TOTALLY GEODESIC SUBMANIFOLDS OF RIEMANNIAN MANIFOLDS AND CURVATURE-INVARIANT SUBSPACES

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1. Introduction

An isometric immersion $\varphi : S \to M$ of a Riemannian manifold S into another Riemannian manifold M is called *totally geodesic* if the geodesics in S are carried into geodesics in M. We call such a pair (S, φ) a totally geodesic submanifold of M. Nevertheless, identifying S and the image $\varphi(S)$, we often call the subset $\varphi(S)$ in M a totally geodesic submanifold. Local problems are generally discussed in such a way. Among submanifolds of a Riemannian manifold, totally geodesic ones are fundamental.

Totally geodesic submanifolds of Riemannian symmetric spaces have been well investigated and it has been shown that they have beautiful and fruitful properties. In particular, due to the (M_+, M_-)-theory by B. Y. Chen and T. Nagano ([3]) this subject has made great progress. The author has a wish to understand well totally geodesic submanifolds of "general" Riemannian manifolds and obtained a few results in this paper.

We are concerned with three problems in this paper.

PROBLEM 1. *For a given subspace V in a tangent space T_pM, find good (or practical) criteria to conclude that there is a totally geodesic submanifold through p whose tangent space at p is V.*

For this problem, we recall a theorem of E. Cartan is section 2 (Theorem 2.1 in this paper), which becomes a remarkable criterion if M is a Riemannian symmetric space. That is, there is a totally geodesic submanifold whose tangent space is V if and only if V is a curvature-invariant subspace with respect to the Riemannian curvature tensor R, i.e.,

$$(1.1) \qquad R(x, y)z \in V \quad \text{for any } x, y, z \in V.$$

Is there such a criterion as above for a wider class of Riemannian manifolds? For example, for homogeneous Riemannian manifolds does there exist a finite number d such that a condition similar to (1.1) for $R, \nabla R, \cdots, \nabla^d R$ implies the existence of totally geodesic submanifolds? In this paper we will show this for

naturally reductive homogeneous spaces (Theorem 2.3). And in section 7, with the same sense as the above question, we make "experiments" classifying totally geodesic surfaces in 3-dimensional Lie groups with left invariant Riemannian metrics (Theorem 7.2 and its observations).

In section 3, to understand totally geodesic submanifolds in another point of view, we introduce a distribution $\mathfrak{D}$ on the Grassman bundle over a Riemannian manifold, which is an extension of the notion of geodesic spray on the unit sphere bundle. We show the relation between totally geodesic submanifolds and integral manifolds of $\mathfrak{D}$ (Theorem 3.1). Applying this relation, we consider the global existence problem of totally geodesic submanifolds (Theorem 3.5 and Theorem 3.6). These results are reformulations of a theorem of H. Reckziegel ([11]) and a theorem of R. Hermann ([7]).

It is a natural idea that "general" Riemannian manifolds will not have any r-dimensional totally geodesic submanifolds for $2 \leq r \leq n-1$. In fact, M. Spivak states in his book ([12], p. 39) as follows: "It seems rather clear that if one takes a Riemannian manifold $(N, \langle , \rangle)$ "at random", then it will not have any totally geodesic submanifolds of dimension >1. But I must admit that I don't know of any specific example of such a manifold". This yields our second problem.

PROBLEM 2. *Show that "general" Riemannian manifolds do not have any r-dimensional totally geodesic submanifolds for $2 \leq r \leq n-1$. Give specific examples of Riemannian manifolds which do not have any totally geodesic submanifolds.*

Our results related to this problem are Theorem 5.4, examples after Proposition 6.6, and an observation of Theorem 7.2.

The following algebraic problem is motivated by Theorem 2.1 and it will play an important role in investigating totally geodesic submanifolds of Riemannian manifolds.

