Q. CHEN
KODAI MATH. J.
20 (1997), 85—91

ON THE FIRST EIGENVALUE OF MINIMAL SUBMANIFOLDS

QING CHEN

§ 1. Let (M, g_{ij}) be a Riemannian manifold. Given a compact domain $D \subset M$ with C^1-boundary, the first eigenvalue of D of the Laplacian under the Dirichlet boundary condition is defined to be the smallest positive number λ_1 such that, for some non identically zero function f, $f|_{\partial M}=0$ and $\Delta f+\lambda_1 f=0$, where Δ is the Laplacian of M and action of Δ on functions is defined by

$$\Delta f=\frac{1}{\sqrt{\det(g_{ij})}}\sum_{i,j}\frac{\partial}{\partial x^i}\left\{\sqrt{\det(g_{ij})}g^{ij}\frac{\partial f}{\partial x^j}\right\},\quad (g^{ij})=(g_{ij})^{-1}.$$

The first eigenvalue of domain D is denoted by $\lambda_1(D)$ in this paper. For the first eigenvalue of minimal submanifolds in a space form, S. Y. Cheng, P. Li and S. T. Yau [CLY] proved

Let $M \to N(c)$ be a minimal immersed n-dimensional submanifold, $N(c)$ is a space form of constant curvature $c=1, 0$ or -1. Suppose D is a C^2 compact domain in M. If $D \subset B(a)$, a geodesic ball of radiusa in $N(c)$, if $c=1$ then assume $a \leqq \pi/2$, otherwise $a<+\infty$, then

$$\lambda_1(D) \geqq \lambda_1(B_a(n, c)),$$

where $B_a(n, c)$ is a geodesic ball of radius a in an n-dimensional space form of constant curvature c.

Motivating by a paper of V. G. Tkachev [T], we generalize above theorem as following.

THEOREM 1. *Let N be a Riemannian manifold with the sectional curvature bounded from above by a constant c. Denote $B_a(p)$ the geodesic ball of N of center p and radius a, $i(p)$ the injective radius of N at p.*

Let M be an n-dimensional immersed minimal submanifold in N. Suppose D is a compact domain with C^1-boundary in M. If $D \subset B_a(p)$ for some $p \in M$, assume either $a<i(p)$ when $c<0$ or $a<\min(i(p), \pi/2\sqrt{c})$ when $c>0$. Then

$$\lambda_1(D) \geqq \lambda_1(B_a(n, c)),$$

Received November 30, 1995; revised March 7, 1997.

where $B_a(n, c)$ is a geodesic ball of radius a in an n-dimensional space form of constant curvature c.

When the domain D has *C^2-boundary,* the theorem can be obtained easily from the heat kernel comparison theorem of S. Markvorson [M], see the remark in section 2.

Theorem 1 gives a lower bound of the first eigenvalue of minimal submanifolds which only depends on the upper bound of sectional curvature of ambient manifold. Recently Coghlanand and Itokawa [CI] obtained a lower bound of injective radius of submanifolds which also depend on the upper bound of sectional curvature of ambient manifold.

Acknowledgment. The paper was finished in the Department of Mathematical Sciences, The University of Tokyo. The author sincerely thanks Professor T. Ochiai for his inspiring guidance and consistent encouragement. The author would like to thank the refree for many useful suggestions, and informing me of Markvorson's results.

§2. Let M be an n-dimensional minimally immersed submanifold in N. Denote the covariant derivative of N and M by D and ∇ respectively. The second fundamental form A of M is a symmetric linear map from $TM \otimes TM$ to the normal bundle of M, defined by

$$A(X, Y) = -D_X Y + \nabla_X Y, \quad \text{for } X, Y \in TM. \tag{1}$$

M is said to be minimal if $\operatorname{tr} A = 0$.

Denote r the distance function of N with respect to a fixed point p_0. Suppose the sectional curvature of N is bounded from above by a constant c. For any point p in cut locus of p_0, r is smooth at p, and the Hessian comparison theorem ([CE]) reads

$$(D^2 r)(X, X)(p) \geqq \left(\frac{\varphi_c'}{\varphi_c} \circ r\right)(p)(|X|^2 - \langle X, Dr\rangle^2), \tag{2}$$

for any $X \in T_p M$. Where $\varphi_c(t)$ is the solution of following

$$\begin{cases} \varphi_c''(t) + c\varphi_c(t) = 0, \\ \varphi_c(0) = 0, \quad \varphi_c'(0) = 1, \end{cases} \tag{3}$$

i.e.

$$\varphi_c(t) = \begin{cases} \dfrac{1}{\sqrt{c}} \sin \sqrt{c}\, t, & \text{if } c > 0; \\ t, & \text{if } c = 0; \\ \dfrac{1}{\sqrt{|c|}} \sinh \sqrt{|c|}\, t, & \text{if } c < 0. \end{cases} \tag{4}$$

In the following, $r|_M$ is still denoted by r.

