X.-L. CHAO
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THE SPECTRAL GEOMETRY OF HARMONIC MAPS INTO $HP^n(c)$

XIAOLI CHAO

§0. Introduction

The spectral geometry of the Laplace-Beltrami operator has developed greatly during the last twenty years. Recently, H. Urakawa use Gilkey's results about the asymptotic expansion of the trace of the heat kernel of a certain differential operator of a vector bundle to research the spectral geometry of harmonic maps into S^n and CP^n. In this paper, inspired by these, we firstly determine a spectral invariant of the Jacobi operator of harmonic maps into HP^n (corollary 3). Using this we obtain some geometric results distinguishing typical harmonic maps, i.e., isometric minimal immersions and Riemannian submersions with minimal fibres.

§1. The spectral invariants of the Jacobi operator

Let (M, g) be a m-dimentional compact Riemmanian manifold without boundary and (N, h) an n-dimentional Riemannian manifold. A smooth map $\phi:(M, g)\to(N, h)$ is said to be harmonic if it is a critical point of the energy $E(\phi)$ defined by

$$E(\phi) = \int_M e(\phi)vg \tag{1}$$

$$e(\phi) = \frac{1}{2}\sum_{i=1}^{m} h(\phi_* e_i, \phi_* e_i) \tag{2}$$

where ϕ_* is the differential of ϕ. Namely, for every vector field V along ϕ

$$\left.\frac{d}{dt}\right|_{t=0} E(\phi_t) = 0.$$

Here $\phi_t: M\to N$ is a one parameter family of smooth maps with $\phi_0=\phi$ and

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$$\left.\frac{d}{dt}\right|_{t=0} \phi_t(x) = V_x \in T_{\phi(x)}N$$

for every point $x \in M$.

The second variation formula of the energy $E(\phi)$ for a harmonic map ϕ is given by

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(\phi_t) = \int_M h(V, J_\phi V) v_g. \tag{3}$$

Here J_ϕ is a differential operator (called the *Jacobi operator*) acting on the space $\Gamma(E)$ of sections of the induced bundle $E = \phi^{-1}TN$. The operator J_ϕ is of the form

$$J_\phi V = \tilde{\nabla}^* \tilde{\nabla} V - \sum_{i=0}^{m} R_h(\phi_* e_i, V)\phi_* e_i, \; V \in \Gamma(E). \tag{4}$$

Here $\tilde{\nabla}$ is the connection of E which is defined by

$$\tilde{\nabla} V = \nabla^h_{\phi_* X} V$$

for $V \in \Gamma(E)$, $X \in TM$, and the Levi-Civita connection ∇^h of (N, h). R_h is the curvature tensor of (N, h) whose sign is the same as $R^{\tilde{\nabla}}$. Note that $\tilde{\nabla}$ is compatible with the metric h. Define the endomorphism L for our E by

$$L(V) = \sum_{\iota=0}^{m} R_h(\phi_* e_i, V)\phi_* e_i, \quad V \in \Gamma(E). \tag{5}$$

Then we have

$$\mathrm{Tr}(L) = \mathrm{Tr}_g(\phi^* \rho_h). \tag{6}$$

We denote also the spectrum of the Jacobi operator J_ϕ of the harmonic map ϕ by

$$\mathrm{Spec}(J_\phi) = \{\lambda_1 \leq \lambda_2 \leq \cdots \lambda_\iota \leq \cdots \uparrow \infty\}. \tag{7}$$

Then the trace$Z(t) = \exp(-t\lambda_i)$ of the heat kernel for the Jacobi operator J_ϕ has the asymptotic expansion

$$Z(t) \sim (4\pi t)^{-m/2} \{a_0 (J_\phi) + a_1(J_\phi)t + a_2(J_\phi)t^2 + \cdots\}.$$

Moreover we have

THEOREM 1 ([U]). *For a harmonic map $\phi : (M^n, g) \to (N^n, h)$,*

$$\begin{aligned} a_0(j_\phi) &= n \,\mathrm{Vol}(M) \\ a_1(j_\phi) &= \frac{n}{6}\int_M \tau_g v_g + \int_M \mathrm{Tr}_g(\phi^* \rho_h) v_g \\ a_2(j_\phi) &= \frac{n}{360}\int_M (5\tau_g^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2) v_g \end{aligned} \tag{8}$$

$$+\frac{1}{360}\int_M (-30\|\phi^* R_h\|^2 + 60\tau_g \mathrm{Tr}_g(\phi^*\rho_h) + 180\|L\|^2)v_g$$

where, for X, $Y \in T_xM$, $(\phi^* R_h)_{X,Y}$ is the endomorphism of $T_{\phi(x)}N$ given by $(\phi^* R_h)_{X,Y} = R_{h\phi_* X, \phi_* Y}$.