PROBLEM 3. *Given a curvature tensor R on $\mathbf{R}^n$, classify curvature-invariant subspaces with respect to R.*

In sections 4 and 5, we study curvature-invariant subspaces. Let $G_r(\mathbf{R}^n)$ be the Grassmann manifold of r-dimensional subspaces in $\mathbf{R}^n$. How are r-dimensional curvature-invariant subspaces characterized in $G_r(\mathbf{R}^n)$? Some answers to this question are given as Theorem 4.1, Theorem 4.4, and Corollary 4.5. In the case of hyperplanes, such characterizations are stated as Proposition 4.6, Proposition 4.7, and Theorem 4.8. Theorem 4.8 is a variation of so-called axioms of planes. That is, the existence of many curvature-invariant subspaces implies that its curvature tensor is of constant sectional curvature. As an application of Theorem 4.8, a slight generation of Theorem 1 in B. Y. Chen ([2]) is obtained as Corollary 4.9. In section 5, we will classify curvature-invariant subspaces with respect to curvature tensors of special form (Propositions 5.1, 5.2 and 5.3). Applying these results, we construct examples of curvature tensors which don't

have any r-dimensional curvature-invariant subspaces of $3 \leq r \leq n-1$ (Theorem 5.4). Comparing with Corollary 4.2, we see that it is different from 2-dimensional case

In sections 6 and 7, we will classify totally geodesic submanifolds of specific and simple Riemannian manifolds. It also has the meaning of testing adequacy of our problems and our results. In section 6, we classify totally geodesic submanifolds of hypersurfaces in R^{n+1} defined by the equations: $a_1(x^1)^2 + a_2(x^2)^2 + \cdots + a_{n+1}(x^{n+1})^2 = 1$ $(a_i \neq 0 \quad i=1, 2, \cdots, n+1)$ or $(x^1)^3 + (x^2)^3 + \cdots + (x^{n+1})^3 = 1$ (Theorem 6.5 and Theorem 6.7). In section 7, Totally geodesic surfaces of 3-dimensional Lie groups with left invariant Riemannian metrics are classified (Theorem 7.2).

2. Preliminaries—A theorem of E. Cartan on the existence of totally geodesic submanifolds of Riemannian manifolds

In this section, we shall recall a theorem of E. Cartan on the existence of totally geodesic submanifolds of Riemannian manifolds and state related results.

Let M be an $n(\geq 3)$-dimensional Riemannian manifold. We introduce some notation. For a vector u in the tangent space T_pM at $p \in M$, we denote by γ_u the geodesic through p whose tangent vector at p is u and denote by $R_u(t)$ the $(1, 3)$-tensor on T_pM obtained by the parallel translation of the curvature tensor at $\gamma_u(t)$ along the geodesic γ_u. We define a $(1, 2)$-tensor $r_u(t)$ on T_pM by $r_u(t)(x, y) = R_u(t)(u, x)y$, $x, y \in T_pM$. Now we can give an answer to Problem 1 in section 1. The following theorem is fundamental in our study, which is due to E. Cartan (cf. R. Hermann ([7])).

THEOREM 2.1. *Let V be a subspace of T_pM. Then the following three conditions are equivalent.*

(1) *There is a totally geodesic submanifold of M through p whose tangent space at p is V.*

(2) *There is a positive number ε such that for any unit vector $u \in V$ and any $t \in (-\varepsilon, \varepsilon)$, the following holds:*

$$(2.1) \qquad R_u(t)(x, y)z \in V \quad for \ any \ x, y, z \in V.$$

(3) *There is a positive number ε such that for any unit vector $u \in V$ and any $t \in (-\varepsilon, \varepsilon)$, the following holds:*

$$(2.2) \qquad r_u(t)(x, y) \in V \quad for \ any \ x, y \in V.$$

Remark. The third condition in Theorem 2.1 is evidently weaker than the second one. The formulation of the third condition is due to K. Tojo ([13]).

The following is immediately seen.