PROPOSITION. *We have, in* $M\cap B_{i(p_0)}(p_0)$,

$$\Delta r \geqq \left(\frac{\varphi_c'}{\varphi_c}\circ r\right)(n-|\nabla r|^2), \tag{5}$$

where Δ *is the Laplacian of* M.

Proof. For a point $p\in M\cap B_{i(p_0)}(p_0)$. Choose $e_1, e_2, \dots, e_n$, an orthonormal basis of T_pM, then

$$\begin{aligned}\Delta r &= \sum_i (\nabla^2 r)(e_i, e_i)\\ &= \sum_i \{(D^2 r)(e_i, e_i)+A(e_i, e_i)\}\\ &\geqq \sum_i \left(\frac{\varphi_c'}{\varphi_c}\circ r\right)(|e_i|^2-\langle e_i, Dr\rangle^2)\\ &= \left(\frac{\varphi_c'}{\varphi_c}\circ r\right)(n-|\nabla r|^2).\end{aligned}$$

This prove the proposition.

Before proceed further we recall some well known facts about eigenfunction of space form. We denote Δ_c the Laplacian of n-dimensional space form $N^n(c)$ of constant curvature c, and ρ the distance function of $N^n(c)$ with respect to a fixed point. It is well known

$$\Delta_c\rho=(n-1)\frac{\varphi_c'}{\varphi_c}\circ\rho, \tag{6}$$

Suppose f is a first eigenfunction of domain $B_a(n, c)\subset N^n(c)$. We can write $f=f(\rho)$ since f is rotationally symmetric. Then by (6), $\Delta_c f+\lambda_1(B_a(n, c))f=0$ and $f|_{\partial B_a(n,c)}=0$ imply that f satisfies

$$\begin{cases}\varphi_c(t)f''(t)+(n-1)\varphi_c'(t)f'(t)+\lambda_1(B(_a(n, c))\varphi_c(t)f=0\\ f'(0)=f(a)=0,\end{cases} \tag{7}$$

and $f(t)>0$ for $t\in[0, a)$. (If there were $b\in[0, a)$ such that $f(b)=0$, then f would be a first eigenfunction of $B_b(n, c)$, contradicts with $B_b(n, c)>B_a(n, c)$).

LEMMA 1 (Corollary 1 of Theorem 4 of [SY]). *Given any compact domain* D *in* M *with* C^1*-boundary, for any positive* C^2 *function* f *defined on* D,

$$\lambda_1(D)\geqq\inf_{p\in D}\left(-\frac{\Delta f}{f}\right).$$

Proof of Theorem 1. Let f be the solution of (7) and put $F=f\circ r$. Then F is a smooth function on the domain $D\subset B_a(p)$ and F is positive in interior of D. We have

$$\Delta F=f'\circ r\Delta r+f''\circ r|\nabla r|^2. \tag{8}$$

We shall show in next section (Theorem 2) that f' and $f'\varphi_c'-f''\varphi_c$ are both negative in $(0, a)$. Submitting (5) into (8) we obtain

$$\begin{aligned}\Delta F&\leqq(n-|\nabla r|^2)\Big(\frac{f'\varphi_c'}{\varphi_c}\Big)\circ r+|\nabla r|^2 f''\circ r\\ &=\Big((n-1)\frac{f'\varphi_c'}{\varphi_c}+f''\Big)\circ r+(1-|\nabla r|^2)\Big(\frac{f'\varphi_c'-f''\varphi_c}{\varphi_c}\Big)\circ r\\ &\leqq-\lambda_1(B_a(n, c))f\circ r\\ &=-\lambda_1(B_a(n, c))F,\end{aligned} \tag{9}$$

where the last inequality is followed by $|\nabla r|^2\leqq|Dr|^2=1$. Thus the theorem is followed by the Lemma 1.