From now on, we assume that the target manifold is quaternionic space form $Q(c)$ with quaternionic sectional curvature c. The Riemannian curvature tensor R of $Q(c)$ is of the form

$$\text{(9)} \qquad R(X, Y)Z = -\frac{c}{4}\Big\{h(Y, Z)X - h(X, Z)Y + \sum_{t=1}^{3}[(Z, J_tY)J_tX - (Z, J_tX)J_tY + 2(X, J_tY)J_tZ]\Big\}$$

where $\{J_1, J_2, J_3\}$ is a canonical local basis of quaternionic Kähler structure of $Q(c)$. Then for a harmonic map ϕ: $(M^n, g) \to Q(c)$, we obtain

$$\text{(10)} \qquad \mathrm{Tr}(L) = 2(n+2)ce(\phi)$$

since $\rho_h = (n+2)ch$. Moreover let $\{e'_1, \ldots, e'_n, J_1e'_1, \ldots, J_1e'_n, J_2e'_1, \ldots, J_2e'_n, J_3e'_1, \ldots, J_3e'_n\}$ be a local orthonormal field on $Q^n(c)$. Then since

$$\|R^{\tilde\nabla}\|^2 = \sum_{i,j=1}^{m}\sum_{k=1}^{n}\{\|R_{h\phi_*e_i,\phi_*e_j}(e'_k)\|^2 + \|R_{h\phi_*e_i,\phi_*e_j}(J_1e'_k)\|^2 + \|R_{h\phi_*e_i,\phi_*e_j}(J_2e'_k)\|^2 + \|R_{h\phi_*e_i,\phi_*e_j}(J_3e'_k)\|^2\}$$

$$\mathrm{Tr}(L^2) = \sum_{i,j=1}^{m}\sum_{k=1}^{n}\Big\{h(R_{h\phi_*e_i,e'_k}(\phi_*e_i), R_{h\phi_*e_j,e'_k}(\phi_*e_j)) + \sum_{t=1}^{3} h(R_{h\phi_*e_i,J_te'_k}(\phi_*e_i), R_{h\phi_*e_j,J_te'_k}(\phi_*e_j))\Big\}$$

by a straightforward computation we obtain

$$\text{(11)} \qquad \begin{aligned} \|R^{\tilde\nabla}\|^2 &= \frac{c^2}{2}\Big(4e(\phi)^2 - \|\phi^* h\|^2 + (2n+1)\sum_t\|\phi^*\Phi_t\|^2\Big) \\ \mathrm{Tr}(L^2) &= \frac{c^2}{4}\Big(4(n+4)e(\phi)^2 + 7\|\phi^* h\|^2 + 3\sum_t\|\phi^*\Phi_t\|^2\Big) \end{aligned}$$

where $\Phi_t(X, Y) = h(X, J_tY)$, for vector field X, Y on $Q(c)$. Hence we have

THEOREM 2. *Let ϕ be a harmonic map of a compact Riemannian manifold (M, g) into a quaternionic space form $Q(c)$. Then the coefficients $a_0(J_\phi)$, $a_1(J_\phi)$ and $a_2(J_\phi)$ of the asymptotic expansion for the Jacobi operator J_ϕ are*

$$
\begin{aligned}
& a_0(J_\phi) = 4n\mathrm{vol}(M) \\
(12) \quad & a_1(J_\phi) = \frac{2n}{3}\int_M \tau_g v_g + 2(n+2)ce(\phi) \\
& a_2(J_\phi) = \frac{n}{90}\int_M (5\tau_g^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2)v_g \\
& \quad + \frac{1}{12}\int_M \left\{2(3n+11)c^2 e(\phi)^2 + 11\|\phi^* h\|^2 - (n-4)\sum_t \|\phi^*\Phi_t\|^2\right\}v_g \\
& \quad + \frac{1}{3}(n+2)c\int_M r_g e(\phi) v_g .
\end{aligned}
$$