COROLLARY 2.2. *Let M be a real analytic Riemannian manifold and V be a subspace of T_pM. Suppose that for $0 \leq i < \infty$, the following holds:*

$$(2.3) \qquad (\nabla^i R)(u, \cdots, u; u, x, y) \in V \quad for \ u, x, y \in V,$$

where $\nabla^i R$ denotes the i-th covariant derivative of R. Then there is a totally geodesic submanifold of M through p whose tangent space at p is V.

In view of Corollary 2.2, we have the following question: Does there exist a finite number d such that the condition (2.3) for $0 \leq i \leq d$ implies the existence of a totally geodesic submanifold? We shall show that there exists such number for naturally reductive homogeneous spaces. Let $M = K/H$ be a naturally reductive homogeneous space with an $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$ and with the $\mathrm{Ad}(H)$-invariant inner product $\langle , \rangle$ on $\mathfrak{m}$ such that

$$(2.4) \qquad \langle [x, y]_{\mathfrak{m}}, z \rangle + \langle y, [x, z]_{\mathfrak{m}} \rangle = 0 \quad for \ x, y, z \in \mathfrak{m}$$

(cf. S. Kobayashi and K. Nomizu [8] Chapter X). As usual, we identify $\mathfrak{m}$ with the tangent space T_oM at the origin o (=the coset H) of M. We denote by $\tau(h)$ the diffeomorphism of M induced by $h \in K$. We put $\varLambda : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by $\varLambda(x)y = 1/2[x, y]_{\mathfrak{m}}$ for $x, y \in \mathfrak{m}$. Then (2.4) means that $\varLambda(x)$ is a skew-symmetric linear endomorphism of $(\mathfrak{m}, \langle , \rangle)$. Therefore $e^{\varLambda(x)}$ is a linear isometry of $(\mathfrak{m}, \langle , \rangle)$. Since the Riemannian connection is a natural torsion-free connection on K/H, the following properties hold ([8], Ch. X):

(i) For each $x \in \mathfrak{m}$, the curve $\gamma_x(t) = \tau(\exp tx)(o)$ is a geodesic with $\gamma_x(0) = o$, $\gamma_x'(0) = x$.

(ii) The parallel translation along γ_x is given as follows;

$$\tau(\exp tx)_* e^{-t\varLambda(x)} : T_oM(=\mathfrak{m}) \to T_{\gamma_x(t)}M.$$

(iii) The $(1, 3)$-tensor $R_x(t)$ on $\mathfrak{m}$ obtained by the parallel translation of the curvature tensor along γ_x is given as follows:

$$R_x(t) = e^{t\varLambda(x)} \cdot R_o.$$

In the above, R_o denotes the curvature tensor at the origin o and $e^{t\varLambda(x)} \cdot$ denotes the action of $e^{t\varLambda(x)}$ on the space $\mathfrak{R}(\mathfrak{m})$ of curvature tensors on $\mathfrak{m}$. The formulas (ii) and (iii) are explicitly proved in K. Tojo ([13]). For each $u \in \mathfrak{m}$, we denote by $\mathfrak{R}_u$ the smallest subspace of $\mathfrak{R}(\mathfrak{m})$ which satisfies $R_o \in \mathfrak{R}_u$ and $\varLambda(u) \cdot \mathfrak{R}_u \subset \mathfrak{R}_u$. We define $d(u)$ by $d(u) = \dim \mathfrak{R}_u$. Trivially we have $d(u) \leq \dim \mathfrak{R}(\mathfrak{m})$.

THEOREM 2.3. *Let $M = K/H$ be a naturally reductive homogeneous space and V be a subspace of $\mathfrak{m}$ ($= T_oM$). Suppose that for each unit vector $u \in V$, the following holds:*

$$(2.5) \qquad (\nabla^i R)(u, \cdots, u; u, x, y) \in V \quad for \ 0 \leq i \leq d(u), \ and \ x, y \in V.$$

Then there is a totally geodesic submanifold of M through o whose tangent space at o is V.

