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ON SECTIONAL AND RICCI CURVATURES OF SEMI-RIEMANNIAN SUBMERSIONS*

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Abstract

O'Neill introduced a notion of Riemannian submersion [7]. In this paper we give a new notion of semi-Riemannian submersion and want to investigate some geometric properties concerned with sectional and Ricci curvatures of this submersion.

1. Introduction

The theory of Riemannian submersion was firstly introduced by O'Neill ([7]) and its geometric properties have been studied by many differential geometers (Besse [1], Escobales Jr. [2], [3], Gray [4], Magid [5], Nakagawa and Takagi [6], and Takagi and Yorozu [11]). In this paper we introduce a new notion of a semi-Riemannian submersion which is more general than the notion of Riemannian submersion and want to investigate its geometric properties.

The main purpose of section 2 is to give the notion of semi-Riemannian submersion which contains the concepts of both Riemannian and indefinite Riemannian (or said to be pseudo-Riemannian) submersions and to construct some fundamental formulas for this submersion.

In section 3 we will give a typical example of semi-Riemannian submersion of pseudo-hyperbolic space H_n^{m+n}.

Now in section 4 the sectional curvature of semi-Riemannian submersion will be defined and the sufficient conditions for the horizontal distribution $\mathcal{D}_H$ of the minimal semi-Riemannian submersion to be totally geodesic and integrable will be studied in terms of sectional curvature.

Finally, in section 5 we also define the notion of Ricci curvature of the semi-Riemannian submersion and want to investigate some geometric properties for the horizontal distribution $\mathcal{D}_H$ of the minimal semi-Riemannian submersion to be totally geodesic and integrable in terms of Ricci curvature. Moreover, we will give another example of minimal semi-Riemannian submersion which is not totally geodesic.

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2. Preliminaries

Let M be an $(m+n)$-dimensional connected semi-Riemannian manifold of index $r+s(0 \leq r \leq m,\ 0 \leq s \leq n)$, which is denoted by M^{m+n}_{r+s} and let B be an n-dimensional connected semi-Riemannian manifold of index s, which is denoted by B^n_s. A *semi-Riemannian submersion* π: $M \to B$ is a submersion of semi-Riemannian manifolds M and B such that

(1) The fiber $\pi^{-1}(b)$, $b \in B$, are semi-Riemannian submanifolds of M.

(2) The differential $d\pi$ of π preserves scalar products of vectors normal to fibers.

For a semi-Riemannian submersion $\pi: M \to B$ vectors tangent to fibers are said to be *vertical* and those normal to fibers are said to be *horizontal*. Any vector field X on M can be decomposed as

$$X = X' + X'',$$

where X' (resp. X'') denotes a vertical (resp. horizontal) part of X. We define two tensors T and A of type (1, 2) on M by

$$\begin{cases} T(X,\ Y) = (\nabla_{X'}Y'')' + (\nabla_{X'}Y')'', \\ A(X,\ Y) = (\nabla_{X''}Y'')' + (\nabla_{X''}Y')'', \end{cases} \tag{2.1}$$

for any vector fields X and Y on M, where ∇ denotes the Levi-Civita connection on M. They are called *integrability tensors* for the semi-Riemannian submersion $\pi: M \to B$. We choose a local field $e_1, \ldots, e_{m+n}$ of orthonormal frames adapted to the semi-Riemannian metric of M in such a way that, restricted to the fiber $\pi^{-1}(b)$, $b \in B$, $e_1, \ldots, e_m$ is a local field of orthonormal frames adapted to a semi-Riemannian metric of $\pi^{-1}(b)$ induced from that on the semi-Riemannian manifold M. The following convention on the range of indices will be used throughout this paper:

$$A,\ B,\ C,\ D,\ E,\ F,\ \ldots = 1,\ \ldots,\ m+n;$$
$$i,\ j,\ k,\ l,\ \ldots = 1,\ \ldots,\ m;$$
$$\alpha,\ \beta,\ \gamma,\ \delta,\ \ldots = m+1,\ \ldots,\ m+n,$$

where m denotes the dimension of fibers. The summation Σ is taken over all repeated indices, unless otherwise stated. Then we have $< e_A,\ e_B > = \varepsilon_A\delta_{AB}$, where $<,>$ denotes the scalar product on M. The dual coframe field is denoted by $\{\omega_A\}$. The connection form ω_{AB} are characterized by the structure equations of M:

$$\begin{cases} d\omega_A + \sum \varepsilon_B \omega_{AB} \wedge \omega_B = 0, \\ \omega_{AB} + \omega_{BA} = 0, \end{cases} \tag{2.2}$$

$$\begin{cases} d\omega_{AB} + \sum \varepsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega_{AB}. \\ \Omega_{AB} = -\frac{1}{2}\sum \varepsilon_C \varepsilon_D R_{ABCD}\omega_C \wedge \omega_D, \end{cases} \tag{2.3}$$

where Ω_{AB} denotes the curvature form of M and R_{ABCD} are components of the

Riemannian curvature tensor R with respect to the semi-Riemannian metric. The Levi-Civita connection ∇ on M is given by

(2.4) $$\nabla_{e_A} e_B = \sum \varepsilon_C \omega_{CB}(e_A) e_C.$$

We define two tensors h and A_0 of type (1, 2) on M by

$$h(X, Y) = (\nabla_{Y'} X')'', \quad A_0(X, Y) = -(\nabla_{Y''} X'')',$$

for any vector fields X and Y on M. They are also called *integrability tensors* for the semi-Riemannian submersion $\pi: M \to B$. The integrability tensor h restricted to a fiber means the second fundamental form of the fiber. It follows from (2.2) and (2.4) that

$$h(e_i, e_j) = \sum \varepsilon_\alpha \omega_{\alpha i}(e_j) e_\alpha, \quad A_0(e_\alpha, e_\beta) = \sum \varepsilon_j \omega_{\alpha j}(e_\beta) e_j.$$