COROLLARY 3. *Let ϕ, ϕ' be two harmonic maps of a compact Riemannian manifold (M, g) with constant scalar curvature into $Q(c)(c \neq 0)$. Assume that*

$$\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$$

Then we have

$$E(\phi) = E(\phi')$$

and

$$
\begin{aligned}
(13) \quad & \int_M \left\{2(3n+11)c^2 e(\phi)^2 + 11\|\phi^* h\|^2 - (n-4)\sum_t \|\phi^*\Phi_t\|^2\right\}v_g \\
& = \int_M \left\{2(3n+11)c^2 e(\phi)^2 + 11\|\phi'^* h\|^2 - (n-4)\sum_t \|\phi'^*\Phi_t\|^2\right\}v_g .
\end{aligned}
$$

For analogous results for the Jacobi operator associated with minimal submanifolds or Riemannian foliations see [D] [H] and [NTV].

§2. Isometric minimal immersions into $HP^n(c)$

Let M be a submanifold of $HP^n(c)$

(1) M is called *quaternionic* if $JT_pM \subset T_pM$ for all $J \in \mathcal{T}_p$, $p \in M$.

(2) M is called *totally real* if $JT_pM \perp T_pM$ for all $J \in \mathcal{T}_p$, $p \in M$.

(3) M is called *totally complex* if there exists a one-dimensional subspace V of $\mathcal{T}_p$ such that $JT_pM \subset T_pM$ for all $J \in V$ and $JT_pM \perp T_pM$ for all $J \in V^\perp \subset \mathcal{T}_p$, $p \in M$.

Where $\mathcal{T}$ is a quaternionic Kähler structure of $HP^n(c)$, i.e., a rank 3 vector subbundle of $\mathrm{End}(THP^n(c))$ with the following properties:

(1) For each $p \in Q(c)$ there exists an open neighborhood $U(p)$ of p and sections J_1, J_2, J_3 of $\mathcal{T}$ over $HP^n(c)$ such that for all $i \in \{1, 2, 3\}$:

(i) $J_i^2 = -id$, $\langle J_iX, Y\rangle = -\langle X, J_iY\rangle \ \forall X, Y \in TU(p)$

(ii) $J_iJ_{i+1} = J_{i+2} = -J_{i+1}J_i (i \bmod 3)$

(2) $\mathcal{T}$ is a parallel subbundle of $\mathrm{End}(THP^n(c))$.

THEOREM 4. *Let ϕ, ϕ' be isometric minimal immersion of a compact Riemmannian manifold (M, g) into quaternionic projective space $(HP^n(c), h)$.* Assume that $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$. *If ϕ is totally real (resp. quaternionic), then so is ϕ'.*

Proof. Since ϕ and ϕ' are isometric immersions, we have.

$$e(\phi) = e(\phi') = \dim(M)/2$$
$$\|\phi^*h\|^2 = \|\phi'^*h\|^2 = \dim(M).$$

Then, by Corollary 3, the condition $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$ yields

$$\int_M \sum_t \|\phi^*\Phi_t\|^2 \, v_g = \int_M \sum_t \|\phi'^*\Phi_t\|^2 v_g.$$

(i) If ϕ is totally real, i.e. $\|\phi^*\Phi_t\|^2 = 0$, $(t = 1, 2, 3)$, then we have

$$\|\phi'^*\Phi_t\|^2 = 0, \ \forall t.$$

On the other hand, from the definition of Φ_t, we get

$$\begin{aligned} 0 = \|\phi'^*\Phi_t\|^2 &= \sum_{i,j=1}^m h(\phi'_* e_j, J_t\phi'_* e_j)^2 \\ &= \sum_{j=1}^m h(PJ_t\phi'_* e_j, J_t\phi'_* e_j) \\ &= \sum_{j=1}^m h(PJ_t\phi'_* e_j, PJ_t\phi'_* e_j), \ \forall t \end{aligned}$$

$$\Leftrightarrow PJ_t\phi'_* e_j = 0, \ j = 1, \ldots, m, \ \forall t$$
$$\Leftrightarrow h(\phi'_* X, J_t\phi'_* Y), \text{ for all } X, Y \in TM, \ \forall t$$
$$\Leftrightarrow \phi' \text{ is totally real}$$

where $\{e_i, i = 1 \ldots, m\}$ is an orthonormal basis of T_xM, $x \in M$, $\dim(M) = m$. P is the orthogonal projection of $T_{\phi'(x)}N$ onto $\phi'_* T_xM$ with respect to h.

