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ON THE SCHWARZIAN DIFFERENTIAL EQUATION $\{w, z\}=R(z, w)$

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Abstract

It is showed in this note that if the Schwarzian differential equation $(*)$ $\{w, z\}=R(z, w)=P(z, w)/Q(z, w)$, where $P(z, w)$ and $Q(z, w)$ are polynomials in w with meromorphic coefficients, possesses an admissible solution $w(z)$, then $w(z)$ satisfies a first order equation of the form $(**)$ $(w')^2+B(z, w)w'+A(z, w)=0$, where $B(z, w)$ and $A(z, w)$, are polynomials in w having small coefficients with respect to $w(z)$, or by a suitable Möbius transformation $(*)$ reduces into $\{w, z\}=P(z, w)/(w+b(z))^2$ or $\{w, z\}=c(z)$. Furthermore, we study the equation $(**)$.

1. Introduction

We are concerned with the Schwarzian differential equation

$$\{w, z\}=\left(\frac{w''}{w'}\right)'-\frac{1}{2}\left(\frac{w''}{w'}\right)^2=R(z, w)=\frac{P(z, w)}{Q(z, w)}, \tag{1.1}$$

where $P(z, w)$ and $Q(z, w)$ are polynomials in w having meromorphic coefficients with $\deg_w P(z, w)=p$ and $\deg_w Q(z, w)=q$, respectively. Moreover, we assume that they are relatively prime.

We studied the Schwarzian equation $\{w, z\}^m=R(z, w)$ in [2, Theorems 1-3]. The Malmquist-Yoshida type theorem to the Schwarzian equation was obtained. Furthermore, we determined the form of the Schwarzian equation that possesses an admissible solution especially when $R(z, w)$ is independent of z. However, it might be difficult to get the similar assertion in the case when $R(z, w)$ is not independent of z. We treat the Schwarzian equation only when $m=1$, say, the equation (1.1). We also consider the first order equation

$$(w')^2+2B(z, w)w'+A(z, w)=0, \tag{1.2}$$

where $B(z, w)$ and $A(z, w)$ are polynomials in w having meromorphic coefficients. In this note, we use standard notations in the Nevanlinna theory (see e.g., [1], [5], [6]). Let $f(z)$ be a meromorphic function. Here, the word "meromorphic" means meromorphic in $|z|<\infty$. As usual, $m(r, f)$, $N(r, f)$, and $T(r, f)$ denote

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the proximity function, the counting function, and the characteristic function of $f(z)$, respectively. Let $n_{(M}(r, f)$ be the number of poles of order at least M for a meromorphic function $f(z)$ in $|z|\leqq r$ according to its multiplicity. The integrated counting function $N_{(M}(r, f)$ is defined in the usual way.

We define the counting function concerning common zeros of two meromorphic functions $f(z)$ and $g(z)$. Let $n(r, 0; f)_g$ be the number of common zeros of $f(z)$ and $g(z)$ in $|z|\leqq r$, each counted according to the multiplicity of the zero of $f(z)$. The counting function $N(r, 0, f)_g$ is defined in the usual way. The integrated counting function $\overline{N}(r, 0; f)_g$ $(=\overline{N}(r, 0; g)_f)$ counts distinct common zeros of $f(z)$ and $g(z)$.

A function $\varphi(r)$, $0\leqq r<\infty$, is said to be $S(r, f)$ if there is a set $E\subset \boldsymbol{R}^+$ of finite linear measure such that $\varphi(r)=o(T(r, f))$ as $r\to\infty$ with $r\notin E$.

A meromorphic function $a(z)$ is *small* with respect to $f(z)$ if $T(r, a)=S(r, f)$. In the below, $\mathcal{M}=\{a(z)\}$ denotes a given finite collection of meromorphic functions. A transcendental meromorphic function $f(z)$ is *admissible* with respect to $\mathcal{M}$ if $T(r, a)=S(r, f)$ for any $a(z)\in\mathcal{M}$.

Let $c\in\boldsymbol{C}\cup\{\infty\}$. We call z_0 a c-point of $f(z)$ if $f(z_0)-c=0$. Suppose that a transcendental meromorphic function $f(z)$ is admissible with respect to $\mathcal{M}$. A c-point z_0 of $f(z)$ is an *admissible c-point* with respect to $\mathcal{M}$ if $a(z_0)\neq 0$, ∞ for any $a(z)\in\mathcal{M}$.

Suppose $N(r, c; f)\neq S(r, f)$ for a $c\in\boldsymbol{C}\cup\{\infty\}$. Let P be a property. We denote by $n_P(r, c; f)$ the number of c-points in $|z|\leqq r$ that admit the property P. The integrated counting function $N_P(r, c; f)$ is defined in the usual fashion. If

$$N(r, c; f)-N_P(r, c; f)=S(r, f),$$

then we say that *almost all* c-points admit the property P.

