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PROJECTIVE SPACES IN A WIDER SENSE, II

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Introduction

In [2] we studied a family of compact irreducible symmetric spaces with some property which all projective spaces have in common. We call the spaces projective spaces in a wider sense. The family contains two conspicuous kinds of spaces: Grassmann manifolds $G(r, nr)$ and the symmetric space EIII with the exceptional type in the sense of E. Cartan. In these spaces the intersection number of two *lines* in the general position is one.

In this paper we study the projective transformations of EIII (Propositions 4.1–4.5) and, at last, give an embedding map of EIII into E_6 explicitely (Theorem 4.2). In §1 the known facts are quoted from I. Yokota [7], [8]. In §2 we define a projective transformation $\Pi_{A,B}(\kappa)$. In §3 the polar sets in EIII are studied. The one is the oriented Grassmann manifold $G^{OR}(2, 8)$ with the dimension 16. It plays a role of line in the projective plane EIII. In §4 an important transformation $\phi(A, B; \kappa)$ is introduced by modifying $\Pi_{A,B}(\kappa)$. This clarifies the structure of the plane EIII.

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§1. Exceptional Jordan algebra J^C over C

We quote the known facts (Lemmas 1.1–1.6) from Yokota [7], [8]. Note that zero divisors exist in the Cayley algebra $\mathfrak{C}^C$ over the field C of complex numbers. Being different from the Cayley plane, this fact gives a new polar set (see. Lemma 3.2, ②), which plays a geometrically important role in EIII. Our main product in J^C is $X \triangle Y$.

In $\mathfrak{C}^C$ there are two kinds of conjugation: let $\{e_i\}$ be a basis of $\mathfrak{C}$ (over the field $\boldsymbol{R}$ of real numbers) and e_0 the unit element.

(1) $x = \sum \xi_i e_i \to \bar{x} = \xi_0 e_0 - \sum_{i \neq 0} \xi_i e_i$,

(2) $x = \sum \xi_i e_i \to \tilde{x} = \sum \tilde{\xi}_i e_i$ (for $\xi_i \in C$),

where $\xi \to \tilde{\xi}$ is the usual conjugation in C. The following properties hold: for x, $y \in \mathfrak{C}^C$,

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(1) $\overline{xy} = \bar{y}\bar{x}$,

(2) $x^2y = x(xy)$, $\qquad yx^2 = (yx)x$.

This algebra has zero divizors; for example, $x\bar{x} = 0$ for $x = e_0 + ie_1$, where $i = \sqrt{-1}$ in C. An inner product (x, y) is defined by $(x, y) = \frac{1}{2}(x\bar{y} + y\bar{x})$.

We denote by J^C the exceptional Jordan algebra over C: each element in J^C has the 3×3 hermitian form

$$X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \qquad (\xi_i \in C,\ x_i \in \mathfrak{C}^C).$$

J^C is closed under the usual Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$, which satisfies

(1) $X \circ Y = Y \circ X$,

(2) $X^2 \circ (X \circ Y) = X \circ (X^2 \circ Y)$.

We write the above X as $X = X(\xi, x)$. In J^C two non-degenerate inner products (X, Y) and $\langle X, Y\rangle$ are introduced:

(1) $\mathrm{tr}(X) = \xi_1 + \xi_2 + \xi_3$, $\qquad (X, Y) = \mathrm{tr}(X \circ Y)$,

$\mathrm{tr}(X, Y, Z) = (X, Y \circ Z)$,

(2) $X \times Y = \frac{1}{2}(2X \circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\mathrm{tr}(Y) - (X, Y))E)$,

$(X, Y, Z) = (X, Y \times Z)$, $\qquad \det X = \frac{1}{3}(X, X, X)$,

(3) $\langle X, Y\rangle = \mathrm{tr}(\tau X \circ Y)$, $\qquad X \triangle Y = \tau(X \times Y)$,

where E is the 3×3 unit matrix and τ is defined by $\tau X = X(\tilde{\xi}, \tilde{x})$ for $X = X(\xi, x)$.

We denote by $\mathrm{Iso}_C(J^C)$ the group of C-linear automorphisms α in J^C, i.e., α satisfies $\alpha(\xi X) = \xi\alpha X$ ($\xi \in C$) and $\alpha(X \circ Y) = \alpha X \circ \alpha Y$. The compact simple Lie group E_6 can be given via the complex Lie group E_6^C (cf. [7]):

(1) $E_6^C := \{\alpha \in \mathrm{Iso}_C(J^C) \mid (\alpha X, \alpha Y, \alpha Z) = (X, Y, Z)\}$,

(2) $E_6 = \{\alpha \in E_6^C \mid \langle \alpha X, \alpha Y\rangle = \langle X, Y\rangle\}$.

