subset S(V)$, $\mathcal{U}=S(V)\setminus S(V)^{(T)}$ of $S(V)$ such that*

$$\operatorname{cat}_G(\mathcal{U})\geqq\frac{(\sum_{i=1}^r l_i)-1}{2}.$$

Proof. Since $\dim\mathcal{H}(3, l)=2l+1$ ([BtD]), $\dim S(V)=\sum_1^r(2l_i+1)-1=\sum_1^r 2l_i+r-1$. On the other hand $V^T\cong\boldsymbol{R}^r$, which gives $\dim S(V)^T=r-1$.

Moreover the only not zero-dimensional subgroups of $G=SO(3)$ are $T=S^1=SO(2)\subset O(2)$, which shows that all isotropy groups on $S(\mathcal{H}(3, l))\backslash S(\mathcal{H}(3, l)^{(T)})$ are finite if $l>1$. This force the same property for V. Substituting the above correspondingly to the numerator and denominator of the formula of Corollary 1.7 we get the statement. □

In a similar, but a little bit more complicated way we can study the case of an arbitrary odd $n=2m+1$. The above description of $\mathcal{H}(n, l)$ shows that $\mathcal{H}(n, l)=\mathcal{H}_0\oplus\mathcal{H}_1$, where $\mathcal{H}_0$ consists of the harmonics containing even degrees of ξ and $\mathcal{H}_1$ consists of the harmonics containing odd degree of ξ. It is known that $\mathcal{H}_0\cong\mathcal{P}(n-1, l)$, $\mathcal{H}_1\cong\mathcal{P}(n-1, l-1)$ not only as vector spaces but also as representations of $G=SO(n)$ ([BtD]). One can check that the maximal torus of $SO(n)$ is equal to $T=T^m$ and $\mathcal{H}(n, l)^T\cong\mathcal{P}(m, l/2)$, or $\mathcal{H}(n, l)^T\cong\mathcal{P}(m, (l-1)/2)$ if l is even or odd respectively. Since

$$\dim\mathcal{P}(n, l)=\binom{n+l-1}{l},$$

we have

$$\dim\mathcal{H}(n, l)=\binom{2m+l}{l}+\binom{2m+l-1}{l-1},$$

and

$$\dim\mathcal{H}(n, l)^T=\binom{m+l/2}{l/2} \quad \text{if } l \text{ is even, or}$$

$$\dim\mathcal{H}(n, l)^T=\binom{m+(l-1)/2}{(l-1)/2} \quad \text{if } l \text{ is odd,}$$

where in the round brackets are the Newton symbols.

Since $\dim SO(n)=m(2m+1)$, and $\dim T\subset SO(n)=m$, we have $\delta(SO(n))=2m^2$. The above equalities lead to the following consequence of Corollary 1.7.

2.2 THEOREM. *Let $\mathcal{H}(n, l)$ be l-th irreducible representation of the group $G=SO(n)$, $n=2m+1$. Then for the open invariant set $\mathcal{U}=S(\mathcal{H}(n, l))\backslash S(\mathcal{H}(n, l))^{(T)}$ we have*

$$\operatorname{cat}_G(\mathcal{U})\geqq\frac{\binom{2m+l}{l}+\binom{2m+l-1}{l-1}-\binom{m+[l/2]}{[l/2]}-2m^2}{1+m(2m+1)}.$$

Proof. In respect of above the statement is a direct consequence of Corollary 1.7, since $d(\mathcal{U})\leqq d(G)$. □

Note that right hand side of the inequality of Theorem 2.2 is greater than the corresponding term in the estimate of $\operatorname{cat}_G S(V_l)_{(e)}$ in [Ba-Br]. We are left with a task to show our main result.

2.3 THEOREM. *Let V_l be the l-th real irreducible representation of the group $G=SO(3)$ and $l=2q+1$ is odd. Then*

$$\mathrm{cat}_G(S(V_l)) \geqq \frac{2l+1}{4},$$

or equivalently

$$\mathrm{cat}_G(S(V_l)) \geqq q+1=\left[\frac{l}{2}\right]+1.$$

Proof. It is known that $\dim V_l=2l+1$ and $\dim(V_l)^T=1$, thus $\dim S(V_l)^T=0$ ([Bar2], [CLM]). Moreover the maximal torus of $SO(3)$ is equal to $T=SO(2)=S^1$ and $N(T)=O(2)$ ([Bre])). Furthermore if $l=2q+1$ is odd then $\dim(V_1)^{O(2)}=0$, thus $S(V_l)^{O(2)}=\emptyset$. Consequently we can apply Theorem 1.10, which gives the desired estimate, since $\dim SO(3)=3$. □

Theorem 2.3 gives an interesting example of a G-space for which the G-category is strictly greater than the G-genus of the same space. The equivariant genus of a G-space X, $X^G=\emptyset$, is, by definition (cf. [Bar2]), the smallest number $k\geqq 0$ such that there exists a G-map $f: X \to (G/H_1)* \cdots *(G/H_k)$, $H_i \neq G$, and it is denoted by $\mathrm{genus}_G(X)$.

2.4 COROLLARY. *If $l>103$ is an odd number then for the l-th real irreducible representation of the group $G=SO(3)$ we have*

$$\mathrm{cat}_G(S(V_l)) > \mathrm{genus}_G(S(V_l)).$$

Moreover the right hand side of the above inequality is a constant independent on l but the left tends to infinity as $l \to \infty$.

