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NOTE ON BP-THEORY FOR EXTENSIONS OF CYCLIC GROUPS BY ELEMENTARY ABELIAN p-GROUPS

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Introduction

Let $BP^*(-)$ be the Brown-Peterson cohomology theory and $K(m)^*(-)$ the Morava K-theory. It is conjectured [K-Y], [H-K-R], [H] that $BP^{\mathrm{odd}}(BG)=K(m)^{\mathrm{odd}}(BG)=0$ for finite groups and even compact Lie groups G. In this note we show that the conjecture is affirmative for the cases G are extensions

(0.1) $$0 \longrightarrow (Z/p)^n \longrightarrow G \longrightarrow Z/p^s \longrightarrow 0.$$

We first show $H^{\mathrm{odd}}(BZ/p^s; BP^*(B(Z/p)^n)=0$ and hence $BP^{\mathrm{odd}}(BG)=0$. Using a result of Tezuka-Yagita [T-Y], we next see $K(m)^{\mathrm{odd}}(BG)=0$.

This note is motivated by the Kriz' study $K(m)^*(BG)$ for $p=3$, $n=4$ and $s=2$. (Recently he announced the similar result as above.) The author thank to Bjon Schuster and Geoffrey Falk for some arguments.

§1. $BP^*(B(Z/p)^n)$

It is well known [L], [J-W] that $BP^*(B(Z/p)^n)\cong\otimes^n{}_{BP^*}BP^*(B(Z/p))$ and $BP^*(BZ/p)\cong BP^*[[y]]/([p](y))$ where $[p](y)=y+_{BP}+\cdots+_{BP}y=py+v_1y^p+\cdots$ is the p-th sum of the formal group law for BP-theory with the coefficient $BP^*=Z_{(p)}[v_1, \ldots]$. We will study more detail in this section.

Recall the Milnor operation $Q_0=\beta$, $Q_n=Q_{n-1}\rho^{p^{n-1}}-\rho^{p^{n-1}}Q_{n-1}$ (for $p=2$, $Q^0=Sq^1$, $Q_n=Q_{n-1}Sq^{2^n}-Sq^{2^n}Q_{n-1}$) and let us write $H^*(BZ/p; Z/p)\cong Z/[y]\otimes\Lambda(x)$, $\beta x=y$ (for $p=2$, let $y=x^2$). Then $Q_nx=y^{p^n}$. Recall $P(i)^*(-)$ is the complex oriented cohomology theory with the coefficient $P(i)^*=BP^*/(p, \ldots, v_{n-1})=Z/p[v_n, \ldots]$ (see [J-W] for details). In the spectral sequence

$$E_2^{*,*}=H^*(X; P(i)^*) \Longrightarrow P(i)^*(X),$$

the first non zero differential is $d_{2p^i-1}(x)=v_i\otimes Q_i(x)$ for all $x\in H^*(X; Z/p)$.

Consider Atiyah-Hirzebruch spectral sequences

$$E_2^{*,*}(X)=H^*(X; BP^*) \Longrightarrow BP^*(X)$$
$$E_2^{*,*}(X\times BZ/p)=H^*(X\times BZ/p; BP^*) \Longrightarrow BP^*(X\times BZ/p).$$

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LEMMA 1.1. *Assume that there is a filtration such that* $\mathrm{gr}\, E_\infty^{*,*}(X)=\bigoplus_{i=0}^{n-1} P(i+1)^* Q_0 \cdots Q_i G_i$ *for some* $G_i \subset H^*(X; Z/p)$. *Then there is a filtlation such that*

$$\mathrm{gr}\, E_\infty^{*,*}(X\times BZ/p)=\bigoplus_{i=0}^{n} P(i+1)^* Q_0 \cdots Q_i G_i'$$

with $G_i'=G_i\otimes Z/p\{1, y, \dots, y^{p^{i+1}-1}\}\oplus G_{i-1}\otimes Z/p[y]\{x\}$ *here* $Z/p\{a, \dots\}$ *means the free* Z/p*-module generated by* $a, \dots$.