In fact, by the definition we get

$$h(e_i, e_j) = (\sum \varepsilon_C \omega_{Ci}(e_j) e_C)'' = \sum \varepsilon_\alpha \omega_{\alpha i}(e_j) e_\alpha.$$

On the other hand, it is seen by Gray [4] that the integrability tensor A_0 satisfies the following relation:

(2.5) $$A_0(e_\alpha, e_\beta) = -A_0(e_\beta, e_\alpha) = -\frac{1}{2}[e_\alpha, e_\beta]',$$

and hence we get

$$A_0(e_\alpha, e_\beta) = -(\sum \varepsilon_C \omega_{C\alpha}(e_\beta) e_C)' = \sum \varepsilon_j \omega_{\alpha j}(e_\beta) e_j.$$

Thus the only components h^A_{BC} (resp. A^B_{CD}) of h (resp. A_0) which may not vanish are

$$h^\alpha_{ij} = \omega_{\alpha i}(e_j), \quad (\text{resp. } A^\iota_{\alpha\beta} = \omega_{\alpha i}(e_\beta)).$$

Accordingly the connection form $\omega_{\alpha\iota}$ are given by

(2.6) $$\omega_{\alpha\iota} = \sum \varepsilon_j h^\alpha_{ij} \omega_j + \sum \varepsilon_\beta A^\iota_{\alpha\beta} \omega_\beta.$$

We may choose a suitable semi-Riemannian metric on the tangent bundle TM of M and decompose TM as a direct product of a vertical distribution $\mathcal{D}_V$ and a horizontal one $\mathcal{D}_H$, where the vertical (resp. horizontal) distribution is defined by an assignment of any point x in M with a tangent space (resp. the orthonormal subspace) to a fiber through x. A distribution $\mathcal{D}$ is said to be *integrable* if $[X, Y]$ belong to $\mathcal{D}$ whenever vector fields X and Y belong to $\mathcal{D}$. Since the vertical distribution $\mathcal{D}_V$ is defined by $\omega_\alpha = 0$ and it is integrable, by Cartan's lemma we have

(2.7) $$h^\alpha_{ij} = h^\alpha_{ji}.$$

Since the integrability tensor A_0 is also skew-symmetric, we get

(2.8) $$A^i_{\alpha\beta} = A^\iota_{\beta\alpha}.$$

The semi-Riemannian submersion $\pi: M \to B$ is said to be *minimal* if each fiber is minimal, i.e., if it satisfies $\Sigma \varepsilon_j h^\alpha_{jj} = 0$. The semi-Riemannian submersion

$\pi: M \to B$ is said to be *totally geodesic* if each fiber is totally geodesic, i.e., if it satisfies $h_{ij}^{\alpha} = 0$. By (2.5) the horizontal distribution $\mathcal{D}_H$ is integrable if and only if

$$A^{\iota}_{\alpha\beta} = 0. \tag{2.9}$$

Now, for a tensor field $T = (T^{A_1 \cdots A_r}_{B_1 \cdots B_s})$ on M, we define the covariant derivative $T^{A_1 \cdots A_r}_{B_1 \cdots B_s C}$ by

$$\sum \varepsilon_C T^{A_1 \cdots A_r}_{B_1 \cdots B_s C} \omega_C = dT^{A_1 \cdots A_r}_{B_1 \cdots B_s} - \sum \varepsilon_C T^{A_1 \cdots A_{a-1} C A_{a+1} \cdots A_r}_{B_1 \cdots B_s} \omega_{CA_a} - \sum \varepsilon_C T^{A_1 \cdots A_r}_{B_1 \cdots B_{b-1} C B_{b+1} \cdots B_s} \omega_{CB_b}. \tag{2.10}$$

Then, from the definition of (h^A_{BCD}), (A^A_{BCD}) and (2.6), it follows that

$$h^{\iota}_{ijk} = -\sum \varepsilon_\alpha h^{\alpha}_{ij} h^{\alpha}_{\iota k}, \quad h^{\iota}_{ij\alpha} = -\sum \varepsilon_\beta h^{\beta}_{ij} A^{\iota}_{\beta\alpha},$$
$$h^{\alpha}_{\beta ij} = \sum \varepsilon_k h^{\alpha}_{ki} h^{\beta}_{kj}, \quad h^{\alpha}_{\beta i\gamma} = \sum \varepsilon_k h^{\alpha}_{ki} A^{k}_{\beta\gamma},$$
$$h^{A}_{\beta\gamma C} = h^{\iota}_{\alpha CD} = h^{\iota}_{C\beta D} = 0, \quad A^{i}_{j\alpha\beta} = -\sum \varepsilon_\gamma A^{\iota}_{\gamma\alpha} A^{J}_{\gamma\beta},$$
$$A^{\iota}_{j\alpha k} = -\sum \varepsilon_\beta A^{\iota}_{\beta\alpha} h^{\beta}_{jk}, \quad A^{\gamma}_{\alpha\beta j} = \sum \varepsilon_l A^{l}_{\alpha\beta} h^{\gamma}_{lj},$$
$$A^{\gamma}_{\alpha\beta\delta} = \sum \varepsilon_l A^{l}_{\alpha\beta} A^{l}_{\gamma\delta}, \quad A^{C}_{ijD} = A^{\alpha}_{iCD} = A^{\alpha}_{CjD} = 0,$$

$$h^{\alpha}_{i\beta j} = \sum \varepsilon_k h^{\alpha}_{ik} h^{\beta}_{kj}, \tag{2.11}$$

$$h^{\alpha}_{i\beta\gamma} = \sum \varepsilon_k h^{\alpha}_{ik} A^{k}_{\beta\gamma}, \tag{2.12}$$

$$A^{\iota}_{\alpha j\beta} = -\sum \varepsilon_\gamma A^{\iota}_{\alpha\gamma} A^{J}_{\gamma\beta}, \tag{2.13}$$

$$A^{\iota}_{\alpha jk} = -\sum \varepsilon_\beta A^{\iota}_{\alpha\beta} h^{\beta}_{jk}. \tag{2.14}$$