Let $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $E_1 \triangle E_1 = 0$ holds. We see later that E_1 is a base point in the symmetric space EIII. Define an involutive automorphism σ in J^C by

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \quad \to \quad \begin{pmatrix} \xi_1 & -x_3 & -\bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ -x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}.$$

Let $E_6(\sigma)=\{\alpha\in E_6 \mid \alpha\sigma=\sigma\alpha\}$. Then one has $\sigma\in E_6(\sigma)$ and $E_6(\sigma)=(U(1)\times Spin(9))/Z_4$.

LEMMA 1.1. *If $\alpha\in E_6\ (\sigma)$, then there exists $\xi\in \boldsymbol{C}$, $|\xi|=1$, such that $\alpha E_1=\xi E_1$.*

LEMMA 1.2. *The compact symmetric space EIII can be characterized by the three forms:*

(1) $EIII=\{X\in J^{C} \mid X\Delta X=0,\ X\neq 0\}/C^*$,

(2) $EIII=\{\alpha E_1 \mid \alpha\in E_6\}/C^*$,

(3) $EIII=E_6/E_6(\sigma)$,

where $C^*:=C-\{0\}$ and the notation "$/C^*$" means the equivalence relation by C^*.

LEMMA 1.3. *In J^{C} the following properties hold: for $\eta,\ \mu,\ \xi\in C$,*

(1) $\langle \xi X, Y\rangle=\tilde{\xi}\langle X, Y\rangle$, $\quad\langle X, \xi Y\rangle=\xi\langle X, Y\rangle$,

(2) $\langle X, Y\rangle=\widetilde{\langle Y, X\rangle}$, $\quad\tau(X\times Y)=\tau X\times\tau Y$,

(3) $\langle X\Delta Y, Z\rangle=(X, Y, Z)$ $\quad$ (*symmetric for* $X, Y, Z\in J^{C}$),

(4) $\langle B\Delta(A\Delta X), U\rangle=\langle X, A\Delta(B\Delta U)\rangle$,

(5) $\eta A\Delta(\mu B\Delta\xi X)=\tilde{\eta}\,\mu\,\xi\,(A\Delta(B\Delta X))$,

(6) $(X\Delta X)\Delta(X\Delta X)=(\det X)X$.

Proof. (4) is derived from (3). (6) is essentially due to H. Freudenthal ([3], p. 220). The remaining proofs are easy. □

For $\alpha\in \mathrm{Iso}_C\,(J^{C})$ we define the contragradient form α^* by

$$\langle \alpha^* X,\ Y\rangle=\langle X,\ \alpha Y\rangle \qquad (\text{for all } Y\in J^{C}).$$

LEMMA 1.4. *For $\alpha,\ \beta\in \mathrm{Iso}_C(J^{C})$, one has*

(1) $(\alpha^*)^*=\alpha$, $\quad$ (2) $(\alpha\beta)^*=\beta^*\alpha^*$,

(3) $1^*=1$ *(identity map)*, $\quad$ (4) $(\alpha^*)^{-1}=(\alpha^{-1})^*$.

LEMMA 1.5.

(1) $E_6^{C}=\{\alpha\in \mathrm{Iso}_C(J^{C}) \mid \alpha X\Delta\alpha Y=(\alpha^*)^{-1}(X\Delta Y)\}$,

(2) $E_6=\{\alpha\in \mathrm{Iso}_C(J^{C}) \mid \alpha(X\Delta Y)=\alpha X\Delta\alpha Y,\ \langle\alpha X, \alpha Y\rangle=\langle X, Y\rangle\}$.

Proof. By (3) in Lemma 1.3 we can show (1) because $(\alpha X, \alpha Y, \alpha Z)=(X, Y, Z)$ is equivalent to $\alpha X\Delta\alpha Y=(\alpha^*)^{-1}(X\Delta Y)$. The equation $\langle\alpha X, \alpha Y\rangle=\langle X, Y\rangle$ implies $(\alpha^*)^{-1}=\alpha$. This gives (2). □

COROLLARY 1.1. *For $\alpha\in E_6$ one has $(\alpha^*)^{-1}=\alpha$.*

LEMMA 1.6. *If* $\alpha \in E_6^C$ *then* $\alpha^* \in E_6^C$ *holds.*

Proof. Using (6) in Lemma 1.3 we can verify this assertion. The method is the same as Lemma 2.4 in ([8], p. 42) essentially. □

§2. Transformation $\Pi_{A,B}(\kappa)$

We define a transformation $\Pi_{A,B}(\kappa)$ in J^C, which was first introduced by H. Freudenthal ([4], p. 277) in the exceptional Jordan algebra over $\boldsymbol{R}$. It will have later the meaning of a projective transformation in EIII.