We define an admissible solution of the equation

$$\Omega(z, w, w', \dots, w^{(n)})=\sum_{J\in\mathcal{J}}\Phi_J=\sum_{J\in\mathcal{J}}c_J(z)w^{j_0}(w')^{j_1}\cdots(w^{(n)})^{j_n}=0, \tag{1.3}$$

where $\mathcal{J}$ is a finite set of multi-indices $J=(j_0, j_1, \dots, j_n)$, and $c_J(z)$ are meromorphic functions. Let $\mathcal{M}_{(1.3)}$ be the collection of the coefficients of $\Omega(z, w, w', \dots, w^{(n)})$ in (1.3), say, $\mathcal{M}_{(1.3)}:=\{c_J(z)|J\in\mathcal{J}\}$. A meromorphic solution $w(z)$ of the equation (1.3) is an *admissible solution* if $w(z)$ is admissible with respect to $\mathcal{M}_{(1.3)}$.

We now state the results below.

THEOREM 1.1. *Suppose that the Schwarzian equation* (1.1) *possesses an admissible solution* $w(z)$. *Then* $w(z)$ *satisfies a Riccati equation, a first order differential equation of the form* (1.2), *or the equation* (1.1) *is one of the following forms*:

$$\{w, z\}=\frac{P(z, w)}{(w+b(z))^2}, \tag{1.4}$$

$$\{w, z\}=c(z), \tag{1.5}$$

where $b(z)$, $c(z)$ *are small functions with respect to* $w(z)$. *In the case* $w(z)$ *satisfies a first order differential equation* (1.2), *by a suitable transformation* $u=1/(w-\tau)$, $\tau\in\boldsymbol{C}$, *we see that* $u(z)$ *satisfies a first order differential equation of the form* (1.2) *with* $\deg_u B(z, u)\leqq 1$, $\deg_u A(z, u)=3$.

THEOREM 1.2. *Suppose that* $\deg_w B(z, w)\leqq 1$ *and* $\deg_w A(z, w)=3$ *in* (1.2)

$$\begin{cases} B(z, w)=b_1(z)w+b_0(z) \\ A(z, w)=a_3(z)w^3+a_2(z)w^2+a_1(z)w+a_0(z). \end{cases} \tag{1.6}$$

If the equation (1.2) *possesses an admissible solution* $w(z)$, *then by a suitable Möbius transformation with meromorphic small coefficients with respect to* $w(z)$

$$y=\frac{\alpha(z)w+\beta(z)}{\gamma(z)w+\delta(z)}, \quad \alpha(z)\delta(z)-\beta(z)\gamma(z)\not\equiv 0, \tag{1.7}$$

the equation (1.2) *reduces into one of the following types*:

$$(y')^2=a(z)(y-e_1)(y-e_2)(y-e_3), \tag{1.8}$$

$$(y'+b(z)y)^2=a(z)y(1+c(z)y)^2, \tag{1.9}$$

$$\left(y'-\frac{a'(z)}{2a(z)}y\right)^2=y(y^2-a(z)), \tag{1.10}$$

$$\left(y'-\frac{a'(z)}{3a(z)}y\right)^2=y^3-a(z), \tag{1.11}$$

where $a(z)$, $b(z)$, $c(z)$ *are small meromorphic functions with respect to* $w(z)$ *and* e_1, e_2, e_3 *are distinct constants.*

Remark 1.1. Put $g=y^2/a$ in (1.10) and put $h=y^3/a$ in (1.11). Then we see that $g(z)$ and $h(z)$ respectively satisfy the binomial equations

$$(g')^4=16a(z)g^3(g-1)^2 \quad \text{and} \quad (h')^6=729a(z)h^4(h-1)^3.$$

We can find the Malmquist-Yosida-Steinmetz-He-Laine theorem to binomial equation, for instance, in Laine [5, Theorem 10.3, p. 194].

2. Preliminary Lemmas

In this section, we prepare some lemmas to prove Theorems 1.1 and 1.2.