Proof. At first, we recall the fundamental fact on the curve in the m-dimensional Euclidean space $\boldsymbol{R}^m$. Let $c : I \to \boldsymbol{R}^m$ be a curve defined on an open interval I of $\boldsymbol{R}$ into $\boldsymbol{R}^m$. We say that c has *constant osculating rank* r if for all $t \in I$, its higher order derivatives $c'(t), \cdots, c^{(r)}(t)$ are linearly independent and $c'(t), \cdots, c^{(r+1)}(t)$ are linearly dependent in $\boldsymbol{R}^m$. It is a fundamental fact that if c has constant osculating rank r, there exist smooth functions $a_1, \cdots, a_r : I \to \boldsymbol{R}$ such that

$$c(t) = c(0) + a_1(t)c'(0) + \cdots + a_r(t)c^{(r)}(0) \quad \text{for all } t \in I.$$

We return to a naturally reductive homogeneous space M. For a unit vector $u \in V$, $R_u(t) = e^{tA(u)} \cdot R_o$ is a curve in $\Re(\mathfrak{m})$. Since $e^{tA(u)}$ is a 1-parameter subgroup of the group of linear isometries of $\Re(\mathfrak{m})$, the curve $R_u(t)$ has constant osculating rank r. Therefore we have

$$R_u(t) = R_u(0) + a_1(t)R_u'(0) + \cdots + a_r(t)R_u^{(r)}(0).$$

We note that $R_u^{(i)}(0)(x, y)z = (\nabla^i R)(u, \cdots, u ; x, y, z)$. Hence from the condition $(\nabla^i R)(u, \cdots, u ; u, x, y) \in V$ for $0 \leq i \leq r$ and $x, y \in V$, it follows that $r_u(t)(x, y) \in V$ for $x, y \in V$ and all t. On the other hand, since $R_u^{(i)}(0) = A(u)^i \cdot R_o$, $\Re_u$ coincides with the subspace of $\Re(\mathfrak{m})$ spanned by $R_o, A(u) \cdot R_o, \cdots, A^{(r)}(u) \cdot R_o$. In particular, we have $r = d(u) - 1$ or $d(u)$. $\qquad\square$

3. An extension of geodesic spray and the existence of totally geodesic submanifolds

Let M be an n-dimensional Riemannian manifold and P be the bundle of orthonormal frames over M with structure group $O(n)$. We fix an integer r, $1 \leq r \leq n-1$ and denote by $G_r(\boldsymbol{R}^n) = O(n)/O(r) \times O(n-r)$ the Grassmann manifold of r-dimensional subspaces of $\boldsymbol{R}^n$. Let $G_r(M)$ be the set of all r-dimensional subspaces of the tangent spaces at all points of M, i.e.,

$$G_r(M) = \{(p ; V) \mid p \in M, V \text{ is an } r\text{-dimensional subspace of } T_p M\}.$$

Then it is the associated fibre bundle with P with standard fibre $G_r(\boldsymbol{R}^n)$ which is expressed by $P \times_{O(n)} G_r(\boldsymbol{R}^n)$ or by $P/O(r) \times O(n-r)$. The bundle $G_r(M)$ is called the Grassmann bundle over M and sometimes simply denote by E in this section. We use the following symbols for associated projections:

$$\pi : P \longrightarrow M,$$

$$\pi_E : E \longrightarrow M,$$

$$\mu : P \longrightarrow E.$$

Here the projection μ is explicitly described as $\mu(u)=(\pi(u)\,;\,\{u(e_1),\,\cdots,\,u(e_r)\}_R)$ for $u\in P$, where $\{e_1,\,\cdots,\,e_n\}$ is the natural basis of R^n and $\{u(e_1),\,\cdots,\,u(e_r)\}_R$ denotes the subspace of $T_{\pi(u)}M$ spanned by orthonormal vectors $u(e_1),\,\cdots,\,u(e_r)$.