Moreover, by the exterior derivatives of (2.6) and by means of (2.2), (2.3) and (2.10)–(2.14), we have

$$R_{\alpha ijk} = h^{\alpha}_{ijk} - h^{\alpha}_{ikj} + A^{\iota}_{\alpha jk} - A^{\iota}_{\alpha kj}, \tag{2.15}$$

$$R_{\alpha ij\beta} = h^{\alpha}_{ij\beta} - h^{\alpha}_{i\beta j} + A^{\iota}_{\alpha j\beta} - A^{\iota}_{\alpha\beta j}, \tag{2.16}$$

$$R_{\alpha i\beta\gamma} = h^{\alpha}_{i\beta\gamma} - h^{\alpha}_{i\gamma\beta} + A^{\iota}_{\alpha\beta\gamma} - A^{\iota}_{\alpha\gamma\beta}. \tag{2.17}$$

Next, by virtue of (2.2), (2.3) and (2.10) we have the Ricci formulas for the second covariant derivatives of h as the following

$$h^{A}_{BCDE} - h^{A}_{BCED} = \sum \varepsilon_F (h^{F}_{BC} R_{AFDE} + h^{A}_{FC} R_{BFDE} + h^{A}_{BF} R_{CFDE}).$$

3. Examples

In this section typical examples of semi-Riemannian submersion of an $(m+n)$-dimensional pseudo-hyperbolic space H^{m+n}_n are considered.

Let $\boldsymbol{C}$ or $\boldsymbol{H}$ be the field consisting of complex numbers or quaternion numbers. They are simply denoted by $\boldsymbol{K}$. In $\boldsymbol{K}^{n+1}$ with the standard basis, a semi-Hermitian form F is defined by

$$F(z, w) = -\sum_{i=1}^{r} z_i\bar{w}_i + \sum_{j=r+1}^{n+1} z_j\bar{w}_j,$$

where $z=(z_1, \ldots, z_{n+1})$ and $w=(w_1, \ldots, w_{n+1})$ are in $\boldsymbol{K}^{n+1}$. The complex or quaternion semi-Euclidean space $(\boldsymbol{K}^{n+1}, F)$ is simply denoted by $\boldsymbol{K}_r^{n+1}$. The scalar product $g'(z, w)$ is given by $ReF(z, w)$ is a semi-Riemannian metric of index dr in $\boldsymbol{K}_r^{n+1}$, where $d=2$ or $d=4$ according as $\boldsymbol{K}=\boldsymbol{C}$ or $\boldsymbol{K}=\boldsymbol{H}$. Let H_{dr-1}^{dn+d-1} be a real hypersurface of $\boldsymbol{K}_r^{n+1}$, $r \geq 1$, defined by

$$H_{dr-1}^{dn+d-1} = \{z \in \boldsymbol{K}_r^{n+1}\colon F(z, z) = -1\},$$

and let g be a semi-Riemannian metric of H_{dr-1}^{dn+d-1} induced from the semi-Riemann metric g'. Then (H_{dr-1}^{dn+d-1}, g) is the semi-Riemannian manifold of constant sectional curvature -1 and with index $dr-1$, which is called a *unit pseudo-hyperbolic space*. For the unit pseudo-hyperbolic space H_{dr-1}^{dn+d-1} with index $dr-1$ the tangent space $T_z(H_{dr-1}^{dn+d-1})$ at each point z can be identified (through the parallel displacement in $\boldsymbol{K}_r^{n+1}$) with $\{w \in \boldsymbol{K}_r^{n+1}\colon ReF(z, w)=0\}$.

Let T_z' be the orthogonal complement of the vector iz in $T_z(H_{2r}^{2n+1})$ or the vectors iz, jz and kz in $T_z(H_{4r}^{4n+3})$, where we denote by i an imaginary unit in C and by 1, i, j and k a basis for $\boldsymbol{H}$ so that they satisfy $i^2=j^2=k^2=-1$, $ij=-ji=k$, $jk=-kj=i$ and $ki=-ik=j$. Let H_{d-1}^{d-1} be the multiplicative group of these numbers of absolute value 1. Then H_{dr}^{dn+d-1} can be considered a principal fiber bundle over a pseudo-hyperbolic $\boldsymbol{K}$-space $H_{r-1}^n\boldsymbol{K}$ with group H_{d-1}^{d-1} and the projection π. Furthermore there is a connection such that T_z' is the horizontal subspace at z which is invariant under the H_{d-1}^{d-1}-action. The metric g_0 of constant holomorphic sectional curvature -4 is given by $g_{0_b}(X, Y)=g_z(X^*, Y^*)$ for any tangent vectors X and Y in $T_b(H_{r-1}^n\boldsymbol{K})$, where z is any point in the fiber $\pi^{-1}(b)$ and X^* and Y^* are vectors in T_z' such that $d\pi X^*=X$ and $d\pi Y^*=Y$.

On the other hand, complex structures $I\colon w\mapsto iw$, $J\colon w\mapsto jw$ and $K\colon w\mapsto kw$ in T_z' is compatible with the action of H_{d-1}^{d-1} and induce almost complex structures I, J and K on $H_{r-1}^n\boldsymbol{K}$ such that $d\pi\circ i=I\circ d\pi$, $d\pi\circ j=J\circ d\pi$ and $d\pi\circ k=K\circ d\pi$. Thus $H_{r-1}^n\boldsymbol{K}$ is a pseudo-hyperbolic space over $\boldsymbol{K}$ of constant holomorphic sectional curvature -4 and it is seen that the principal H_{d-1}^{d-1}-bundle H_{dr}^{dn+d-1} over $H_{r-1}^n\boldsymbol{K}$ with projection π is a semi-Riemannian submersion with the fundamental tensors I, J and K. A distribution $\mathfrak{D}$ determined by the subspace spanned by iz, jz and kz at any point z is integrable. In fact, we have

$$(3.1) \qquad \nabla_{iz}(jz) = j\nabla_{iz}(z) = jiz = -kz,$$

because j is parallel and H_{dr-1}^{dn+d-1} is totally umbilic in $\boldsymbol{K}_r^{n+1}$. This shows that $[iz, jz]=-2kz$. Since the others hold similarly, it means that the distribution $\mathfrak{D}$ is integrable. On the other hand, (3.1) implies that the maximal integral submanifold of $\mathfrak{D}$ is totally geodesic. Thus the semi-Riemannian submersions have totally geodesic time-like fibers H_{d-1}^{d-1}.