LEMMA 2.1 [2, Theorem 2, pp. 261-262]. *If the equation* (1.1) *possesses an admissible solution* $w(z)$, *then the denominator* $Q(z, w)$ *of* $R(z, w)$ *must be one of the followings*:

$$Q(z, w)=c(z)(w+\tilde{b}_1(z))^2(w+\tilde{b}_2(z))^2, \tag{2.1}$$

(2.2) $$Q(z, w)=c(z)(w^2+\tilde{a}_1(z)w+\tilde{a}_0(z))^2,$$

(2.3) $$Q(z, w)=c(z)(w+b(z))^2,$$

(2.4) $$Q(z, w)=c(z)(w+b(z))^2(w-\tau_1)(w-\tau_2),$$

(2.5) $$Q(z, w)=c(z)(w+b(z))^2(w-\tau_1),$$

(2.6) $$Q(z, w)=c(z)(w-\tau_1)(w-\tau_2)(w-\tau_3)(w-\tau_4),$$

(2.7) $$Q(z, w)=c(z)(w-\tau_1)(w-\tau_2)(w-\tau_3),$$

(2.8) $$Q(z, w)=c(z)(w-\tau_1)(w-\tau_2),$$

(2.9) $$Q(z, w)=c(z)(w-\tau_1),$$

(2.10) $$Q(z, w)=c(z),$$

where $c(z)$, $\tilde{a}_1(z)$, $\tilde{a}_0(z)$ *are meromorphic functions,* $|\tilde{a}_1'|+|\tilde{a}_2'|\neq 0$, $\tilde{b}_1(z)$, $\tilde{b}_2(z)$, $b(z)$ *are nonconstant meromorphic functions, and* τ_j, $j=1, \dots, 4$ *are distinct constants.*

LEMMA 2.2 [4, Theorem 1 (ii)]. *Suppose that the equation* (1.3) *possesses an admissible solution* $w(z)$ *that satisfies* $N_{(M}(r, w)=S(r, w)$ *for some* $M>0$. *Let* $G(z, w)$ *be an irreducible polynomial in* w *having small coefficients with respect to* $w(z)$. *If* $F(z, w)$ *is a polynomial in* w *having small coefficients with respect to* $w(z)$ *such that* $F(z, w)$ *and* $G(z, w)$ *are relatively prime, then*

(2.11) $$N(r, 0; G)_F=S(r, w).$$

LEMMA 2.3. *Let* $f(z)$ *be a transcendental meromorphic function and let* $\Omega(z, f, f', \dots, f^{(n)})$ *be a differential polynomial in* f *of total degree* $\gamma_\Omega\leqq q$ *having small coefficients with respect to* $f(z)$. *Define*

$$h(z):=\Omega(z, f(z), f'(z), \dots, f^{(n)}(z))\Big/\prod_{j=1}^{q}(f(z)-\tau_j),$$

where $\tau_1, \dots, \tau_q$ *are distinct complex constants. Then*

$$m(r, h)\leqq \sum_{j=1}^{q} m\left(r, \frac{1}{f-\tau_j}\right)+S(r, f).$$

The proof of Lemma 2.3 is the same as that of the original proof except for obvious modifications (see Steinmetz [8, Lemma 3, pp. 48-49]).

LEMMA 2.4. *Let* $f(z)$ *be a transcendental meromorphic function.*

(i) *Let* $K(z, f)$ *be a rational function in* f *having small coefficients with respect to* $f(z)$, *i.e.,* $K(z, f):=F(z, f)/G(z, f)$ *where* $F(z, f)$ *and* $G(z, f)$ *are relatively prime polynomials in* f. *If* $m(r, K)=S(r, f)$ *where* $K(z)=K(z, f(z))$, *then*

$$m\left(r, \frac{1}{G}\right)=S(r, f), \quad \textit{where } G(z)=G(z, f(z)).$$

(ii) *Let* $G(z, f)$ *be a polynomial in* f *of degree* k *having small coefficients with respect to* $f(z)$ *such that* $m(r, 1/G)=S(r, f)$, *where* $G(z):=G(z, f(z))$. *Let* $\Omega(z, f, f', \dots, f^{(n)})$ *be a differential polynomial in* f *of total degree* $\gamma_\Omega \leqq k$ *having small coefficients with respect to* $f(z)$. *Then we have*

$$m\left(r, \frac{\Omega}{G}\right)=S(r, f), \quad \textit{where } \Omega(z):=\Omega(z, f(z), f'(z), \dots, f^{(n)}(z)).$$

Proof of Lemma 2.4. (i) Set $d=\deg_f K(z, f)$. Since $m(r, K)=S(r, f)$, by Mokhon'ko's Theorem (see e.g., Laine [5, Theorem 2.25, pp. 29-34]), we have $dT(r, f)=N(r, K)+S(r, f)$. Set $\deg_f F=d_1$ and $\deg_f G=d_2$. If $d_1>d_2$ so that $d=d_1$, then

$$\begin{aligned} N(r, K) &\leqq (d_1-d_2)N(r, f)+N\left(r, \frac{1}{G}\right)+S(r, f) \\ &\leqq (d_1-d_2)T(r, f)+N\left(r, \frac{1}{G}\right)+S(r, f). \end{aligned}$$

Hence, we get $d_2T(r, f)\leqq N(r, 1/G)+S(r, f)$, which proves our assertion in the case $d_1>d_2$. If $d_1\leqq d_2$, then

$$N(r, K)\leqq N\left(r, \frac{1}{G}\right)+S(r, f).$$

Therefore, we also have proved our assertion in the case $d_1\leqq d_2$.