Remark. 1. When D is of C^2-boundary, Theorem 1 is an easy corollary of a result of Markvorson [M], we show this as follow:

Let $p(t, x, y)$ and $\bar{p}(t, x, y)$ be the heat kernels of D and the corresponding domain in the space form respectively. Then by the result of Markvorson [M], we have

$$p(t, x, y)\leqq\bar{p}(t, x, y).$$

Let ϕ_1 be the first eigenfunction of D, and $\bar{\lambda}_i$, $\bar{\phi}_i$, the i-th eigenvalue and i-th eigenfunction of the corresponding domain in the space form. Then

$$\begin{aligned}e^{-\lambda_1 t}\phi_1(x)^2&\leqq p(t, x, x)\\ &\leqq\bar{p}(t, 0)\\ &=\sum e^{-\bar{\lambda}_i t}\bar{\phi}_i(x)^2.\end{aligned}$$

If $\lambda_1<\bar{\lambda}_1$, then

$$\phi_1(x)^2\leqq\sum e^{(\lambda_1-\bar{\lambda}_i)t}\bar{\phi}(x)^2 \longrightarrow 0$$

as $t\to+\infty$, it contradicts to the fact that $\phi_1(x)\neq 0$ in the interior of D. So $\lambda_1\geqq\bar{\lambda}_1$.

2. By Hopf's maximal principle, Lemma 1 holds when D is a compact domain with piecewise smooth boundary. Hence Theorem 1 is still valid in this case.

§3. Consider following equation with two parameters λ, μ.

$$\text{(10)} \qquad \begin{cases} \varphi_c(t)f''(t)+\mu\varphi_c'(t)f'(t)+\lambda\varphi_c(t)f(t)=0, & t\in[0, a), \\ f(0)=\xi_0>0, \quad f'(0)=0, \\ \mu>0. \end{cases}$$

Let f be the solution of (10), we have

LEMMA 2. *If f is positive in $(0, a)$ and $\lambda>0$, then f' is negative in $(0, a)$.*

Proof. Since $\varphi_c'(0)=1$ and $\varphi_c(t)/t\to 1$ as $t\to 0$, by (10)

$$\begin{aligned} 0&=f''(0)+\mu\lim_{t\to 0}\frac{\varphi_c'(t)}{\varphi_c(t)}f'(t)+\lambda\xi_0 \\ &=f''(0)+\mu f''(0)+\lambda\xi_0, \end{aligned}$$

i.e.

$$\text{(11)} \qquad f''(0)=-\frac{\lambda\xi_0}{1+\mu}<0.$$

Thus $f(t)$ is strictly decreasing in some neighbourhood of origin, since $f'(0)=0$. If f take a local minimum at $t_0\in(0, a)$, then $f'(t_0)=0$ and by (10) $f''(t_0)=-\lambda f(t_0)<0$, a contradiction. So f is non-increasing function in $(0, a)$, i.e. $f'\leqq 0$ in $(0, a)$.

To prove the lemma we suppose there is a $t_1\in(0, a)$ such that $f'(t_1)=0$, then t_1 is a local maximum of $f'(t)$ which implies $f''(t_1)=0$. Using equation (10) again we get $f(t_1)=0$, contradiction. This proves the lemma.

LEMMA 3. *If $f(t)$ is a solution of (10), then $f_1(t)=f'(t)/\varphi_c(t)$ is a solution of (10) with two new parameters $\bar{\mu}=\mu+2$, $\bar{\lambda}=\lambda-c(\mu+1)$ and*

$$f_1(0)=\frac{\lambda\xi_0}{\mu+1}, \quad f_1'(0)=0.$$

Proof. As in the proof of Lemma 2

$$\lim_{t\to 0}f_1(t)=-f''(0)=\frac{\lambda\xi_0}{\mu+1}.$$

Differentiation of equation (10) yields

$$\text{(12)} \qquad \varphi_c f'''+(\mu+1)\varphi_c'f''+\mu\varphi_c''f'+\lambda\varphi_c f'+\lambda\varphi_c'f=0.$$

Divide (12) by t and put $t\to 0$, since

$$\lim_{t\to 0}\frac{\varphi_c''(t)}{t}=c\lim_{t\to 0}\frac{\varphi_c(t)}{t}=c,$$

and $\varphi_c'(0)=1$, $f'(0)=0$, we get

$$\begin{aligned}0&=f'''(0)+\lim_{t\to 0}\Big((\mu+1)\frac{f''(t)}{t}+\lambda\frac{f(t)}{t}\Big)\\&=f'''(0)+\lim_{t\to 0}\Big((\mu+1)\frac{f''(t)-f''(0)}{t}+\lambda\frac{f(t)-f(0)}{t}\Big)\\&=(\mu+2)f'''(0),\end{aligned}$$

i.e.