$$\begin{array}{ccc} H^{d-1}_{d-1} & \rightarrow & H^{dn+d-1}_{dr-1} \\ & & \downarrow \pi \\ & & H^{n}_{r-1}\boldsymbol{K} \end{array}$$

In particular, we consider the case $r = 1$. Then there exist totally geodesic space-like submersions $\pi: H^{2n+1}_1 \rightarrow H^n\boldsymbol{C}$ and $\pi: H^{4n+3}_3 \rightarrow H^n\boldsymbol{H}$ whose basic manifold is Riemannian.

4. Sectional curvatures

Let $M = M^{m+n}_{r+s}$ be an $(m+n)$-dimensional semi-Riemannian manifold of index $r+s$ and $B = B^n_s$ be an n-dimensional semi-Riemannian manifold of index s. We denote by P_D and P_I the set of all definite plane sections and all non-degenerate plane sections, respectively. For any non-degenerate plane section P_I the sectional curvature is denoted by $K(P_I)$. Let $\pi: M \rightarrow B$ be a semi-Riemannian submersion. Then we have

$$R_{\alpha ij\beta} = h^{\alpha}_{ij\beta} - \sum \varepsilon_k h^{\alpha}_{ik} h^{\beta}_{kj} + \sum \varepsilon_\gamma A^{\iota}_{\alpha\gamma} A^{j}_{\beta\gamma} - A^{\iota}_{\alpha\beta j} \tag{4.1}$$

by means of (2.8), (2.11), (2.13) and (2.16). Assume that the semi-Riemannian submersion $\pi: M \rightarrow B$ is minimal. Then it is easily seen that we have

$$\sum \varepsilon_j h^{\alpha}_{jj\beta} = 0,$$

from which the following

$$\sum \varepsilon_j R_{\alpha jj\beta} = -\sum \varepsilon_j \varepsilon_k h^{\alpha}_{jk} h^{\beta}_{kj} + \sum \varepsilon_j \varepsilon_\gamma A^{j}_{\alpha\gamma} A^{j}_{\beta\gamma} - \sum \varepsilon_j A^{j}_{\alpha\beta j}$$

is derived. Since the left hand side and the first two terms of the right hand side are symmetric with respect to indices α and β and the last one is skew-symmetric, we have

$$\sum \varepsilon_j R_{\alpha jj\beta} = -\sum \varepsilon_j \varepsilon_k h^{\alpha}_{jk} h^{\beta}_{kj} + \sum \varepsilon_j \varepsilon_\gamma A^{j}_{\alpha\gamma} A^{j}_{\beta\gamma}. \tag{4.2}$$

THEOREM 4.1. *Let $\pi: M^{m+n}_m \rightarrow B^n$ be a semi-Riemannian submersion. If $K(P_I) \geq 0$ and if the submersion is minimal, then it is totally geodesic and horizontal distribution is integrable.*

Proof. By (4.2) we get

$$\sum \varepsilon_j R_{\alpha jj\alpha} = -\sum \varepsilon_j \varepsilon_k h^{\alpha}_{jk} h^{\alpha}_{kj} + \sum \varepsilon_j \varepsilon_\gamma A^{j}_{\alpha\gamma} A^{j}_{\alpha\gamma} \leq 0,$$

because of $\varepsilon_j = -1$ and $\varepsilon_\gamma = 1$. By the assumption $K(P_I) \geq 0$ we get $\varepsilon_j \varepsilon_\alpha R_{\alpha jj\alpha} \geq 0$. Thus we get $h^{\alpha}_{ij} = 0$ and $A^{\iota}_{\alpha\beta} = 0$ for any indices. □

Similarly, using (4.2) one can prove the following:

COROLLARY 4.2. *Let $\pi: M^{m+n}_n \rightarrow B^n_n$ be a semi-Riemannian submersion. If $K(P_I) \leq 0$ and if the submersion is minimal, then it is totally geodesic and the horizontal distribution is integrable.*

Certain semi-Riemannian submersions like those in Theorem 4.1 and Corollary 4.2 have simple geometric situation. The distribution $\mathfrak{D}$ is said to be *parallel* if the vector field $\nabla_X Y$ belong to $\mathfrak{D}$ whenever a vector field Y belongs to $\mathfrak{D}$. Let $\pi: M \rightarrow B$ be a semi-Riemannian submersion with totally geodesic fibers. We assume that the horizontal distribution $\mathfrak{D}_H$ is integrable. Then by (2.4) and (2.6) we have

$$\nabla_{eA} e_\beta = \sum \varepsilon_\gamma \omega_{\gamma\beta}(e_A) e_\gamma,$$

which means that the horizontal distribution is parallel. Thus the vertical distribution orthogonal to $\mathfrak{D}_H$ is also parallel and hence one finds the following:

THEOREM 4.3. *Let $\pi: M_m^{m+n} \rightarrow B^n$ be a semi-Riemannian submersion. If it is totally geodesic and if the horizontal distribution is integrable, then the total space M is locally decomposed into the product manifold $F \times B$.*

Remark 1. Let $\pi: M_m^{m+n} \rightarrow B^n$ be a semi-Riemannian submersion. If it is totally geodesic, the mixed sectional curvature $K(U, X)$ is always non-positive, where U (resp. X) is a vertical vector (resp. a horizontal vector).