(ii) Let τ_j, $j=1, 2, \dots, \gamma_\Omega$ be distinct complex constants such that $G(z, \tau_j)\neq 0$ and $m(r, 1/(f-\tau_j))=S(r, f)$. Since $\gamma_\Omega\leqq k$, we have

$$N\left(r, \frac{\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)}{G}\right)=N\left(r, \frac{1}{G}\right)+S(r, f). \tag{2.12}$$

By our assumption $m(r, 1/G)=S(r, f)$, and by the first fundamental theorem and Mokhon'ko's theorem

$$N\left(r, \frac{1}{G}\right)=T(r, G)-m\left(r, \frac{1}{G}\right)+O(1)=kT(r, f)+S(r, f). \tag{2.13}$$

Hence, by Mokhon'ko's theorem and Lemma 2.3, we get

$$\begin{aligned} m\left(r, \frac{\Omega}{G}\right) &\leqq m\left(r, \frac{\Omega}{\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)}\right)+m\left(r, \frac{\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)}{G}\right)+S(r, f) \\ &\leqq \sum_{j=1}^{\gamma_\Omega} m\left(r, \frac{1}{f-\tau_j}\right)+kT(r, f)-N\left(r, \frac{\prod_{j=1}^{\gamma_\Omega}(f-\tau_j)}{G}\right)+S(r, f). \end{aligned} \tag{2.14}$$

The assertion follows from (2.12), (2.13) and (2.14). □

Here, we refer to the lemmas on a representable double poles, (see [3, Theorem 2.6]). Let $f(z)$ be a transcendental meromorphic function and let $r_1(z)$, $r_2(z)$, $a_0(z)$, $a_1(z)$, ..., $a_5(z)$ be small functions with respect to $f(z)$. Let z_0 be a double pole of $f(z)$. We call z_0 a *strongly representable double pole in the first sense of* $f(z)$ *by* $r_1(z)$, $r_2(z)$, $a_0(z)$, $a_1(z)$, ..., $a_5(z)$, if $f(z)$ is written in a neighbourhood of z_0 as

$$f(z)=\frac{r_2(z_0)}{(z-z_0)^2}+\frac{r_1(z_0)}{z-z_0}+a_0(z_0)+\cdots+a_5(z_0)(z-z_0)^5+O(z-z_0)^6, \quad \text{as } z\to z_0.$$

For the sake of simplicity, we abbreviate it SD1-*pole*. We denote by $n_{\langle SD1\rangle}(r, f)$ the number of the SD1-poles. The integrated counting function $N_{\langle SD1\rangle}(r, f)$ is defined in terms of $n_{\langle SD1\rangle}(r, f)$ in the usual way.

LEMMA 2.5. *Let $w(z)$ be a transcendental meromorphic function and let $r_1(z)$, $r_2(z)$, $a_0(z)$, $a_1(z)$, ..., $a_5(z)$ be small functions with respect to $w(z)$. If*

$$m(r, w)+(N(r, w)-N_{\langle SD1\rangle}(r, w))=S(r, w),$$

then $w(z)$ satisfies a differential equation of the form (1.2) *with* $\deg_w B(z, w)\leqq 1$, $\deg_w A(z, w)=3$.

Before we state Lemma 2.6, we write (1.2) as

$$(w'+B(z, w))^2=B(z, w)^2-A(z, w)=D(z, w). \tag{2.15}$$

Moreover, we write $D(z, w)$ as

$$\begin{aligned} D(z, w)&=d_3(z)w^3+d_2(z)w^2+d_1(z)w+d_0(z)\\ &=d_3(z)(w-\eta_1(z))(w-\eta_2(z))(w-\eta_3(z)), \end{aligned} \tag{2.16}$$

where $d_j(z)$, $j=0, 1, 2, 3$ are meromorphic functions, $\eta_j(z)$, $j=1, 2, 3$, are algebroid functions.

LEMMA 2.6. *Suppose that the equation* (2.15) *possesses an admissible solution $w(z)$. Let $\eta(z)$ be a root of the equation $D(z, \eta)=0$. If $\eta(z)$ is a simple root, then $\eta(z)$ satisfies the equation*

$$\eta'+B(z, \eta)=0. \tag{2.17}$$

Lemma 2.6 is originally proved by Steinmetz [8, p. 51] under the condition that the coefficients are not transcendental. We will follow his proof. To do this, we refer to the following Malmquist-Yosida type theorem, see Steinmetz [7], Laine [5, Theorem 13.1].