$$f'''(0)=0. \tag{13}$$

On the other hand we compute directly that

$$\begin{aligned}f_1'(t)&=\frac{-\varphi_c(t)f''(t)+\varphi_c'(t)f'(t)}{\varphi_c^2(t)},\\f_1''(t)&=\frac{-\varphi_c^2(t)f'''(t)+2\varphi_c(t)\varphi_c'(t)f''(t)+(\varphi_c(t)\varphi_c''(t)-2(\varphi_c'(t))^2)f'(t)}{\varphi_c^3(t)}.\end{aligned}\tag{14}$$

So that

$$\begin{aligned}f_1'(0)&=\lim_{t\to 0}\frac{-\varphi_c(t)f''(t)+\varphi_c'(t)f'(t)}{\varphi_c^2(t)}\\&=\lim_{t\to 0}\frac{\varphi_c''(t)f'(t)+\varphi_c'(t)f''(t)-f'''(t)\varphi_c(t)-f''(t)\varphi_c'(t)}{2\varphi_c(t)\varphi_c'(t)}\\&=-\frac{1}{2}f'''(0)=0,\end{aligned}$$

and

$$\begin{aligned}&\varphi_c f_1''+(\mu+2)\varphi_c' f_1'+(\lambda-c(\mu+1))\varphi_c f_1\\&=\frac{1}{\varphi_c^2}\{-\varphi_c^2 f'''+2\varphi_c\varphi_c' f''+(\varphi_c\varphi_c''-2(\varphi_c')^2)f'+(\mu+2)(\varphi_c')^2 f'-(\mu+2)\varphi_c\varphi_c' f''\}\\&\qquad\qquad -(\lambda-c(\mu+1))f'\\&=-f'''-\mu\frac{\varphi_c'}{\varphi_c}f''+\mu\Big(\frac{\varphi_c'}{\varphi_c}\Big)^2 f'+\frac{\varphi_c''}{\varphi_c}f'+(c(\mu+1)-\lambda)f'\\&=-\frac{d}{dt}\Big(f''+\mu\frac{\varphi_c'}{\varphi_c}f'+\lambda f\Big)+\lambda f'+(\mu+1)\frac{\varphi_c''}{\varphi_c}f'+(c(\mu+1)-\lambda)f'\\&=0,\end{aligned}$$

where the last equality is followed by equation (10) and $\varphi_c''+c\varphi_c=0$. This complete the proof of the lemma.

Combining with Lemma 2 and Lemma 3 we have

THEOREM 2. *Let f be a solution of* (10) *with $\mu=n-1$ and $\lambda=\lambda_1(B_a(n, c))$, and f is positive in* $[0, a)$. *If $c>0$ we assume $a<\pi/2\sqrt{c}$. Then f' and $-\varphi_c f''+\varphi_c' f'$ are both negative in* $(0, a)$.

Proof. The negativity of f' is followed from Lemma 2. By Lemma 3, $f_1=-f'/\varphi_c$ is a solution of (10) with $\mu=n+1$ and $\lambda=\lambda_1(B_a(n, c))-cn$, and f_1 is positive in $[0, a)$. Notice $\lambda_1(B_a(n, c))-cn>0$, since when $c>0$ by assumption (cf. [Ch]),

$$\lambda_1(B_a(n, c))>\lambda_1(B_{\frac{\pi}{2\sqrt{c}}}(n, c))=cn\,.$$

So by Lemma 2 again we see $(d/dt)f_1<0$ in $(0, a)$, which implies $-\varphi_c f''+\varphi_c' f'<0$ in $(0, a)$, we complete the proof.

REFERENCES

[Ch] S.-Y. CHENG, Eigenvalue comparison theorems and its geometric application, Math. Z., **143** (1975), 289-297.

[ChY] S.-Y. CHENG AND S.-Y. YAU, Differential equations on Riemannian manifold and their geometric application, Comm. Pure Appl. Math., **28** (1975), 333-354.

[CI] L. COGHLAN AND Y. ITOKAWA, An injective radius estimate and stability of minimal submanifolds, preprint.

[CLY] S.-Y. CHENG, P. LI AND S.-T. YAU, Heat equations on minimal, submanifolds and their applications, Amer. J. Math., **106** (1984), 1033-1065.

[GW] R.E. GREENE AND H. WU, Function Theory on Manifolds Which Possess a Pole, Lecture Notes in Math., **699**, Springer-Verlag, 1979.

[M] S. MARKVORSON, On the heat kernal comparison theorems for minimal submanifolds, Proc. Amer. Math. Soc., **97**, 479-482.

[T] V.G. TKACHEV, A sharp lower bound for the first eigenvalue on a minimal surface, Math. Notes. (Trans.), **54** (1994), 835-840.

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA
HEFEI, ANHUI, P.R. CHINA