Now, an m-dimensional semi-Riemannian manifold of index r and of constant curvature c is called a *semi-Riemannian space form*, which is denoted by $M_r^m(c)$. Let $\pi: M_m^{m+n}(c) \rightarrow B^n$ be a minimal semi-Riemannian submersion. We denote by S the square of the length of the second fundamental form of the fiber, that is, $S = \Sigma \varepsilon_i \varepsilon_j \varepsilon_\alpha h_{ij}^\alpha h_{ij}^\alpha = \Sigma h_{ij}^\alpha h_{ij}^\alpha$, because of $\varepsilon_i = -1$ and $\varepsilon_\alpha = 1$. Then by (4.2) we obtain

$$S = \sum \varepsilon_\alpha \varepsilon_\beta \varepsilon_j A_{\alpha\beta}^j A_{\alpha\beta}^j \; - \; mnc.$$

From this equation we can conclude the following properties: For a minimal semi-Riemannian submersion $\pi: M_m^{m+n}(c) \rightarrow B^n$

(1) $c \geq 0$ implies that $c = 0$, $h_{ij}^\alpha = 0$ and $A_{\alpha\beta}^j = 0$ for any indices i, j, α and β.

(2) $c < 0$ implies $S \leq -mnc$, where the equality holds if and only if $A_{\alpha\beta}^j = 0$ for all indices α, β and j.

Therefore we can state the following lemmas:

LEMMA 4.4. *Let $\pi: M_m^{m+n}(c) \rightarrow B^n$ be a semi-Riemannian submersion. If the submersion is minimal and if $c \geq 0$, then $c = 0$. Moreover it is totally geodesic and the horizontal distribution is integrable.*

LEMMA 4.5. *Let $\pi: M_m^{m+n}(c) \rightarrow B^n$ be a semi-Riemannian submersion. If the submersion is minimal and if $c < 0$, then $S \leq -mnc$, where the equality holds if and only if the horizontal distribution is integrable.*

Thus we can prove the following:

THEOREM 4.6. *Let M be an $(m+n)$-dimensional semi-Riemannian space form $M_m^{m+n}(c)$ of index m and B be an n-dimensional Riemannian manifold. If $c > 0$, then there are no minimal semi-Riemannian submersions $\pi: M \rightarrow B$.*

Similarly, the following properties can be verified:

LEMMA 4.7. *Let $\pi: M_n^{m+n}(c) \to B_n^n$ be a semi-Riemannian submersion. If the submersion is minimal and if $c \leq 0$, then $c = 0$. Moreover it is totally geodesic and the horizontal distribution is integrable.*

THEOREM 4.8. *Let M be an $(m+n)$-dimensional semi-Riemannian space form $M_n^{m+n}(c)$ of index n and B be an n-dimensional semi-Riemannian manifold of index n. If $c < 0$, then there are no minimal semi-Riemannian submersions $\pi: M \to B$.*

5. Ricci curvatures

Let M be an $(m+n)$-dimensional semi-Riemannian manifold of index $r+s$ $(r, s \geq 0)$ and B be an n-dimensional semi-Riemannian manifold of index s. Let $\pi: M \to B$ be a semi-Riemannian submersion. We choose a local field $\{e_A\}$ of orthonormal frames, restricted to the fiber $\pi^{-1}(b)$, $b \in B$, $\{e_j\}$ is a local field of orthonormal frames of $\pi^{-1}(b)$. By means of the semi-Riemannian submersion, $\{e''_\alpha = d\pi(e_\alpha)\}$ is a local field of orthonormal frames on B. The dual coframe field is denoted by $\{\omega''_\alpha\}$ on B with respect to $\{e''_\alpha\}$. It is easily seen that we get

$$\omega_\alpha = \pi^* \omega''_\alpha. \tag{5.1}$$

The connection forms $\{\omega''_{\alpha\beta}\}$ are characterized by the structure equations of B:

$$\begin{cases} d\omega''_\alpha + \sum \varepsilon_\beta \omega''_{\alpha\beta} \wedge \omega''_\beta, \\ \omega''_{\alpha\beta} + \omega''_{\beta\alpha} = 0, \end{cases} \tag{5.2}$$

$$\begin{cases} d\omega''_{\alpha\beta} + \sum \varepsilon_\gamma \omega''_{\alpha\gamma} \wedge \omega''_{\gamma\beta} = \Omega''_{\alpha\beta}, \\ \Omega''_{\alpha\beta} = -\dfrac{1}{2} \sum \varepsilon_\gamma \varepsilon_\delta R''_{\alpha\beta\gamma\delta} \omega''_\gamma \wedge \omega''_\delta, \end{cases} \tag{5.3}$$

where $\Omega''_{\alpha\beta}$ denotes the curvature form of B and $R''_{\alpha\beta\gamma\delta}$ are component of the Riemannian curvature tensor R'' of B. Differentiating (5.1) exteriorly we get $d\omega_\alpha = d(\pi^*\omega''_\alpha) = \pi^*(d\omega''_\alpha)$ and hence, using (2.2), (2.6), (2.7), (5.1) and (5.2) we get

$$\sum \varepsilon_\beta (\omega_{\alpha\beta} - \pi^* \omega''_{\alpha\beta}) \wedge \omega_\beta - \sum \varepsilon_j \varepsilon_\beta A^j_{\alpha\beta} \omega_j \wedge \omega_\beta = 0. \tag{5.4}$$

We put

$$\omega_{\alpha\beta} - \pi^* \omega''_{\alpha\beta} = \sum \varepsilon_C W_{\alpha\beta C} \omega_C, \quad \omega_{\alpha j} = \sum \varepsilon_C W_{\alpha j C} \omega_C. \tag{5.5}$$