LEMMA 2.7. *Let $P(z, w, w', \ldots, w^{(n)})$ be a differential polynomial in w with meromorphic coefficients and let $R(z, w)$ be a rational function in w having meromorphic coefficients. If the differential equation*

$$P(z, w, w', \dots, w^{(n)})=R(z, w)$$

possesses an admissible solution, then $R(z, w)$ reduces to a polynomial in w.

Proof of Lemma 2.6. Differentiating (2.15), and combining (2.15) with the obtained equation we get

$$(2.18)\qquad \begin{aligned}&(2w''+2B_z(z, w)+2B_w(z, w)w'-D_w(z, w))^2\\ &=\frac{(D_z(z, w)-D_w(z, w)B(z, w))^2}{D(z, w)}.\end{aligned}$$

Write $D(z, w)=(w-\eta)\Delta(z, w)$, where $\Delta(z, \eta(z))\not\equiv 0$. We denote by $R(z, w)$ the right-hand side of (2.18). We actually compute

$$\begin{aligned}R(z, w)=&\frac{(\eta'(z)+B(z, w))^2\Delta(z, w)}{w-\eta(z)}-2(\eta'(z)+B(z, w))(\Delta_z(z, w)-\Delta_w(z, w)B(z, w))\\ &+\frac{(w-\eta(z))(\Delta_z(z, w)-\Delta_w(z, w)B(z, w))^2}{\Delta(z, w)}.\end{aligned}$$

By means of Lemma 2.7, $R(z, w)$ must be a polynomial in w. Since $w-\eta(z)$ and $\Delta(z, w)$ are mutually prime polynomials in w, there exists a polynomial $Q(z, w)$ in w such that

$$(\eta'(z)+B(z, w))^2\Delta(z, w)=Q(z, w)(w-\eta(z)).$$

From the reasoning $\Delta(z, \eta(z))\not\equiv 0$, we obtain $\eta'(z)+B(z, \eta(z))\equiv 0$. □

LEMMA 2.8. *Let $a(z)$, $b_1(z)$, $b_0(z)$, $\eta_j(z)$, $j=1, 2, 3$ be meromorphic functions. Suppose that the equation*

$$(2.19)\qquad (w'+b_1(z)w+b_0(z))^2=a(z)(w-\eta_1(z))(w-\eta_2(z))(w-\eta_3(z))$$

possesses an admissible solution $w(z)$. Then by a suitable Möbius transformation with small meromorphic coefficients with respect to $w(z)$

$$y=\frac{\alpha(z)w+\beta(z)}{\gamma(z)w+\delta(z)},\quad \alpha(z)\delta(z)-\beta(z)\gamma(z)\not\equiv 0,$$

the equation (2.19) *reduces to the type* (1.8) *or* (1.9).

We note that Lemma 2.8 is originally proved by Steinmetz [8, pp. 51-52]. In fact, if $\eta_1=\eta_2=\eta_3$, then by $y=1/(w-\eta_1(z))$ the equation (2.19) reduces to

$$(y'-b_1(z)y)^2=a(z)y.$$

In case, $\eta_1=\eta_3$, $\eta_1\neq\eta_2$, then by $y=(w-\eta_1(z))/(w-\eta_2(z))$ we get

$$(y')^2=a(z)(\eta_2(z)-\eta_1(z))(y-1)y^2.$$

Finally we consider the case $\eta_j \neq \eta_k$, $j \neq k$. By Lemma 2.7, $\kappa=(\eta_1(z)-\eta_3(z))/(\eta_2(z)-\eta_3(z))$ is a constant, $\kappa \neq 0, 1$. Put $y=(w-\eta_1(z))/(w-\eta_2(z))$. Then $y(z)$ satisfies

$$(y')^2=a(z)(\eta_2(z)-\eta_3(z))y(y-1)(y-\kappa).$$

3. Proof of Theorem 1.1

Proof of Theorem 1.1. We may assume that $\deg_w P(z, w)=\deg_w Q(z, w)$. Moreover, we can suppose that almost all poles of $w(z)$ are simple and we have

$$m(r, w)+N_1(r, w)=S(r, w), \tag{3.1}$$

(see [2, Proof of Lemma 1, p. 266]). By means of Lemma 2.2,

$$N(r, 0; Q)_P=S(r, w), \tag{3.2}$$

where $Q(z):=Q(z, w(z))$ and $P(z):=P(z, w(z))$. Furthermore, we know that the existence of an admissible solution implies that $Q(z, w)$ must be of the form (2.1)-(2.10) in the statements of Lemma 2.1. We prove Theorem 1.1 separately according to the cases above.