(ii) If ϕ is quaternionic, then

$$J_t\phi_* TM \subset \phi_* TM, \ \forall t.$$

Hence

$$\int_M \sum_t \|\phi'^*\Phi_t\|^2 v_g = \int_M \sum_t \|\phi^*\Phi_t\|^2 v_g = 3m \operatorname{vol}(M).$$

On the other hand, since

$$\begin{aligned} 0 \le \|\phi'^*\Phi_t\|^2 &= \sum_{i,j+1}^m h(PJ_t\phi'_* e_j, PJ_t\phi'_* e_j) \\ &\le \sum_{i,j+1}^m h(J_t\phi'_* e_j, J_t\phi'_* e_j) = m, \ \forall t \end{aligned}$$

we get, for each t

$$\begin{aligned} &\|\phi'^{*}\Phi_t\|^2 = m \\ \Leftrightarrow\ &PJ_t\phi'_{*}e_j = J_t\phi'_{*}e_j,\ j = 1, \ldots, m \\ \Leftrightarrow\ &J_t\phi'_{*}e_j \subset \phi'_{*}T_xM,\ \forall x \in M. \end{aligned}$$

Then ϕ' is also quaternionic.

3. Spectral characterization of harmonic Riemannian submersions

In this section, we study spectral characterization of harmonic Riemannian submersions among the set of all harmonic morphisms.

A smooth map $\phi\colon M \to N$ is a harmonic morphisms if for every harmonic function r on open subset U in N, $v \circ \phi$ is a harmonic function on $\phi^{-1}(U)$ provided that $\phi^{-1}(U) \neq \phi$.

LEMMA 5 ([F] or [I]) (i). *If* $\dim(M) < \dim(N)$, *every harmonic morphism is constant.*

(ii) *If* $\dim(M) > \dim(N)$, *a smooth map* $\phi\colon (M, g) \to (N, h)$ *is a harmonic morphism if and only if* ϕ *is horizontal weakly conformal and harmonic.*

Here a smooth $\phi\colon (M, g) \to (N, h)$ is horizontal weakly conformal if (i) the differential $\phi_{*x}\colon T_xM \to T_{\phi(x)}N$ is surjective at the point x with $e(\phi)(x) \neq 0$, and (ii) there exists a smooth function λ on M such that if $e(\phi)(x) \neq 0$, the pull back ϕ^*h satisfies

$$\phi^*h(X, Y) = \lambda^2(x)g(X, Y),\ X, Y \in H_x$$

where H_x is the orthogonal complement of the kernel of the differential ϕ_{*x} with respect to g_x, $x \in M$. It is known that the set $\{x \in M : e(\phi)(x) \neq 0\}$ is open and dense in M and the function λ^2 is given by

$$\lambda^2 = 2e(\phi)\dim(N)^{-1}$$

and $\|\phi^*h\|^2 = \dim(N)\lambda^4$. A smooth map $\phi\colon (M, g) \to (N, h)$ is a Riemannian submersion if it is horizontal weakly conformal with $\lambda = 1$, i.e., $e(\phi) = \dim(N)/2$, everywhere M.

Now we have

THEOREM 6. *Let* (M, g) *be a compact Riemannian manifold whose scalar curvature is constant.* ϕ, ϕ' *be harmonic morphisms of* (M, g) *into* (HP^n, h) *with* $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$. *If* ϕ *is Riemannian submersion, then so is* ϕ'.

Proof. At each point $x \in M$ with $e(\phi)(x) \neq 0$, we can define a linear transformation $\tilde{J}_t$ of H_x into itself such that $J_t \circ \phi_* = \phi_* \circ \tilde{J}_t$ and $\tilde{J}_t^2 = -I$, $t = 1, 2, 3$, where $\{J_1, J_2, J_3\}$ is a canonical basis of quaternionic Kähler structure of HP^n. Then

$$\begin{aligned} g(\tilde{J}_t X, \tilde{J}_t Y) &= g(X, Y) \\ g(\tilde{J}_t X, Y) &= 0, \ X, Y \in H_x, \ \forall t \\ \tilde{J}_t \circ \tilde{J}_{t+1} &= \tilde{J}_{t+1} \circ \tilde{J}_t = \tilde{J}_{t+2}, \ (t \mod 3). \end{aligned}$$