Then by (2.6), (5.4) and (5.5) we have

$$\begin{cases} W_{\alpha\beta\gamma} = 0, \quad W_{\alpha\beta j} = A^j_{\alpha\beta}, \\ W_{\alpha j i} = h^\alpha_{ij}, \quad W_{\alpha j \beta} = A^j_{\alpha\beta}, \end{cases} \tag{5.6}$$

where we have used the fact that $W_{\alpha\beta\gamma}$ is skew-symmetric with respect to α and β. A tensor W whose components are given by $W_{\alpha BC}$ is called the *structure* *tensor* for the semi-Riemannian submersion $\pi : M \to B$. We denote by $W_{\alpha BCD}$ components of the covariant derivative of the structure tensor W with respect to the connection form $\pi^* \omega''_{\alpha\beta}$. Then it is defined by

$$\sum \varepsilon_D W_{\alpha BCD} \omega_D = dW_{\alpha BC} - \sum \varepsilon_\delta W_{\delta BC} \pi^* \omega''_{\delta\alpha} - \sum \varepsilon_D (W_{\alpha DC} \omega_{DB} + W_{\alpha BD} \omega_{DC}). \tag{5.7}$$

Taking account of (2.6), (5.6) and (5.7) we can easily obtain

$$\begin{cases} W_{\alpha\beta\gamma\iota} = \sum \varepsilon_j (A^j_{\alpha\beta} h^\gamma_{ji} + A^j_{\alpha\gamma} h^\beta_{ji}), \\ W_{\alpha\beta\gamma\delta} = \sum \varepsilon_j (A^j_{\alpha\beta} A^j_{\gamma\delta} + A^j_{\alpha\gamma} A^j_{\beta\delta}). \end{cases} \tag{5.8}$$

Differentiating (5.5) exteriorly we get

$$d(\omega_{\alpha\beta} - \pi^* \omega''_{\alpha\beta}) = \sum \varepsilon_C d(W_{\alpha\beta C} \omega_C).$$

Accordingly, making use of the structure equations (2.2), (2.3) for M and the structure equations (5.1), (5.2) and (5.3) for B together with (5.5) and (5.7), we can directly obtained the following Ricci formula:

$$W_{\alpha\beta CD} - W_{\alpha\beta DC} = R_{\alpha\beta CD} - \delta_{C\gamma} \delta_{D\delta} R''_{\alpha\beta\gamma\delta}. \tag{5.9}$$

Thus, from (5.8) and (5.9) we have for the semi-Riemannian submersion $\pi : M \to B$

$$R_{\alpha\beta\gamma\delta} - R''_{\alpha\beta\gamma\delta} = \sum \varepsilon_j (2A^j_{\alpha\beta} A^j_{\gamma\delta} + A^j_{\alpha\gamma} A^j_{\beta\delta} - A^j_{\alpha\delta} A^j_{\beta\gamma}), \tag{5.10}$$

$$R_{\alpha\beta\beta\alpha} - R''_{\alpha\beta\beta\alpha} = -3 \sum \varepsilon_j A^j_{\alpha\beta} A^j_{\alpha\beta}. \tag{5.11}$$

Remark 1. The above equations (5.10) and (5.11) in the Riemannian submersion are already obtained by Besse [1], Escobales Jr. [3], Gray [4], and O'Neill [7].

For the non-degenerate plane spanned by vectors u and v at any point on the semi-Riemannian manifold B the sectional curvature of the plane section is denoted by $K''(u, v)$.

LEMMA 5.1. *For a semi-Riemannian submersion $\pi : M^{m+n}_m \to B^n$ $(n \geq 2)$, we have*

$$K(e_\alpha, e_\beta) \geq K''(d\pi e_\alpha, d\pi e_\beta) \circ \pi,$$

where the equality holds if and only if $A^j_{\alpha\beta} = 0$ for any index j.

Proof. By the assumption of the semi-Riemannian submersion we have $\varepsilon_j = -1$, which implies that (5.11) is equivalent to

$$R_{\alpha\beta\beta\alpha} - R''_{\alpha\beta\beta\alpha} = -3 \sum \varepsilon_j A^j_{\alpha\beta} A^j_{\alpha\beta} \geq 0.$$

Since the sectional curvature of the plane section spanned by e_α and e_β (resp. $d\pi e_\alpha$ and $d\pi e_\beta$) is given by

$$K(e_\alpha, e_\beta) = R_{\alpha\beta\beta\alpha}, \; K''(d\pi e_\alpha, d\pi e_\beta) = R''_{\alpha\beta\beta\alpha},$$

we get $K(e_\alpha, e_\beta) \geq K''(d\pi e_\alpha, d\pi e_\beta) \circ \pi$, where the equality holds if and only if $A^j_{\alpha\beta} = 0$ *for any index* j. □

THEOREM 5.2. *For a semi-Riemannian submersion* $\pi: M^{m+n}_m \to B^n$ $(n \geq 2)$ *if* $K(P_D) \leq 0$, *then there exists at least one plane* P'' *in* T_bB, $b \in B$, *such that* $K''(P'') < 0$ *or* B *is locally flat and the horizontal distribution is integrable.*

Proof. Suppose that there does not exist a plane section P'' such that $K''(P'') < 0$. Then, for any point $b \in B$ and for any plane section P'' in T_bB we have $K''(P'') \geq 0$. Accordingly Lemma 5.1 means that

$$K(e_\alpha, e_\beta) \geq K''(d\pi e_\alpha, d\pi e_\beta) \circ \pi \geq 0$$

for any indices α and β. Since the plane section spanned by e_α and e_β is definite, by the assumption we get $K(e_\alpha, e_\beta) \leq 0$ for any indices α and β, which means that B is locally flat and $A^j_{\alpha\beta} = 0$ for any indices. □

From now on we assume that the semi-Riemannian submersion $\pi: M \to B$ is minimal. Then we have the formula (4.2) given in section 4. By virtue of this formula we can prove

THEOREM 5.3. *Let* M^{m+1}_1 *be a Lorentzian manifold satisfying the strongly energy condition and* $\pi: M^{m+1}_1 \to B^1_1$ *be a semi-Riemannian submersion of codimension one and with space-like fibers. If it is minimal, then it is totally geodesic.*

Proof. By the assumption of codimension we have $\dim B = 1$ and each fiber is a space-like hypersurface, which implies that B is time-like. The assumption for the strongly energy condition means that the Ricci tensor $\mathrm{Ric}(e_\alpha) = \mathrm{Ric}(e_\alpha, e_\alpha)$ in the direction of the time-like vector e_α of M satisfies $\mathrm{Ric}(e_\alpha) = R_{\alpha\alpha} \geq 0$, where $\alpha = m+1$.