First we treat the case $Q(z, w)$ is of the form (2.1) or (2.2). It follows from (1.1) that almost all zeros of $Q(z)$ are zeros of $w'(z)$. We have that almost all zeros of $Q(z)$ are double zeros, (see [2, Lemma 2(i), p. 264]). Define

$$\varphi_1(z):=\frac{w'}{(w+b_1(z))(w+b_2(z))}, \quad \text{if } Q(z, w) \text{ is of the form (2.1)},$$

$$\varphi_2(z):=\frac{w'}{w^2+a_1(z)w+a_0(z)}, \quad \text{if } Q(z, w) \text{ is of the form (2.2)}.$$

Then almost all zeros of $Q(z)$ are regular points of $\varphi_j(z)$, $j=1, 2$. It follows from (3.1) that almost all poles of $w(z)$ are also regular points of $\varphi_j(z)$, $j=1, 2$. Hence we get $N(r, \varphi_j)=S(r, w)$, $j=1, 2$. Using the theorem on the logarithmic derivative, from (1.1) we have that $m(r, R)=S(r, w)$ in each case. By Lemma 2.4(i), we get $m(r, Q)=S(r, w)$. By virtue of Lemma 2.4(ii), we conclude that $m(r, \varphi_j)=S(r, w)$, $j=1, 2$. Therefore, $\varphi_j(z)$, $j=1, 2$, are small functions with respect to $w(z)$ in both cases. This implies that $w(z)$ satisfies a Riccati equation in each case.

We see that if $Q(z, w)$ has a factor $(w-\tau)$, then almost all τ-points of $w(z)$ are of multiplicity two. Thus $w'(z)$ has a simple zero at these τ-points, (see [2, Proof of Lemma 2(ii), p. 267]).

Next, we treat the case $Q(z, w)$ is of the form (2.4) or (2.6). We define

$$\phi_1(z):=\frac{(w')^2}{(w+b_1(z))^2(w-\tau_1)(w-\tau_2)}, \quad \text{if } Q(z, w) \text{ is of the form (2.4)},$$

$$\phi_2(z):=\frac{(w')^2}{(w-\tau_1)(w-\tau_2)(w-\tau_3)(w-\tau_4)}, \quad \text{if } Q(z, w) \text{ is of the form (2.6)}.$$

Then almost all zeros of $Q(z)$ are regular point of $\psi_j(z)$, $j=1, 2$, and almost all poles of $w(z)$ are also regular points, which means that $N(r, \psi_j)=S(r, w)$, $j=1, 2$. By the same arguments in the first case, we get $m(r, \psi_j)=S(r, w)$, $j=1, 2$. Hence we conclude that $w(z)$ satisfies a binomial equation which is a special form of (1.2).

Finally we consider the case $Q(z, w)$ is of the form (2.5), (2.7), (2.8) or (2.9). We know that almost all τ_1-points of $w(z)$ are double points without defect in each case. It gives that if we put $u=1/(w-\tau_1)$ in (1.1), then almost all poles of $u(z)$ are double poles and we have in each case

$$m(r, u)+(N(r, u)-N_{(2)}(r, u))=S(r, u). \tag{3.3}$$

It is easy to see that when $Q(z, w)$ is of the form (2.5), (2.7), (2.8) or (2.9) the equation (1.1) transforms into the following equations, respectively:

$$\{u, z\}=\frac{P_1(z, u)}{(u+\tilde{b}(z))^2}, \quad \text{if } Q(z, w) \text{ is of the form (2.5)}, \tag{3.4}$$

$$\{u, z\}=\frac{P_2(z, u)}{(u-\sigma_1)(u-\sigma_2)}, \quad \text{if } Q(z, w) \text{ is of the form (2.7)}, \tag{3.5}$$

$$\{u, z\}=\frac{P_3(z, u)}{u-\sigma_1}, \quad \text{if } Q(z, w) \text{ is of the form (2.8)}, \tag{3.6}$$

$$\{u, z\}=P_4(z, u), \quad \text{if } Q(z, u) \text{ is of the form (2.9)}, \tag{3.7}$$

where $P_j(z, u)$, $j=1, 2, 3, 4$, are polynomials in u having small coefficients with respect to $u(z)$ and $\deg_u P_1(z, u)=3$, $\deg_u P_2(z, u)=3$, $\deg_u P_3(z, u)=2$, $\deg_u P_1(z, u)=1$, $\tilde{b}(z)$ is a non-constant small function with respect to $u(z)$, σ_j, $j=1, 2$ are constants $\sigma_1\neq\sigma_2$. Let z_0 be an admissible pole of $u(z)$. We write $u(z)$ in a neighbourhood of z_0 as

$$u(z)=\frac{r_2}{(z-z_0)^2}+\frac{r_1}{z-z_0}+a_0+\cdots+a_5(z-z_0)^5+O(z-z_0)^6, \quad \text{as } z\to z_0. \tag{3.8}$$