Proof. Consider the spectral sequence

$$EE_2^{*,*}=H^*(BZ/p\,; \mathrm{gr}\, E_\infty^{*,*}(X)) \Longrightarrow \mathrm{gr}\, E_\infty^{*,*}(X\times BZ/p).$$

The EE_2-term is a direct sum of $P(i+1)^*\otimes Q_0 \cdots Q_i G_i \otimes Z/p[y]\otimes \Lambda(x)$. Since $d_{2p^{i+1}-1}x=v_{i+1}\otimes Q_{i+1}x=v_{i+1}y^{p^{i+1}}$ in the above $P(i+1)^*$-module, we get

$$\begin{aligned}&\mathrm{gr}\, E_\infty^{*,*}(X\times BZ/p)=EE_\infty^{*,*}\\ &=\bigoplus_{i=0}^{n}(P(i+1)^*\{1, \dots, y^{p^{i+1}-1}\}\oplus P(i+2)^*[[y]]\{y^{p^{i+1}}=Q_{i+1}x\})Q_0 \cdots Q_i G_i.\end{aligned}$$

Therefore taking G' as stated in this lemma, we have

$$\mathrm{gr}\, E_\infty^{*,*}=\bigoplus_{i=0}^{n} P(i+1)^* Q_0 \cdots Q_i G_i'. \qquad \text{q.e.d.}$$

Let us write $H^*(B(Z/p)^n; Z/p)=S_n\otimes \Lambda_n$ with $S_n=Z/p[y_1, \dots, y_n]$ and $\Lambda_n=\Lambda(x_1, \dots, x_n)$ with $\beta x_i=y_i$.

COROLLARY 1.2. $\mathrm{gr}\, BP^*(B(Z/p)^n)\cong\bigoplus_{i=0}^{n-1} P(i+1)^*(\mathrm{Im}\, Q_0 \cdots Q_i \cap S_n)$.

For non negative sequence $I=(i_1, \dots, i_n)$, denote by y_I the element $y_1^{i_1} \cdots y_n^{i_n}$ in $H^*(B(Z/p)^n; Z/p)$ or in $BP^*(B(Z/p)^n)$. Define length $l(I)\geqq m$ if there is a subsequence $(i_{s0}, \dots, i_{sm})\subset I$ such that $p^k\leqq i_{sk}$ for all $0\leqq k\leqq m$. Since

$$Q_0 \cdots Q_m(x_1 \cdots x_{m+1})=y_1 y_2^p \cdots y_{m+1}^{p^m} \quad \text{module } \{y_I \mid l(I)<m\},$$

we can easily prove

$$\{y_I \mid l(I)=m\}=\mathrm{Im}\, Q_0 \cdots Q_m \cap S_n \quad \text{modulo } \{y_I \mid l(I)<m\}.$$

LEMMA 1.3. *Taking filtration by* $l(I)$, *we have*

$$\mathrm{gr}\, BP^*(B(Z/p)^n)=\bigoplus_{i=0}^{n-1} P(i+1)^*\{y_I \mid l(I)=i\}.$$

§ 2. $BP^*(BG)$

We consider the spectral sequence

$$E_2^{*,*}=H^*(B(Z/p^s); BP^*(B(Z/p)^n)) \Longrightarrow BP^*(BG). \tag{2.1}$$

induced from the extension (0.1). Let b be the generator of Z/p^s. The action of b on $(Z/p)^n$ is represented by an element in $U\subset GL_n(Z/p)$; upper triangular matrices with diagonal entry=1. Moreover changing basis in $(Z/p)^n$, the action b is represented as a Jordan's normal form, that is,

(2.2) $$by_1=y_1,\quad by_i=y_i+_{BP}\varepsilon_i y_{i-1}\quad \text{for } i\geqq 2 \text{ and } \varepsilon_i=0 \text{ or } 1.$$

The E_2-term of the spectral sequence (2.1) is expressed as

$$E_2^{j,*}\cong\begin{cases}\operatorname{Ker}(1-b) & \text{for } j=0\\ \operatorname{Ker}(1-b)/\operatorname{Im} N & \text{for } j=\text{even}>0\\ \operatorname{Ker} N/\operatorname{Im}(1-b) & \text{for } j=\text{odd}>0\end{cases}$$

where $N=1+b+\cdots+b^{p^s-1}$. We will prove;

LEMMA 2.3. $\operatorname{Ker} N/\operatorname{Im}(1-b)=0$.