We define a derivation $D_{X,U}(A)$ in E_6^C by

$$D_{X,U}(A) = \tau U \circ (X \circ A) - X \circ (\tau U \circ A) - (X \circ \tau U) \circ A + \frac{1}{3}(\tau U, X)A.$$

Put $L_x(A) = X \circ A$. Then $[L_{\tau U}, L_x]$ is a derivation of the complex exceptional Lie group F_4^C. Set $S = X \circ \tau U - \frac{1}{3}(\tau U, X)E$. Since the trace of S is 0, L_s is a derivation of E_6^C. Hence $D_{X,U} = [L_{\tau U}, L_x] - L_s$ is also a derivarion of E_6^C. This satisfies the following identity. The proof is due to H. Freudenthal ([3], p. 220) essentially.

LEMMA 2.1. *For* $X, U, A \in J^C$, *one has*

$$U \Delta (X \Delta A) = \frac{1}{2} D_{X,U}(A) + \frac{1}{4}\langle U, A\rangle X + \frac{1}{12}\langle U, X\rangle A.$$

DEFINITION. Let $\kappa \in C^*$ and $A, B, X \in J^C$, where $\langle A, B\rangle \neq 0$. We define

$$\Pi_{A,B}(\kappa)X = X + \frac{1-\kappa}{\kappa}\frac{\langle B, X\rangle}{\langle B, A\rangle} A - 4\frac{1-\kappa}{\langle B, A\rangle} B\Delta(A\Delta X).$$

LEMMA 2.2. *It holds that* $\langle \Pi_{A,B}(\kappa)X, U\rangle = \langle X, \Pi_{B,A}(\tilde{\kappa})U\rangle$.

Proof. Using (1), (2) and (4) in Lemma 1.3, one has

$$\begin{aligned}\langle \Pi_{A,B}(\kappa)X, U\rangle &= \langle X, U\rangle + \frac{1-\tilde{\kappa}}{\tilde{\kappa}}\frac{\langle X, B\rangle}{\langle A, B\rangle}\langle A, U\rangle \\ &\quad - 4\frac{1-\tilde{\kappa}}{\langle A, B\rangle}\langle X, A\Delta(B\Delta U)\rangle \\ &= \langle X, \Pi_{B,A}(\tilde{\kappa})U\rangle.\end{aligned}$$

□

§3. Polar sets in $E\mathrm{III}$

As the definition of $E\mathrm{III}$, we make use of $E\mathrm{III}=\{X\in \mathfrak{J}^C \mid X\Delta X=0,\ X\neq 0\}/C^*$. Denote the elements in $E\mathrm{III}$ by $[X]$ because they are the equivalence classes by C^*. The polar sets for $[E_1]$ (= the fixed point set of σ) are studied here (cf. [6], p. 42). Since the coefficient field of $\mathfrak{C}^C$ is C, $|x|^2$ $(=x\bar{x})=0$ does not always mean $x=0$. By easy calculation one has the following two lemmas, where $X=X(\xi, x)\in \mathfrak{J}^C$ and $X\neq 0$.

LEMMA 3.1. *For $X\in \mathfrak{J}^C$, it holds that $[X]\in E\mathrm{III}$* if and only if

$$\xi_2\xi_3=|x_1|^2,\qquad \xi_3\xi_1=|x_2|^2,\qquad \xi_1\xi_2=|x_3|^2,$$
$$x_2x_3=\xi_1\bar{x}_1,\qquad x_3x_1=\xi_2\bar{x}_2,\qquad x_1x_2=\xi_3\bar{x}_3.$$

LEMMA 3.2. *The automorphism σ leaves $[X]\in E\mathrm{III}$ fixed if and only if X satisfies one of the following three cases:*

① $X=\xi_1E_1 \qquad (\xi_1\neq 0).$

② $X=\begin{pmatrix} 0 & x_3 & \bar{x}_2\\ \bar{x}_3 & 0 & 0\\ x_2 & 0 & 0\end{pmatrix}, \qquad (x_2x_3=0,\ x_2\bar{x}_2=0,\ x_3\bar{x}_3=0).$

③ $X=\begin{pmatrix} 0 & 0 & 0\\ 0 & \xi_2 & x_1\\ 0 & \bar{x}_1 & \xi_3\end{pmatrix}, \qquad (\xi_2\xi_3=|x_1|^2).$

For the above polars we know the following facts (cf. Atsuyama [1]).

(1) In the case ① the fixed point set of σ consists of one point $[E_1]$. We may regard $[E_1]$ as the North Pole in $E\mathrm{III}$.

(2) In the case ② we have $X\circ X=0$. The set of $[X]$ becomes the compact irreducible symmetric space with the type $D\mathrm{III}(5)$. This is connected and has the dimension 20. This equals the set consisting of the midpoints of the shortest closed geodesics starting from $[E_1]$ (cf. [1], p. 246).

(3) In the case ③ the set of $[X]$ is also connected. This is the real oriented Grassmann manifold $G^{OR}(2, 8)$ with the dimension 16.