We assert that in each case $r_2, r_1, a_0, \dots, a_5$ are written in terms of small functions with respect to $u(z)$, say, z_0 is an SD1-pole. In fact, we put $P_1(z, u)=p_1(z)u^3+p_{12}(z)u^2+p_{11}(z)u+p_{10}(z)$, $P_2(z, u)=p_2(z)u^3+p_{22}(z)u^2+p_{21}(z)u+p_{20}(z)$, $P_3(z, u)=p_3(z)u^2+p_{31}(z)u+p_{30}(z)$ and $P_4(z, u)=p_4(z)u+p_{40}(z)$, $p_j(z)\not\equiv 0$, $j=1, 2, 3, 4$. Using Test-power test, say, substituting (3.8) into the both sides of (3.4), (3.5), (3.6) and (3.7), we compare the coefficients of $(z-z_0)^{-2}$. Then we see that in each case $r_2=-3/(2p_j(z))$, $j=1, 2, 3, 4$. Moreover, comparing the coefficients of $(z-z_0)^m$, $m=-1, 0, \dots, 5$, we get $-2p_j(z_0)r_1=S_{-1}^{(j)}(r_2)$, $-p_j(z_0)a_0=S_0^{(j)}(r_2, r_1)$, $4p_j(z_0)a_1=S_1^{(j)}(r_2, r_1, a_0)$, $15p_j(z_0)a_2=S_2^{(j)}(r_2, r_1, a_0, a_1)$, $34p_j(z_0)a_3=S_3^{(j)}(r_2, r_1, a_0, a_1, a_2)$, $63p_j(z_0)a_4=S_4^{(j)}(r_2, r_1, a_0, a_1, a_2, a_3)$ and $104p_j(z_0)a_5=S_5^{(j)}(r_2, r_1, a_0, a_1, a_2, a_3, a_4)$, $j=1, 2, 3, 4$, where $S_m^{(j)}$ are polynomials in the indicated arguments with the coefficients that are the values of small functions with respect to $u(z)$ at z_0. This implies that $r_2, r_1, a_0, \dots, a_5$ are written in terms of small functions with respect to $u(z)$, say, z_0 is an SD1-pole in each case. Hence, by Lemma 2.5, $u(z)$ satisfies a first order differential equation of the form (1.2). Therefore, by a simple computation, we see that $w(z)$ also satisfies a differential equation of the form (1.2). Hence, we conclude that unless $Q(z, w)$ is of the form (2.3) or (2.10), then $w(z)$ satisfies a first order differential equation of the form (1.2). Then the assertion follows. □

4. Proof of Theorem 1.2

Proof of Theorem 1.2. In the case $B(z, w)\equiv 0$, the equation (1.2) is a binomial equation. Hence, by the Malmquist-Yosida-Steinmetz theorem, the equation (1.2) reduces into (1.8) or (1.9). Therefore, we may assume that $B(z, w)\not\equiv 0$. Lemma 2.8 insists that if the all roots $\eta(z)$ of $D(z, \eta)=0$ are meromorphic, then (2.15) reduces to the equation (1.8) or (1.9). Hence we shall show that $\eta_j(z)$ in (2.16) are meromorphic or (2.15) reduces to (1.9), (1.10) or (1.11). In case there is a double or triple root of $D(z, \eta)=0$, then they are meromorphic. Hence we may assume that $\eta_j(z)$, $j=1, 2, 3$, are all simple roots. It follows from Lemma 2.6 that they satisfy the equation (2.17). We put $u=d_3(z)w+d_2(z)/3$ in (2.15) and (2.16). Then we get the equation

$$(u'+\tilde{B}(z, u))^2=u^3+\tilde{d}_1(z)u+\tilde{d}_0(z),$$

where $\tilde{B}(z, u)$ is a polynomial in u with degree at most 1, and $\tilde{d}_1$, $\tilde{d}_0$ and the coefficients of $\tilde{B}(z, u)$ are small functions with respect to $u(z)$. Hence, we also assume that $d_3(z)\equiv 1$ and $d_2(z)\equiv 0$ in (2.16).