For the proof, we prepare some notations. Given an element $x\in BP^*(B(Z/p)^n)$, we can uniquely write it from Lemma 1.3 as

(2.4) $$x=\sum a_I y_I \quad \text{with } 0\neq a_I\in P(l(I)+1)^*.$$

For each sequence $I=(i_1, \dots, i_n)$, define the moment by $\|I\|=i_1+\cdots+i_n$ and define the lexicographic order $I>I'$ if there is k so that $i_k>i'_k$ and $i_j=i'_j$ for all $j>k$. Let J be the maximal order in which moment $\|J\|$ are smallest of I in (2.4), namely,

(2.5) $$x=a_J y_J \quad \text{modulo (BMSO)}$$

where (BMSO)=(bigger moment and smaller order elements)$=\{a_K y_K \mid \|K\|>\|J\| \text{ or } (\|J\|=\|K\| \text{ and } J>K)\}$.

Proof of Lemma 2.3 for the case $n\leqq p$, $s=1$ and $\varepsilon_i=1$ for $2\leqq i$. First note that

$$by_{i+1}=y_{i+1}+_{BP}y_i=y_{i+1}+y_i \mod \{a_I y_I \mid \|I\|>p\}.$$

So it is immediate that

$$(b-1)y_{i+1}^j=(y_{i+1}+y_i)^j-y_i^j=jy_{i+1}^{j-1}y_i \mod \text{(BMSO)}.$$

By the definition of the order, we can easily show if $i_2\neq 0 \mod p$, then

(2.6) $$(b-1)y_I=i_2 y_I(y_1/y_2) \mod \text{(BMSO)}.$$

Suppose $x\in\operatorname{Ker} N$. By inductive assumption on n, we suppose $j_1\neq 0$. From (2.6), we can take adequate $X\in BP^*(B(Z/p)^n)$ such that

(2.7) $$x-(b-1)X=a_J y_J \mod \text{(BMSO)}, \quad \text{with } j_2=p-1 \mod p.$$

Suppose the above element is non zero, that is $0\neq a_J\in P(l(J)+1)^*$. Let $y_2(p)=\prod_{i\in Z/p}b^i y_2$ so that this element is invariant under b. Let y_J' be element made from y_I exchanging factors y_2^p by $y_2(p)$, that is,

$$y_J'=y_J(y_2(p)/y_2^p)^{[j_2/p]}=y_1^{j_1}y_2(p)^{[j_2/p]}y_2^{p-1}y_3^{j_3}\cdots y_n^{j_n}.$$

Then we have

$$\begin{aligned}Ny_J'&=(y_J'/y_2^{p-1})(y_2^{p-1}+(y_2+y_1)^{p-1}+\cdots+(y_2+(p-1)y_1)^{p-1})\\&=(y_J/y_2^{p-1})(-y_1^{p-1})\quad \text{mod (BOMS)}.\end{aligned}$$

Since $l(y_J(y_1/y_2)^{p-1})\leqq l(y_J)$, we know

$$N\sum a_I y_I=a_J y_J(y_1/y_2)^{p-1}\neq 0\quad \text{mod (BMSO)}.$$

This is a contradiction to $x\in\operatorname{Ker}N$. q.e.d.

Proof of Lemma 2.3 for the case $p+1\leqq n\leqq p^2$, $s=2$ and $\varepsilon_i=1$ for $2\leqq i$. Also suppose $x\in\operatorname{Ker}N$ and $j_1\neq 0$. Notice that $(b^p-1)y_{p+1}=y_1$ mod (BM) and $(b^p-1)y_i=0$ for $i<p+1$. Hence we can take X and X' so that $j_2=j_{p+1}=p-1$ mod p and

$$x-(b-1)X-(b^p-1)X'=a_J y_J'\quad \text{mod (BMSO)}$$

where $y_J'=y_J(y_{p+1}(p)/y_{p+1})^{[j_{p+1}/p]}(y_2(p)/y_2^{p-1})^{[j_2/p]}$ and $y_{p+1}(p)=\prod_{i\in Z/p}b^{ip}y_{p+1}$.