Let ω and θ be the Riemannian connection form and the canonical form on P, respectively; ω is an $\mathfrak{o}(n)$-valued 1-form and θ is an R^n-valued 1-form. With respect to the natural basis $e_1,\,\cdots,\,e_n$ of R^n, we write $\theta=\sum_{i=1}^n\theta^i e_i$. We define a distribution $\tilde{\mathfrak{D}}$ of dimension r on P as follows: For each $u\in P$, we set

$$\tilde{\mathfrak{D}}_u=\{X\in T_uP\mid \omega(X)=0,\ \theta^{r+1}(X)=\cdots=\theta^n(X)=0\}.$$

$\tilde{\mathfrak{D}}_u$ is characterized as the subspace of T_uP which satisfies $\tilde{\mathfrak{D}}_u\subset\mathscr{H}_u$ and $\pi_*(\tilde{\mathfrak{D}}_u)=\{u(e_1),\,\cdots,\,u(e_r)\}_R$, where $\mathscr{H}_u$ denotes the horizontal subspace of T_uP with respect to the Riemannian connection ω. We denote by R_a the right action on P by $a\in O(n)$. We remark that for $a\in O(r)\times O(n-r)$, $R_{a*}\tilde{\mathfrak{D}}_u=\tilde{\mathfrak{D}}_{ua}$ and hence $\mu_{*u}\tilde{\mathfrak{D}}_u=\mu_{*ua}\tilde{\mathfrak{D}}_{ua}$. For each $\sigma=(p\,;\,V)\in E=G_r(M)$, we choose $u\in P$ such that $\mu(u)=\sigma$ and set $\mathfrak{D}_\sigma=\mu_{*u}\tilde{\mathfrak{D}}_u$. By the preceding remark it follows that $\mathfrak{D}_\sigma$ is independent of the choice of $u\in P$ such that $\mu(u)=\sigma$ and hence it is well-defined. Moreover we have $\pi_{E*}\mathfrak{D}_\sigma=\pi_*\tilde{\mathfrak{D}}_u=\{u(e_1),\,\cdots,\,u(e_r)\}_R=V$. Thus a distribution $\mathfrak{D}$ of dimension r on the Grassmann bundle $G_r(M)$ has been defined. We consider the case of $r=1$. Let UM be the unit tangent bundle over M and G be the geodesic spray defined on UM. We denote by ρ the bundle homomorphism of UM onto $G_1(M)$ which is naturally defined. Then we have $\mathfrak{D}_{\rho(x)}=\{\rho_*G_x\}_R$ for all $x\in UM$. So we can view the distribution $\mathfrak{D}$ on $G_r(M)$ as an extended notion of geodesic spray.

We shall show that integral manifolds of the distribution $\mathfrak{D}$ on $G_r(M)$ are in one to one correspondence with r-dimensional totally geodesic submanifolds of M. It is an analogous property to that of geodesic spray.

THEOREM 3.1. *Let $f:S\to M$ be a totally geodesic imbedding of an r-dimensional connected Riemannian manifold S into M and g be the mapping of S into the Grassmann bundle $G_r(M)$ defined by $g(p)=(f(p)\,;\,f_*T_pS)$ at $p\in S$. Then the image $g(S)$ is an integral manifold of the distribution $\mathfrak{D}$.*
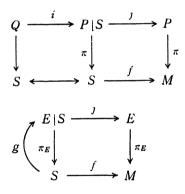
Conversely let S be an integral manifold of $\mathfrak{D}$ on $G_r(M)$. The restriction of the projection $\pi_E:G_r(M)\to M$ to S is a totally geodesic immersion of S into M.

Proof. We will review the theory of submanifolds for the proof of Theorem 3.1. For this we follow [8] Chapter VII and describe submanifolds using the notion of frame bundles.