Let $[P]$ be an arbitrary point in $E\mathrm{III}$. We denote by $L(P)$ the polar set with the type ③ for $[P]$. Thus, if $\alpha[P]=[E_1]$ for some $\alpha\in E_6$, $L(\alpha P)$ is the set of all elements with the type ③ in Lemma 3.2. $L(P)$ is called a *line* (in the sense of projective geometry). Let $[P]$, $[Q]\in E\mathrm{III}$. We say that $[P]$ and $[Q]$ are in the singular position if they lie on a shortest closed geodesic. If not, we say that they are in the general position. Then the following properties hold in $E\mathrm{III}$. By this reason we call $E\mathrm{III}$ a projective plane in the wider sense.

(i) For two points in the general position there exists only one line passing through them. If in the singular position, the set of lines passing through them becomes a connected manifold with the dimension 8 (cf. [1], p. 247–248)

(ii) There exits a duality between the points $[P]$ and the lines $L(P)$, i.e., $[P]\in L(Q)$ if and only if $[Q]\in L(P)$. The correspondence L: $[P]\to L(P)$ gives the polarity.

LEMMA 3.3. *Let $[X_0]\in E\mathrm{III}$ be an arbitrary point with the type ② (in Lemma 3.2) for $[E_1]$. Then $[X]\in E\mathrm{III}$ lies on a shortest closed geodesic passing through $[E_1]$ and $[X_0]$ if and only if there exists $\xi\in \boldsymbol{C}$ such that $[X]=[\xi E_1+X_0]$.*

Proof. (Sufficiency) Since $[X_0]$ lies on a shortest closed geodesic starting from $[E_1]$, we may assume $\xi\neq 0$ (cf. [1], p. 246). Next we define a curve $\gamma(t)=[X(t)]$, $t\in \boldsymbol{R}$, in $E\mathrm{III}$ by

$$X(t)=\begin{pmatrix} 1-t & \frac{t}{\xi}x_3 & \frac{t}{\xi}\bar{x}_2 \\ \frac{t}{\xi}\bar{x}_3 & 0 & 0 \\ \frac{t}{\xi}x_2 & 0 & 0 \end{pmatrix}.$$

In fact $\gamma(t)$ passes in $E\mathrm{III}$ because $X(t)\Delta X(t)=0$. Furthermore the following facts hold:

(1) $\gamma(0)=[E_1]$, $\gamma\left(\frac{1}{2}\right)=[\xi E_1+X_0]$ and $\gamma(1)=[X_0]$.

(2) $\gamma(t)$ is a closed geodesic: Put $Z=\xi^{-1}X_0$. Since the trace of Z is 0, L_z is an infinitesimal element of $E_6^{\boldsymbol{C}}$. $E_6^{\boldsymbol{C}}$ acts in $E\mathrm{III}$. By $X_0\circ X_0=0$, one has

$$\gamma(t)=[\exp(\theta L_z)E_1],$$

where $\theta=\dfrac{2t}{1-t}$. Essentially $\gamma(t)$ is a great circle in the unit sphere $S^{52}\subset J^{\boldsymbol{C}}$ and $E\mathrm{III}$ is a submanifold in $S^{52}/\boldsymbol{C}^*$. Hence, by $\gamma(-\infty)=\gamma(\infty)$, we can see that $\gamma(t)$ is a closed geodesic.

(3) $\gamma(t)$ is the shortest closed geodesic: There exists the following fact. If a geodesic $\omega(t)$ connecting $[E_1]$ and $[X_0]$ satisfies the condition $(*)$, then it is closed and has the shortest length $2\sqrt{3}\pi$, where the metric in $E\mathrm{III}$ is introduced from the Killing-form of the Lie algebra of E_6.

$(*)$ Let $[X]$ be an arbitrary point on $\omega(t)$, where $[X]\neq[E_1]$, $[X_0]$. If $\alpha\in E_6$ leaves $[E_1]$ and $[X]$ fixed, then α leaves $\omega(t)$ fixed pointwise.

We can see that $\gamma(t)$ satisfies $(*)$: Assume that $\alpha E_1=\eta E_1$ and $\alpha(\xi E_1+X_0)=\mu(\xi E_1+X_0)$, where η, $\mu\in\boldsymbol{C}^*$ and $\xi\neq 0$. Then one has $\alpha X_0=\xi(\mu-\eta)E_1+\mu X_0$ and hence $0=\langle E_1, X_0\rangle=\langle\alpha E_1, \alpha X_0\rangle=\eta\xi(\mu-\eta)$. This implies $\mu=\eta$ and $\alpha X_0=\eta X_0$. Therefore, for any $\nu\in\boldsymbol{C}$, $\alpha[\nu E_1+X_0]=[\nu E_1+X_0]$ holds.