Since $\eta_j(z)$, $j=1, 2, 3$, satisfy the equation (2.17), $y(z):=\eta_1(z)+\eta_2(z)+\eta_3(z)$ satisfies the equation

$$y'+b_1(z)y+3b_0(z)=0.$$

From the assumption $d_2(z)\equiv 0$, we have $y(z)\equiv 0$. This implies that $b_0(z)\equiv 0$. Therefore, $\eta_j(z)$ are written as

$$\eta_j(z)=C_je^{-\int b_1(z)dz}, \quad j=1, 2, 3. \tag{4.1}$$

It follows from (4.1) that at least one function element $\eta_j(z)$, which does not vanish, is a meromorphic function if and only if all function elements η_j are meromorphic. From (2.15), we get

$$w''=\frac{U(z, w)w'+V(z, w)}{2D(z, w)}, \tag{4.2}$$

where

$$\begin{aligned} U(z, w)&=-b_1(z)w^3+(d_1'(z)+b_1(z)d_1(z))w+d_0'(z)+2b_1(z)d_1(z)\\ &=(5a_3(z)b_1(z)-a_3'(z))w^3+(4a_2(z)b_1(z)-4b_1(z)^3+2b_1(z)b_1'(z)-a_2'(z))w^2\\ &\quad+(3a_1(z)b_1(z)-a_1'(z))w+2a_0(z)b_1(z)-a_0'(z), \end{aligned} \tag{4.3}$$

$V(z, w)$ are polynomials in w having meromorphic coefficients with respect to $w(z)$. From (4.2) and (2.15), we get

$$-U(z, w)w''w'+(V(z, w)-2U(z, w)B(z, w))w''+U(z, w)^2$$
$$=\frac{(B(z, w)U(z, w)-V(z, w))^2}{D(z, w)}.$$

In view of Lemma 2.7, we see that $V=BU$ or $D|(V-BU)$ as polynomials in w, since η_j, $j=1, 2, 3$ are simple roots. We get from (4.2),

$$\left(w''-\frac{1}{2}H(z, w)\right)^2=\frac{U(z, w)^2}{4D}, \tag{4.4}$$

where $H(z, w)$ is a polynomial in w having small coefficients with respect to $w(z)$. Hence, by Lemma 2.7 and (4.4) we have $D|U$ as polynomials in w. Write $U(z, w)$ as

$$\begin{aligned} &U(z, w)=-b_1(z)D(z, w)+S(z, w), \\ &S(z, w)=s_2(z)w^2+s_1(z)w+s_0(z), \end{aligned} \tag{4.5}$$

with

$$s_2=a_2b_1-b_1^3+a_2'-\frac{a_3'}{a_3}a_2+\frac{a_3'}{a_3}b_1^2-2b_1'b_1,$$

$$s_1=2a_1b_1+a_1'-\frac{a_3'}{a_3}a_1, \quad s_0=3a_0b_1+a_0'-\frac{a_3'}{a_3}a_0.$$

From our assumptions $d_3(z)=-a_3(z)\equiv 1$ and $d_2(z)=b_1(z)^2-a_2(z)\equiv 0$, we get $s_2(z)\equiv 0$ and

$$s_1(z)=2a_1(z)b_1(z)+a_1'(z) \quad\text{and}\quad s_0(z)=3a_0(z)b_1(z)+a_0'(z). \tag{4.6}$$

We have $S(z, w)\equiv 0$, say $s_1(z)\equiv s_2(z)\equiv 0$. Thus from (4.6), we obtain

$$a_1(z)=C_1(e^{-\int b_1(z)\,dz})^2 \quad\text{and}\quad a_0(z)=C_0(e^{-\int b_1(z)\,dz})^3.$$

If $C_1\neq 0$ and $C_0\neq 0$, then $e^{-\int b_1(z)\,dz}$ is meromorphic. This implies that $D(z, w)$ has a meromorphic function element. If $C_1\neq 0$ and $C_0=0$, then the equation (1.2) reduces to the equation (1.10). If $C_1=0$ and $C_0\neq 0$, then the equation (1.2) reduces to the equation (1.11). If $C_1=0$ and $C_0=0$, then the equation (1.2) reduces to the equation (1.9). □

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REFERENCES

[1] W.K. HAYMAN, Meromorphic Functions, Clarendon Press, Oxford, 1964.

[2] K. ISHIZAKI, Admissible solutions of the Schwarzian differential equations, J. Austral. Math. Soc. Ser. A, **50** (1991), 258-278.

[3] K. ISHIZAKI, A third order differential equation and representable poles, Nihonkai Math. J., **4** (1993), 201-220.

[4] K. ISHIZAKI AND N. YANAGIHARA, On admissible solutions of algebraic differential equations, Funkcialaji Ekvacioj, **38** (1995), 433-442.

[5] I. LAINE, Nevanlinna Theory and Complex Differential Equations, W. Gruyter, Berlin-New York, 1992.

[6] R. NEVANLINNA, Analytic Functions, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[7] N. STEINMETZ, Eigenschaften eindeutiger Lösungen gewöhnlicher Differentialgleichungen im Komplexen, Doctoral Dissertation, Karlsruhe, 1978.

[8] N. STEINMETZ, Ein Malmquistscher Satz für algebraische Differentialgleichungen erster Ordnung, J. Reine Angew. Math., **316** (1980), 44-53.

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