Let us write $N'=1+b+\cdots+b^{p-1}$. Then $N=(1+b^p+\cdots+b^{p(p-1)})N'$. By the arguments similar to the proof for the case $n\leqq p$, we see

$$N'y_J'=y_J'(-y_2/y_1)^{p-1}\quad \text{mod (BMSO)}.$$

So we have

$$\begin{aligned}Ny_J'&=-(1+b^p+\cdots+b^{p(p-1)})y_J'(y_1/y_2)^{p-1}\\&=y_J(y_1^2/y_2y_{p+1})^{p-1}\quad \text{mod (BMSO)}.\end{aligned}$$

The length l of the above term is smaller or equal to $l(y_J)$, we also have $Nx\neq 0$ and this is a contradiction. q.e.d.

Proof of Lemma 2.3. The case $p^m+1\leqq n\leqq p^{m+1}$, $s=m+1$ and $\varepsilon_i=1$ for $i\geqq 2$ are proved by the similar arguments. Taking X so that $j_{p^t+1}=-1$ mod p for $0\leqq t\leqq m$ and

$$x-(b-1)X=a_J y_J'\quad \text{mod (BMSO)}$$

with $y_J'=y_J\prod_{t=0}^m(y_{p^t+1}(p)/y_{p^t+1}^p)^{[i_{p^t+1}/p]}$ and $y_{p^t+1}(p)=\prod_{i\in Z/p}b^{ip^t}y_{p^t+1}$. Then we can prove

$$Ny_J'=y_J'(y_1^{m+1}/y_2\cdots y_{p^m+1})^{p-1}\neq 0\quad \text{mod (BMSO)}.$$

So we see Lemma 2.3 for these cases. The cases with some $\varepsilon_i=0$ are proved more easily by using induction on n. q.e.d.

Since $E_1^{\mathrm{odd},*}=0$, the spectral sequence (2.1) collapses. Thus we get

COROLLARY 2.8. *For groups G in* (0.1), $BP^{\mathrm{odd}}(BG)=0$ *and* $BP^{\mathrm{even}}(BG)$ *is multiplicatively generated by elements induced from* $BP^*(B(Z/p)^n)^{Z/p^s}$ *and* $y_b\in BP^2(BZ/p^s)$.

§ 3. $K(m)^*(BG)$

At first we consider the cohomology theory $\tilde{P}(m)^*(-)$ with the coefficient $\tilde{P}(m)^*=BP^*/(v_1, \dots, v_{m-1})=Z_{(p)}[v_m, \dots]$. Note that $\tilde{P}(m)^*/(p)=P(m)^*$ and $\tilde{P}(1)^*=BP^*$. We can easily prove the $\tilde{P}(m)^*$-version of Lemma 1.1 and Lemma 1.3. The arguments in § 2 also work for $\tilde{P}(m)^*$-theory. Hence we can prove $E_2^{\mathrm{odd},*}(\tilde{P}(m))=0$ for the spectral sequence

$$E_2^{*,*}(\tilde{P}(m))=H^*(BZ/p^s\,;\tilde{P}(m)^*(B(Z/p)^n)\Longrightarrow \tilde{P}(m)^*(BG).$$

Using fact that $\tilde{P}(m)^*(G)$ is torsion free if $E_2^{\mathrm{odd},*}(\tilde{P}(m))=0$, we get the following;

THEOREM 3.1 (Tezuka-Yagita, Theorem 2.6 in [T-Y]). *Suppose that there is an extension*

$$1\longrightarrow G'\longrightarrow G\longrightarrow Z/p^s\longrightarrow 0$$

such that $\tilde{P}(m)^*(BG')\cong\tilde{P}(m)^*\otimes_{BP^*}BP^*(BG')$ *and* $H^{\mathrm{odd}}(BZ/p^s\,;\tilde{P}(m)^*(BG'))\cong 0$ *for all* $m\geqq 1$. *Then* $P(m)^*(BG)\cong P(m)^*\otimes_{BP^*}BP^*(BG)$ *and* $K(m)^*(BG)=K(m)^*\otimes_{BP^*}BP^*(BG)$ *for all* $m\geqq 1$.

Thus we get

THEOREM 3.2. *For groups G in* (0.1), *there are isomorphisms*

$$BP^{\mathrm{odd}}(BG)\cong 0,\quad P(m)^*(BG)\cong P(m)^*\otimes_{BP^*}BP^*(BG)$$

and

$$K(m)^*(BG)\cong K(m)^*\otimes_{BP^*}BP^*(BG)$$

for all $m\geqq 1$.

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