On the other hand, (4.2) is reformed as $R_{\alpha\alpha} = -\sum h^\alpha_{jk} h^\alpha_{jk} - \sum A^j_{\alpha\gamma} A^j_{\alpha\gamma} \leq 0$, because of $\varepsilon_j = 1$ and $\varepsilon_\alpha = -1$. Thus, by the strongly energy condition, the equality holds and hence we have $h^\alpha_{ij} = 0$ for any indices. □

Remark 2. Let M be a compact Riemannian manifold whose Ricci curvature is positive semi-definite. It is proved by Oshikiri [9] that if a foliation $(M, g, \mathcal{F})$ of codimension one is minimal, then $\mathcal{F}$ is totally geodesic and the metric g is bundle-like.

Next we study the Ricci curvature and the Einstein condition for the semi-Riemannian submersions. Since the fibers are submanifolds of the total space M, the Riemannian curvature tensor R' satisfies the Gauss equation

$$R'_{ijkl} = R_{ijkl} + \sum \varepsilon_\alpha (h^\alpha_{il} h^\alpha_{jk} - h^\alpha_{ik} h^\alpha_{jl}), \tag{5.12}$$

where R'_{ijkl} are components of the Riemannian curvature tensor R' of the fiber. We denote by R_{AB} (resp. R'_{ij} and $R''_{\alpha\beta}$) the components of the Ricci curvature tensor of M (resp. the fiber and B). Because of

$$R_{ij} = \sum \varepsilon_k R_{kijk} + \sum \varepsilon_\beta R_{\beta ij\beta},$$

we obtain by (2.8), (4.1) and the Gauss equation (5.12)

$$R_{ij} = R'_{ij} + \sum \varepsilon_\beta h^\beta_{ij\beta} - \sum \varepsilon_k \varepsilon_\beta h^\beta_{kk} h^\beta_{ij} + \sum \varepsilon_\beta \varepsilon_\gamma A^i_{\beta\gamma} A^j_{\beta\gamma}. \tag{5.13}$$

Similarly we get

$$\begin{aligned} R_{\alpha\beta} = R''_{\alpha\beta} &- 2\sum \varepsilon_j \varepsilon_\gamma A^j_{\alpha\gamma} A^j_{\beta\gamma} - \sum \varepsilon_i \varepsilon_j h^\alpha_{ij} h^\beta_{ij} \\ &+ \sum \varepsilon_i h^\alpha_{ii\beta} - \sum \varepsilon_i A^i_{\alpha\beta i}, \end{aligned} \tag{5.14}$$

where we have used (4.1) and (5.10). On the other hand, we have

$$R_{i\beta} = \sum \varepsilon_j (h^\beta_{jji} - h^\beta_{ijj}) + 2\sum \varepsilon_\alpha \varepsilon_k h^\alpha_{ik} A^k_{\beta\alpha} + \sum \varepsilon_\alpha A^i_{\alpha\beta\alpha} \tag{5.15}$$

by (2.12) (2.14), (2.15) and (2.17). Now we denote by Ric (resp. Ric′ and Ric″) the Ricci curvature tensor of M (resp. the fiber and B). The Ricci curvature in the direction of e_j of M is denoted by $\mathrm{Ric}(e_j) = \mathrm{Ric}(e_j, e_j)$. From (5.13) one finds the following:

THEOREM 5.4. *For a minimal semi-Riemannian submersion $\pi : M^{m+n}_m \to B^n$ or $\pi : M^{m+n}_n \to B^n_n$ if*

$$\mathrm{Ric}'(e_j) \geq \mathrm{Ric}(e_j)$$

for any index j, then the horizontal distribution is integrable.

Proof. By the assumption and (5.13) we have

$$0 \geq \mathrm{Ric}(e_j) - \mathrm{Ric}'(e_j) = \sum \varepsilon_\beta h^\beta_{jj\beta} + \sum \varepsilon_\beta \varepsilon_\gamma A^j_{\beta\gamma} A^j_{\beta\gamma} \geq \sum \varepsilon_\beta h^\beta_{jj\beta},$$

where the equality holds if and only if $A^j_{\beta\gamma} = 0$ for any indices.

On the other hand, we have $\sum \varepsilon_j \varepsilon_\beta h^\beta_{jj\beta} = 0$, which implies

$$\sum \varepsilon_j \{\mathrm{Ric}(e_j) - \mathrm{Ric}'(e_j)\} = 0.$$

Thus we get $\mathrm{Ric}(e_j) - \mathrm{Ric}'(e_j) = 0$ for any index j. □

We say the horizontal distribution $\mathcal{D}_H$ satisfies the Yang-Mills condition if it satisfies

$$\sum \varepsilon_\beta A^i_{\beta\alpha\beta} = \sum \varepsilon_k \varepsilon_\beta h^\beta_{ik} A^k_{\alpha\beta}$$

for any indices i and α. It is important for Einstein Riemannian submersions (cf. pp. 243 Besse [1]). From the formula between Ricci curvatures we get the following:

PROPOSITION 5.5. *Let $\pi : M^{m+n}_m \to B^n$ (resp. $\pi : M^{m+n}_n \to B^n_n$) be a semi-*

Riemannian submersion with time-like (resp. space-like) totally geodesic fibers. Then M is Einstein if and only if the horizontal distribution $\mathfrak{D}_H$ satisfies the Yang-Mills condition and the Ricci curvatures of the fibers and the base manifold B satisfy

(1) $R'_{ij} = \lambda\varepsilon_i\delta_{ij} - \sum\varepsilon_\alpha\varepsilon_\beta A^i_{\alpha\beta}A^j_{\alpha\beta}$,

(2) $R''_{\alpha\beta} = \lambda\varepsilon_\alpha\delta_{\alpha\beta} + 2\sum\varepsilon_i\varepsilon_\gamma A^i_{\alpha\gamma}A^i_{\beta\gamma}$,

for a constant λ.