Let $f:S\to M$ be an imbedding of an r-dimensional connected manifold S into M. We identify S with the image $f(S)$ by f if there is no danger of confusion; for any point p of S we shall denote $f(p)\in M$ by the same letter p. Then the tangent space T_pS is a subspace of T_pM. Let $P|S$ and $E|S$ be the principal fibre bundle and its associated fibre bundle over S induced by f from the orthonormal frame bundle over P and the Grassmann bundle $E=G_r(M)$ over M, respectively. $P|S$ and $E|S$ are given by

$$P \,|\, S = \{ u \in P \mid \pi(u) \in S \}\,,$$

$$E \,|\, S = \{ \sigma \in E \mid \pi_E(\sigma) \in S \}\,.$$

We denote by the same letter j the injective homomorphism of $P\,|\,S$ into P and that of $E\,|\,S$ into E. Associated with the imbedding f, the mapping $g : S \to E = G_r(M)$ is defined by $g(p) = (f(p) ; f_* T_p S)$ $(=(p ; T_p S)$ under the preceding identification). Clearly we have $\pi_E \circ g = f$ and we can view g as a cross section of S into $E\,|\,S$. A frame $u \in P\,|\,S$ at $p \in S$ is said to be *adapted* if $\{u(e_1), \cdots, u(e_r)\}$ is a basis of $T_p S$ and hence $\{u(e_{r+1}), \cdots, u(e_n)\}$ is a basis of the normal space $T_p^\perp S$. The set Q of adapted frames forms a principal fibre bundle over S with structure group $O(r) \times O(n-r)$. Q is the subbundle of $P\,|\,S$. It corresponds to the cross section g of $E\,|\,S$ ([8] Chapter I Proposition 5.6). That is, the following holds: $Q = \{u \in P\,|\,S \mid \mu(u) = g(\pi(u))\}$, where π and μ denote the projections of $P\,|\,S \to S$ and that of $P\,|\,S \to E\,|\,S$, respectively. We denote by i the injective homomorphism of Q into $P\,|\,S$. Then we have the following diagrams:

$$
\begin{array}{ccccc}
Q & \xrightarrow{\ i\ } & P\,|\,S & \xrightarrow{\ j\ } & P \\
\Big\downarrow & & {\scriptstyle \pi}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\
S & \longleftrightarrow & S & \xrightarrow{\ f\ } & M
\end{array}
$$

$$
g\left(
\begin{array}{ccc}
E\,|\,S & \xrightarrow{\ j\ } & E \\
{\scriptstyle \pi_E}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_E} \\
S & \xrightarrow{\ f\ } & M
\end{array}
\right.
$$

Let $\omega' = j^* \omega$ be the connection on $P\,|\,S$ which is the restriction of the Riemannian connection ω on P. We note that the connection ω' on $P\,|\,S$ is reducible to a connection on the subbundle Q if and only if S is a totally geodesic submanifold of M (cf. [8] Chapter VII Proposition 3.5 or Proposition 8.2). Let $\mathcal{H}'$ be the horizontal distribution on $P\,|\,S$ defined by ω'. A distribution $\mathfrak{D}'$ of dimension r on $E\,|\,S$ is defined as follows. For each $\sigma \in E\,|\,S$ we choose $u \in P\,|\,S$ such that $\mu(u) = \sigma$ and set $\mathfrak{D}'_\sigma = \mu_* \mathcal{H}'_u$. Now we recall that Q is the subbundle of $P\,|\,S$ corresponding to the cross section $g : S \to E\,|\,S$. It is known that the connection ω' on $P\,|\,S$ is reducible to a connection on Q if and only if at any point $p \in S$, $g_* T_p S = \mathfrak{D}'_{g(p)}$ holds ([8] Chapter II Proposition 7.4). We see that $j_* \mathcal{H}'_u = \widetilde{\mathfrak{D}}_u$ for each $u \in Q$. In fact, we have $\omega(j_* \mathcal{H}'_u) = \omega'(\mathcal{H}'_u) = 0$ and $\pi_* j_* \mathcal{H}'_u = f_* \pi_* \mathcal{H}'_u = f_* T_{\pi(u)} S = \{u(e_1), \cdots, u(e_r)\}_R$. Also we see that $j_* \mathfrak{D}'_{g(p)} = \mathfrak{D}_{g(p)}$ for each $p \in S$. In fact, choosing $u \in Q \cap \pi^{-1}(p)$, we have $\mu(u) = g(p)$ and $j_* \mathfrak{D}'_{g(p)} = j_* \mu_* \mathcal{H}'_u = \mu_* j_* \mathcal{H}'_u = \mu_* \widetilde{\mathfrak{D}}_u = \mathfrak{D}_{g(p)}$.