(Necessity) Let H be the subgroup of E_6 which leaves $[E_1]$ and $[X_0]$ fixed. Consider one closed geodesic $\omega(t)$ which has the shortest length and passes through $[E_1]$ and $[X_0]$. Then any other such geodesic is transitive to ω by H, and the set of orbits of $\omega(t)$ by H, $\{h\omega(t)h^{-1} \mid h\in H,\ t\in \boldsymbol{R}\}$, becomes a two-dimensional sphere because the tangent space to the orbit at $\omega(t)$ consists of some single vector ([1], p. 244). Put $\xi = e^{i\theta}$, $\theta\in\boldsymbol{R}$, in $X(t)$ (defined in the proof of sufficiency), and write $\gamma(t;\ \theta)$ instead of $\gamma(t)$. Then, for $\theta\in\boldsymbol{R}$, $\gamma(t;\ \theta)$ is a geodesic passing through $[E_1]$ and $[X_0]$, and the set $\{\gamma(t;\ \theta) \mid t,\ \theta\in\boldsymbol{R}\}$ becomes the sphere. □

Lemma 3.4. *Let* $[X]\in E\mathrm{III}$. *Then* $X\times E_1 = 0$ *if and only if* $[X]$ *lies on a shortest closed geodesic starting from* $[E_1]$.

Proof. Put $X = X(\xi, x)$. Since $[X]\in E\mathrm{III}$, the entries ξ_i and x_i satisfy the condition in Lemma 3.1.

(Sufficiency) By Lemma 3.3 we may set $X = \mu(\xi E_1 + X_0)$, μ, $\xi\in\boldsymbol{C}$. Since $E_1\times E_1 = 0$ and $X_0\times E_1 = 0$, one gets $X\times E_1 = 0$.

(Necessity) Assume $X\times E_1 = 0$. This implies $x_2x_3 = 0$, $x_2\bar{x}_2 = 0$ and $x_3\bar{x}_3 = 0$. Hence, if we put $X_0 = \xi_1E_1 - X$, $[X_0]$ belongs to the polar set with the type ② (in Lemma 3.2) for $[E_1]$. Hence, from Lemma 3.3, $[X]$ lies on a shortest closed geodesic connecting $[E_1]$ and $[X_0]$. □

Proposition 3.1. *Let* $[A]$, $[B]\in E\mathrm{III}$. *Then*

(1) $A\triangle B = 0$ *if and only if* $[A]$ *and* $[B]$ *are lie on a shortest closed geodesic*,

(2) *if* $A\triangle B\neq 0$, *then* $L(A\triangle B)$ *is the unique line passing through* $[A]$ *and* $[B]$.

Proof. (1) By the transitivity of E_6 in $E\mathrm{III}$, there exists $\alpha\in E_6$ such that $\alpha B = \mu E_1$ ($\mu\in C^*$). Since α is also an automorphism for $\triangle$, we have

$$\alpha(A\triangle B) = \alpha A\triangle\alpha B = \tau(\alpha A\times\alpha B) = \tau(\alpha A\times\mu E_1).$$

Hence, by Lemma 3.4, one obtains

$$\begin{aligned} A\triangle B = 0 &\Leftrightarrow \alpha A\times E_1 = 0,\\ &\Leftrightarrow [\alpha A] \text{ lies on a shortest closed geodesic starting from } [E_1],\\ &\Leftrightarrow [A] \text{ and } [B] \text{ are lie on a shortest closed geodesic.}\end{aligned}$$

(2) Let $\alpha B = \mu E_1$ ($\alpha\in E_6$, $\mu\in\boldsymbol{C}^*$). If we put $\alpha A = X(\xi, x)$ and $Y = A\triangle B$, then

$$\tau\alpha Y = \begin{pmatrix} 0 & 0 & 0\\ 0 & \frac{\mu}{2}\xi_3 & -\frac{\mu}{2}x_1\\ 0 & -\frac{\mu}{2}\bar{x}_1 & \frac{\mu}{2}\xi_2 \end{pmatrix}$$

holds by direct calculation. We get $[\alpha Y]\in E\mathrm{III}$ from Lemma 3.1. Furthermore, by Lemma 3.2, $[\alpha Y]\in L(E_1)$ holds. This means $[Y]\in L(B)$. Since the correspondence L gives the polarity between points and lines in $E\mathrm{III}$, we obtain $[B]\in L(Y)$. Similarly $[A]\in L(Y)$ can be shown. Hence, $L(Y)$ passes through $[A]$ and $[B]$. Since $A\triangle B\neq 0$, $[A]$ and $[B]$ are in the general position. Therefore, the line passing through them is determined uniquely ([1], p. 247). □

§4. Projective transformation $\phi(A, B; \kappa)$

We define a transformation $\phi(A, B; \kappa)$ in J^C by modifying $\Pi_{A,B}(\kappa)$. This explains clearly the structure of projective geometry in $E\mathrm{III}$.