Proof. By Theorem 2.3 we have

$$\mathrm{cat}_G(S(V_l)) \geqq \frac{2l+1}{4}.$$

On the other hand Bartsch in [Bar1] gave a sufficient and necessary condition on a group G for the existence a finite-dimensional representation W_0, $W_0^G=\{0\}$, of G such that for any G-ANR X, $X^G=\emptyset$, there exists an equivariant map $f: X \to S(W_0)$. He also checked that $G=SO(3)$ fulfils this condition and derived a form of W_0 for this particular case of $G=SO(3)$ ([Bar2]). It is sufficient to take the complex representation $W_0=V_6 \oplus V_6$, which is representation of complex dimension 26. It is well known (cf. [Bar2]) that $\mathrm{genus}_G(X) \leqq \dim(X/G)+1$, and surely $\mathrm{genus}_G(X) \leqq \mathrm{genus}_G(Y)$ if there is a G-map $f: X \to Y$. Consequently for every $SO(3)$ space X we have

$$\mathrm{genus}_G(X) \leqq 52.$$

This proves the statement. □

Finally we illustrate Theorem 2.3 by an application. For a natural reason we will discuss a nonlinear Dirichlet problem (for the cube domain see [Be-Pa] and [Kr-Ma]).

Let $f:\boldsymbol{R}\to\boldsymbol{R}$ be a C^1-function such that $f(0)=0$, $f'(0)=1$. Let us consider the equation

$$\Delta u=\mu f(u), \quad u\in H_0^1(\Omega), \tag{2.5}$$

where $\Omega=D(0, 1)\subset\boldsymbol{R}^3$ is the unit open disc, and $\mu\in\boldsymbol{R}$ is the parameter. Note that if $u(x)$ is a smooth solution of (2.5) then $u_{|\partial D}=0$, since $u\in H_0^1(\Omega)$. Observe that the group $SO(3)$ acts on $H_0^1(\Omega)$ by the formula $T_g u(x)=u(gx)$, $g\in SO(3)$, $x\in\Omega$. It is well known that problem (2.5) is variational (i.e. the solutions correspond to the critical points of a functional) provided a regularity conditions are posed on the function $F(t)=\int_0^t f(s)ds$. Then the functional is given as

$$\mathcal{J}(u)=\int_\Omega F(u(x))dx,$$

is C^2 on $H:=H_0^1(\Omega)$, $SO(3)$-invariant and the critical points of $\mathcal{J}$ on the sphere S_ε in H correspond to the classical solutions of (2.5). It is known (see for example [M], [Be-Pa]) that the problem of finding solutions of

$$\operatorname{grad}\mathcal{J}(u)=\lambda u, \tag{2.6}$$

is locally reduced to a problem of finding critical points of a invariant function on a G-manifold M_ρ G-homeomorphic to the sphere in the eigenspace V_μ of the Laplacian $-\Delta$ in $H_0^1(\Omega)$, near the point $(0, \lambda)$, $\mu=1/\lambda$ is the corresponding eigenvalue.

On the other hand the spectrum of Laplacian i.e. the solutions of equation

$$-\Delta u=\mu u, \quad u_{|\partial D}=0, \tag{2.7}$$

is completely described (eigenvalues and eigenspaces) (cf. [VL]). By use of the Laplace separating variables we replace equation (2.7) by a pair of boundary problems.

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}+\lambda v=0, \tag{2.7a}$$

where $v=v(\theta, \phi)\in C^\infty(S^2=\partial D)$, θ, ϕ are polar coordinates on the sphere and λ is a real number.

$$(r^2\mathcal{R}')'+(\mu r^2-\lambda)\mathcal{R}=0, \quad |\mathcal{R}(0)|<\infty, \quad \mathcal{R}(1)=0, \tag{2.7b}$$

where $\mathcal{R}$ is a smooth function on $\boldsymbol{R}$, and μ is the parameter of (2.6).

It is well known that for $\lambda=l(l+1)$, $l=0, 1, \dots$ the problem (2.7a) has solutions equal to the spherical functions v_l^m, $m=0, \pm1, \pm2, \dots, \pm l$. In other words the eigenvalue is equal to $l(l+1)$ and the eigenspace spanned by v_l^m, is isomorphic to V_l the l-th irreducible representation of $G=SO(3)$. Each such λ gives a infinite number of distinct eigenvalues

$$\mu_{lj}=(\lambda_j^{(l+1/2)})^2,$$

where $(\lambda_j^{(l+1/2)})$ is the j-th positive root of the Bessel function $J_{l+1/2}(\lambda)$. Moreover the eigenspace corresponding to μ_{lj} is spanned by the functions $\psi_{ljm}=C_{ljm}v_l^m(\theta, \phi)$, thus isomorphic to V_l as a representation of $SO(3)$.

The above leads to the following.

2.8 PROPOSITION. *Let $\mu_0=\mu_{lj}>0$ be an eigenvalue of the problem* (2.6), *for an odd $l=2q+1$. Then for every sufficiently small $\rho>0$ the problem* (2.5)

$$\Delta u=\mu f(u), \quad u\in H_0^1(\Omega),$$

has at least $q+1=[l/2]+1$ distinct $SO(3)$-orbits of solutions (u, μ) such that $\|u\|=\rho$ and μ is close to μ_0.

Proof. It is enough to the mentioned method of Marino, and Benci and Pacella ([M], [Be-Pa], and also [Kr-Ma]) in local, equivariant, variational Liapunov-Schmidt reduction and then apply Theorem 1.10 in view of the form of spectrum of problem (2.6). □

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