Next we assume that the semi-Riemannian submersion $\pi: M \to B$ is minimal and we denote by r (resp. r' or r'') the scalar curvature of M (resp. the fiber or B). Then by the definition we get

$$r - r'' = \sum\varepsilon_\alpha R_{\alpha\alpha} + \sum\varepsilon_j R_{jj} - \sum\varepsilon_\alpha R''_{\alpha\alpha}.$$

Then by (5.13) and (5.14) it is reformed as

$$r - r' - r'' = -\sum\varepsilon_j\varepsilon_\alpha\varepsilon_\beta A^j_{\alpha\beta}A^j_{\alpha\beta} - \sum\varepsilon_\alpha\varepsilon_i\varepsilon_j h^\alpha_{ij}h^\alpha_{ij}, \tag{5.16}$$

where $r' = \sum\varepsilon_i R'_{ii}$ and $r'' = \sum\varepsilon_\alpha R''_{\alpha\alpha}$. Thus we have the followings:

THEOREM 5.6. *For a minimal Riemannian submersion $\pi: M^{m+n} \to B^n$, we have*

$$r \leq r' + r'',$$

where the equality holds if and only if it is totally geodesic and the horizontal distribution is integrable.

COROLLARY 5.7. *For a Riemannian submersion $\pi: M^{m+n} \to B^n$ if there is a point $x \in M$ such that $r(x) > r'(x) + r''(x)$, then it is not minimal.*

Remark 3. The following results are proved by Watson [10]. Let M be a compact Riemannian manifold whose Ricci tensor is positive semi-definite and B be a Riemannian manifold whose Ricci tensor is negative semi-definite. If there is a point on M at which the Ricci tensor is positive definite, then there are no minimal submersions $\pi: M \to B$. In particular, if B is of negative curvature, there are no minimal submersions $\pi: M \to B$.

From (5.13) and (5.16) we have

$$\sum\varepsilon_\alpha\{\mathrm{Ric}(e_\alpha) - \mathrm{Ric}''(d\pi e_\alpha)\} = 2(r - r' - r'') + \sum\varepsilon_\alpha\varepsilon_i\varepsilon_j h^\alpha_{ij}h^\alpha_{ij}.$$

Thus we prove the following:

LEMMA 5.8. *Let $\pi: M^{m+n}_m \to B^n$ be a semi-Riemannian submersion. If it is minimal, then*

$$\sum\varepsilon_\alpha\{\mathrm{Ric}(e_\alpha) - \mathrm{Ric}''(d\pi e_\alpha)\} \geq 2(r - r' - r''),$$

where the equality holds if and only if it is totally geodesic.

As a direct consequence of (5.16) and Lemma 5.8 we get

THEOREM 5.9. *Let* $\pi: M_m^{m+n} \to B^n$ *be a semi-Riemannian submersion. If it is minimal and if* $\mathrm{Ric}(e_\alpha) \leq \mathrm{Ric}''(d\pi e_\alpha)$ *and* $r - r'' - r'' \geq 0$, *then it is totally geodesic and the horizontal distribution is integrable.*

Example. An example of minimal semi-Riemannian submersion $\pi: M_m^{m+n} \to B^n$ which is not totally geodesic is here constructed.

Let $\{f_A\}$ be the set of smooth positive functions on $\boldsymbol{R}^n$. Let M_m^{m+n} (resp. M_n^{m+n}) be an $(m+n)$-dimensional semi-Riemannian manifold of index m (resp. index n) defined by

$$\begin{aligned} M &= M_m^{m+n} \text{ (resp. } M_n^{m+n}) \\ &= \{(x, y) \in \boldsymbol{R}^m \times \boldsymbol{R}^n : g = (g_{AB}),\ g_{AB}(x, y) = \varepsilon_A f_A{}^2(y)\delta_{AB}\}, \end{aligned}$$

where $\varepsilon_j = -1$, $\varepsilon_\alpha = 1$ (resp. $\varepsilon_j = 1$, $\varepsilon_\alpha = -1$). Also, let $B = B^n$ (resp. B_n^n) be an n-dimensional Riemannian (resp. semi-Riemannian) manifold defined by

$$B = \{y \in \boldsymbol{R}^n : g'' = (g''_{\alpha\beta}),\ g''_{\alpha\beta} = \varepsilon_\alpha f_\alpha{}^2(y)\delta_{\alpha\beta}\}.$$

Then, for the natural projection $\pi: M \to B$, it is a semi-Riemannian submersion whose fibers are defined by a fixed point $y \in \boldsymbol{R}^n$. For the natural coordinate system $\{x_A\}$ the natural basis $\{\partial/\partial x_A\}$ satisfies

$$\begin{cases} g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \varepsilon_i f_i{}^2 \delta_{ij}, \\ g(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_j}) = 0, \\ g(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}) = \varepsilon_\alpha f_\alpha{}^2 \delta_{\alpha\beta}. \end{cases}$$

Accordingly an orthonormal basis $\{e_A\}$ is given by

$$e_i = \frac{1}{f_i} \frac{\partial}{\partial x_i},\quad e_\alpha = \frac{1}{f_\alpha} \frac{\partial}{\partial x_\alpha}.$$

Thus, calculating $\nabla_{e_i} e_j$ we can get

$$h_{ij}^\alpha = -\frac{\varepsilon_i}{f_i f_\alpha} \frac{\partial f_i}{\partial x_\alpha} \delta_{ij}.$$

Consequently we have

$$\sum \varepsilon_j h_{jj}^\alpha = -\frac{1}{f_\alpha} \frac{\partial}{\partial x_\alpha} \log \prod f_j.$$

This shows that if the functions $f_1, \ldots, f_m$ satisfy the condition $\prod f_j = \text{constant}$, then the submersion is minimal, but in general not totally geodesic.

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