LEMMA 4.1. *One has* $A\triangle(U\triangle(A\triangle X))=\frac{1}{4}\langle A, U\rangle A\quad X$ *for* $[A]$, $[X]\in E\mathrm{III}$ *and* $U\in J^C$.

Proof. This identity is independent of choosing the representative elements of $[A]$ and $[X]$. Put $B=A\triangle X$. If $B=0$, the identity is trivial. Hence we may assume $B\neq 0$. And we shall prove

$$A\triangle(U\triangle B) = \frac{1}{4}\langle A, U\rangle B.$$

Note that $[A]\in L(B)$ holds by (2) of Proposition 3.1. If we take $\alpha\in E_6$ such that $\alpha B=\mu E_1$ $(\mu\in C^*)$, the above equation becomes

$$\alpha A\triangle(\alpha U\triangle E_1) = \frac{1}{4}\langle \alpha A, \alpha U\rangle E_1.$$

Since $\alpha A\in L(E_1)$ holds, we may set

$$\tau\alpha A = \begin{pmatrix} 0 & 0 & 0\\ 0 & \xi_2 & a_1\\ 0 & \bar{a}_1 & \xi_3\end{pmatrix}.$$

Then, for $\alpha U=V(\eta, y)$ with $\eta_i\in C$ and $y_i\in\mathfrak{C}^C$, we obtain

$$\begin{aligned}\alpha A\triangle(\alpha U\triangle E_1) &= \frac{1}{4}(\xi_2\eta_2+\xi_3\eta_3+2(a_1, u_1))E_1\\ &= \frac{1}{4}\langle \alpha \mathrm{A}, \alpha U\rangle E_1.\end{aligned}$$

□

COROLLARY 4.1. *If* $[A]\in L(B)$, *one has that* $A\triangle(U\triangle B)=\frac{1}{4}\langle A, U\rangle B$ *for* $U\in J^C$.

LEMMA 4.2. *For* $[A]\in E\mathrm{III}$ and $U\in J^{C}$, *it holds that* $(A\Delta U)\Delta(A\ \ U)=\frac{1}{4}\langle U\Delta U,\ A\rangle A$.

Proof. Consider $\alpha\in E_6$ such that $\alpha A=\mu E_1$ $(\mu\in C^*)$ and put $\alpha U=V(\xi,x)$. Then, by direct calculation, we can see

$$(\mu E_1\Delta V)\Delta(\mu E_1\Delta V)=\frac{1}{4}\mu^2(\xi_2\xi_3-x_1\bar{x}_1)E_1=\frac{1}{4}\langle V\Delta V,\ \mu E_1\rangle\mu E_1. \qquad \square$$

DEFINITION. For $[A]$, $[B]\in E\mathrm{III}$ with $\langle A, B\rangle\neq 0$, we define a transformation ϕ in J^C by $\phi(A, B;\ e^z)=e^{-z}\Pi_{A,B}(e^{3z})$, $z\in C$.

PROPOSITION 4.1. *The following properties hold*:

(1) $\left.\frac{d}{dz}\phi\right|_{z=0}=\frac{6}{\langle B,\ A\rangle}D_{A,B}$,

(2) $\phi(A, B;\ \mu)\ \phi(A, B;\ \nu)=\phi(A, B;\ \mu\nu)$,

(3) $\phi(A, B;\ e^z)=\exp\left(\frac{6}{\langle B,\ A\rangle}D_{A,B}\right)$,

(4) $\phi(A, B;\ \kappa)\in E_6^C$,

(5) $\phi(A, B;\ \kappa)^*=\phi(B, A;\ \tilde{\kappa})$, $\qquad \phi(A, B;\ \kappa)^{-1}=\phi(A, B;\ \kappa^{-1})$,

(6) $\alpha\phi(A, B;\ \kappa)\alpha^{-1}=\phi(\alpha A,\ (\alpha^*)^{-1}B;\ \kappa)$ $\qquad$ *for* $\alpha\in E_6^C$.

Proof. We can calculate (1) directly. (2): From $A\Delta A=0$, one has $\phi(A, B;\mu)A=\mu^{-4}A$. By Lemma 4.1 and (4) of Lemma 1.3, we get $\phi(A, B;\ \mu)(B\Delta(A\Delta X))=\mu^2B\Delta(A\Delta X)$. (In Lemma 4.1, the condition $[X]\in E\mathrm{III}$ is unnecessary because linear combinations $\Sigma\ \xi X$ span J^C where $\xi\in C$ and $[X]\in E\mathrm{III}$). We obtain (2) by these two identities. (3): If we fix $[A]$, $[B]\in E\mathrm{III}$, ϕ is an one-parameter group and it satisfies the initial condition (1). Hence (3) holds. We get (4) from (3) because $D_{A,B}$ is a derivation of E_6^C. (5): The first identity can be obtained by $\langle X,\ \Pi_{A,B}(\nu)Y\rangle=\langle\Pi_{B,A}(\tilde{\nu})X,\ Y\rangle$. The second one holds by (2) and $\phi(A, B;\ 1)=\mathrm{id}$. Finally one can show (6) by $\langle B,\ \alpha^{-1}X\rangle=\langle(\alpha^*)^{-1}B,\ X\rangle$, $\langle B,\ A\rangle=\langle(\alpha^*)^{-1}B,\ \alpha A\rangle$ and $\alpha(B\Delta(A\Delta\alpha^{-1}X))=((\alpha^*)^{-1}B)\Delta(\alpha A\Delta X)$. $\square$