So we can choose $\{e_i, \tilde{J}_1 e_i, \tilde{J}_2 e_i, \tilde{J}_3 e_i, i=1, \ldots, n\}$ as an orthonormal basis of (H_x, g_x). Then we have

$$\begin{aligned} \|\phi^*\Phi_1\|^2 = \sum_{i,j} \{&\phi^*\Phi_1(e_i, e_j)^2 + 2\phi^*\Phi_1(e_i, \tilde{J}_1 e_j)^2 + \phi^*\Phi_1(\tilde{J}_1 e_i, \tilde{J}_1 e_j)^2 \\ &+ 2\phi^*\Phi_1(\tilde{e}_i, \tilde{J}_2 e_j)^2 + 2\phi^*\Phi_1(\tilde{J}_1 e_i, \tilde{J}_2 e_j)^2 + \phi^*\Phi_1(\tilde{J}_2 e_i, \tilde{J}_2 e_j)^2 \\ &+ 2\phi^*\Phi_1(\tilde{e}_i, \tilde{J}_3 e_j)^2 + 2\phi^*\Phi_1(\tilde{J}_1 e_i, \tilde{J}_3 e_j)^2 + 2\phi^*\Phi_1(\tilde{J}_2 e_i, \tilde{J}_2 e_j)^2 \\ &+ \phi^*\Phi_1(\tilde{J}_3 e_i, \tilde{J}_3 e_j)^2\} \\ = \sum_{i,j} \{&h(\phi_* e_i, \phi_* \tilde{J}_1 e_j)^2 + 2h(\phi_* e_i, \phi_* e_j)^2 + h(\phi_* \tilde{J}_1 e_i, \phi_* e_j)^2 \\ &+ 2h(\phi_* e_i, \phi_* \tilde{J}_3 e_j)^2 + 2h(\phi_* \tilde{J}_1 e_i, \phi_* \tilde{J}_3 e_j)^2 + h(\phi_* \tilde{J}_2 e_i, \phi_* \tilde{J}_3 e_j)^2 \\ &+ 2h(\phi_* e_i, \phi_* \tilde{J}_2 e_j)^2 + 2h(\phi_* \tilde{J}_1 e_i, \phi_* \tilde{J}_2 e_j)^2 + 2h(\phi_* \tilde{J}_2 e_i, \phi_* \tilde{J}_2 e_j)^2 \\ &+ h(\phi_* \tilde{J}_3 e_i, \phi_* \tilde{J}_2 e_j)^2\} \\ = \|&\phi^* h\|^2. \end{aligned}$$

Similarly, we have $\|\phi^*\Phi_2\|^2 = \|\phi^* h\|^2$, $\|\phi^*\Phi_3\|^2 = \|\phi^* h\|^2$. Since $\mathrm{Spec}(J_\phi) = \mathrm{Spec}(J_{\phi'})$ and ϕ is a Riemannian submersion, then, by Corollary 3, we have

$$E(\phi') = E(\phi)$$

$$\begin{aligned} &\int_M \left\{2(3n + 11)c^2 e(\phi')^2 + 11\|\phi'^* h\|^2 - (n - 4) \sum_t \|\phi'^*\Phi_t'\|^2\right\} v_g \\ &\quad = \int_M \left\{2(3n + 11)c^2 e(\phi)^2 + 11\|\phi^* h\|^2 - (n - 4) \sum_t \|\phi^*\Phi_t\|^2\right\} v_g \end{aligned}$$

$$\begin{aligned} e(\phi) &= 2n, \ \|\phi^* h\|^2 = 4n \\ e(\phi') &= 2n\lambda^2, \ \|\phi'^* \mathrm{h}\|^2 = 4n\lambda^4. \end{aligned}$$

From these, we get

$$\int_M \lambda^2 = \int_M v_g$$

$$\int_M \lambda^4 = \int_M v_g.$$

Therefore we get $\lambda = 1$ everywhere M by the Cauchy-Schwarz inequality.

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Department of Mathematics
Hangzhou University
Hangzhou, 310028, China

My current address:
Department of Mathematics and Mechanics
Southeast University
Nanjing, 210018, China