PROPOSITION 4.2. $\phi(A, B;\ \kappa)\in E_6$ *if and only if* $[A]=[B]$, $|\kappa|=1$.

Proof. (Necessity) Put $\phi=\phi(A, B;\ \kappa)$ simply. By $\langle\phi A,\ \phi A\rangle=\langle A,\ A\rangle$ and $\phi A=\kappa^{-4}A$, one gets $|\kappa|=1$. Let $[X]$ $(\in E\mathrm{III})$ satisfy $A\Delta X=0$. Then, by Proposition 3.1, $[A]$ and $[X]$ lie on a shortest closed geodesic and the type of X is of ② (in Lemma 3.2) for $[A]$. Moreover we obtain $\langle B,\ X\rangle=0$ from $\langle\phi X,\ \phi X\rangle=\langle X,\ X\rangle$. This means that B has the type ① or ③ for $[A]$ because X is an arbitrary element with the type ②. If the type of B is of ③, then $\langle A, B\rangle = 0$. But this contradicts the definition of $\phi(A, B; \kappa)$. Hence $[A] = [B]$ holds.

(Sufficiency) The definition of $\phi(A, B; \kappa)$ is independent of choosing the representative elements of $[A]$ and $[B]$. Hence, from the transitivity by E_6 in EIII, we may assume $A = B = E_1$. Then, for $X = X(\xi, x)$ and $Y = Y(\eta, y)$, one has

$$(\phi(E_1, E_1; \kappa)^*)^{-1} = \phi(E_1, E_1; \kappa) \qquad \text{(by (5) of Proposition 4.1)}$$

and

$$\begin{aligned}\langle \phi X, \phi Y\rangle &= |\kappa|^{-4}(\tilde{\xi}_1\eta_1 + \tilde{\xi}_2\eta_2 + \tilde{\xi}_3\eta_3) \\ &\quad + 2|\kappa|^4((\tilde{x}_1, y_1) + (\tilde{x}_2, y_2) + (\tilde{x}_3, y_3)) \\ &= \langle X, Y\rangle.\end{aligned}$$

This gives $\phi \in E_6$ by (2) of Lemma 1.5. □

PROPOSITION 4.3. *Let $\alpha \in E_6$ satisfy $\alpha E_1 = \xi E_1$ ($\xi \in C^*$, $|\xi| = 1$). Then α commutes with σ.*

Proof. This is the converse of Lemma 1.1. Since $\sigma = \phi(E_1, E_1; -1)$, we obtain $\alpha\phi(E_1, E_1; -1)\alpha^{-1} = \phi(\alpha E_1, \alpha E_1; -1) = \sigma$ and hence $\alpha\sigma = \sigma\alpha$.

PROPOSITION 4.4. *$\phi(A, B; \kappa)$ satisfies the following properties:*
(1) *ϕ leaves $[A]$ fixed.*
(2) *ϕ fixes the line $L(B)$ pointwise.*
(3) *The image of $[X]$ by ϕ lies on a line passing through $[A]$ and $[X]$.*

Proof. We know (1) by direct calculation. (2): Let $[C]$ be an arbitrary point in $L(B)$. The definition of $\phi(A, B; \kappa)$ implies $\langle A, B\rangle \neq 0$ and hence $[A] \not\in L(B)$. Especially $[A] \neq [C]$. Let l be a line passing through them. If a point $[X] \in l$ satisfies $B \triangle (A \triangle X) = 0$, one has $A \triangle X = 0$ because $0 = A \triangle (B \triangle (A \triangle X)) = \frac{1}{4}\langle A, B\rangle A \triangle X$ by Lemma 4.1. Thus $[A]$ and $[X]$ lie on a shortest closed geodesic from (1) of Proposition 3.1, and $[X]$ is in the singular position for $[A]$. In l, the set of points, in the singular position for $[A]$, becomes a connected submanifold with the dimension 14. On the other hand, since l is 16 dimensional as a submanifold in EIII, there exists $[Y] \in l$ such that $A \triangle Y \neq 0$. Hence $L(A \triangle Y) = l$ holds because the line, passing through $[A]$ and $[Y]$, is determined uniquely by (2) of Proposition 3.1. At the same time we have $B \triangle (A \triangle Y) \neq 0$. This means that the line $L(B \triangle (A \triangle Y))$ passes through $[B]$ and $[A \triangle Y]$. The duality of L asserts $\{[B \triangle (A \triangle Y)]\} = L(B) \cap L(A \triangle Y)$ in EIII. This implies $[B \triangle (A \triangle Y)] = [C]$ because $[C] \in L(B) \cap l$. Hence there exists $\mu \in C^*$ such that $C = \mu B \triangle (A \triangle Y)$. By a similar calculation to (2) of Proposition 4.1, we get,

$$\phi(A, B; \kappa)C = \phi(A, B; \kappa)(\mu B \triangle (A \triangle Y)) = \kappa^2 \mu B \triangle (A \triangle Y) = \kappa^2 C.$$

(3): If $A \triangle X \neq 0$, the line $L(A \triangle X)$, passing through $[A]$ and $[X]$, is determined uniquely. First we have $[A \triangle \phi X] = [A \triangle X]$ by the direct calculation of $A \triangle \phi X$. This asserts $L(A \triangle \phi X) = L(A \triangle X)$. If $A \triangle X = 0$, $[A]$ and $[X]$ lie on a shortest closed geodesic. Let l be an arbitrary line passing through them. Let N be the subset of l consisting of the points in the singular position for $[A]$. We know that N is a compact connected submanifold with the dimension 14. Since the dimension of l is 16 and $[X] \in N$, there exists a sequence $\{X_n\}$ in l such that $X = \lim X_n$ and $A \triangle X_n \neq 0$. Then $L(A \triangle X_n) = l$ holds for each n. Therefore we obtain $\phi X = \phi (\lim X_n) = \lim \phi(X_n) \in l$. □

PROPOSITION 4.5. *The group E_6^C preserves the incidence relation, i.e., if $[X] \in L(Y)$, $\alpha \in E_6^C$ satisfies $[\alpha X] \in L((\alpha^*)^{-1} Y)$.*

Proof. Since E_6^C acts on EIII, we show that $\alpha \in E_6^C$ satisfies $[\alpha X] \in L((\alpha^*)^{-1} Y)$ for $[X] \in L(Y)$. By the transitivity of E_6 in EIII, we may assume $Y = E_1$. Then $[X]$ has the type ③ (in Lemma 3.2) for $[E_1]$. This implies $\langle X, E_1 \rangle = 0$. Let $[Z]$ be an arbitrary element in EIII with the type ② for $[E_1]$. Then Z satisfies $\langle Z, X \rangle = 0$ and $Z \triangle E_1 = 0$. Hence we get $(\alpha^*)^{-1} Z \triangle (\alpha^*)^{-1} E_1 = \alpha(Z \triangle E_1) = 0$. Since Z is arbitrary, this means that the set of $[(\alpha^*)^{-1} Z]$ makes a polar set with the type ② for $[(\alpha^*)^{-1} E_1]$. Finally we can see that αX is an element with the type ③ for $(\alpha^*)^{-1} E_1$ because $\langle \alpha X, (\alpha^*)^{-1} E_1 \rangle = 0$ and $\langle \alpha X, (\alpha^*)^{-1} Z \rangle = 0$. Therefore $[\alpha X] \in L((\alpha^*)^{-1} E_1)$ holds. □

THEOREM 4.1. *EIII becomes a symmetric space in the sense of O. Loos* [5]: *If we define a product in EIII by $[A] \cdot [B] = [\phi(A, A; -1)B]$, the followings hold.*

(1) $[A] \cdot [A] = [A]$,

(2) $[A] \cdot ([A] \cdot [B]) = [B]$,

(3) $[A] \cdot ([B] \cdot [C]) = ([A] \cdot [B]) \cdot ([A] \cdot [C])$,

(4) $[A]$ *is an isolated fixed point in the set of points $[X]$ such that* $[A] \cdot [X] = [X]$.

Proof. Since $\sigma = \phi(E_1, E_1; -1)$ holds and σ is the geodesic symmetry at $[E_1]$ in EIII, we can see (4) in the case of $[A] = [E_1]$. The remainings of proof are easy. □

THEOREM 4.2. *The* map Φ: $[A] \to \phi(A, A; -1)$ gives an embedding of the symmetric space EIII *into the group E_6. Then Φ is a homomorphism for the reflection products in EIII and in E_6.*

Proof. If $\Phi(A) = \Phi(B)$, we obtain $[A] = [B]$ because $\phi(A, A; -1)$ has the isolated fixed point $[A]$. If we define a product $\alpha \cdot \beta = \alpha \beta^{-1} \alpha$ usually, then Φ satisfies $\Phi([A] \cdot [B]) = \Phi(A) \cdot \Phi(B)$. □

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