Now we shall prove Theorem 3.1. Let $f : S \to M$ be a totally geodesic imbedding. Then the connection ω' on $P|S$ is reducible to a connection on the subbundle Q. From this it follows that for each point $p \in S$, $g_* T_p S = \mathfrak{D}'_{g(p)}$ holds. Since $j_* \mathfrak{D}'_{g(p)} = \mathfrak{D}_{g(p)}$, the image $g(S)$ is an integral manifold of the distribution $\mathfrak{D}$.

Conversely let S be an integral manifold of the distribution $\mathfrak{D}$. We denote by f the restriction of π_E to S. Since for each $\sigma \in G_r(M)$, π_{E*} restricted to the subspace $\mathfrak{D}_\sigma$ is injective, f is an immersion. For each $\sigma = (p \, ; V) \in S \subset G_r(M)$, we have $f(\sigma) = \pi_E(\sigma) = p$ and $f_*(T_p S) = \pi_{E*}(\mathfrak{D}_\sigma) = V$. Therefore the mapping $g : S \to G_r(M)$ associated with f is exactly an inclusion mapping. We proceed backward and see that f is totally geodesic.   □

Now we shall study the global existence problem of totally geodesic submanifolds. Our results stated here have been already obtained by H. Reckziegel ([11]) and R. Hermann ([7]). We will state them by another way, applying the preceding theorem.

We need to consider integral manifolds of a distribution which is not necessarily involutive. Even in this case, we can show the existence of maximal integral manifolds, exactly following the proof in the case of an involutive distribution (performed in C. Chevalley [4] Chapter III § VIII). Let N be a manifold and $\mathfrak{D}$ be a distribution of dimension r on N. We don't assume that $\mathfrak{D}$ is involutive. We state the following preparatory lemma which is essentially a consequence of 2.1 Affinity lemma in P. Dombrowski ([5]).

LEMMA 3.2. *Let S be an integral manifold of $\mathfrak{D}$. For each point $p \in S$, there exists a neighborhood U of p in N which satisfies the following property. If I is an open interval of R containing 0 and $\gamma : I \to N$ is a C^∞-curve in N such that $\gamma(0) = p$, $\gamma(I) \subset U$, and $\gamma'(t) \in \mathfrak{D}_{\gamma(t)}$ for any $t \in I$, then $\gamma(I)$ is contained in S and γ is a C^∞-mapping of I into S.*

By this lemma, we obtain the following proposition, which corresponds to Chapter III § VII Proposition 1 in [4].

PROPOSITION 3.3. *Let S and S' be two integral manifolds of $\mathfrak{D}$ containing a point p in common. Then there exists an integral manifold containing p such that it is an open submanifold of both S and S'.*

After preparing this proposition, we can proceed by the same way as the case of an involutive distribution. Let $\mathscr{I}$ be the subset of N defined by

$$\mathscr{I} = \{p \in N \mid \text{there exists an integral manifold of } \mathfrak{D} \text{ containing } p\}.$$

We shall define a topology on $\mathscr{I}$. Let $\mathscr{O}$ be the family of those subsets of $\mathscr{I}$ which may be represented as unions of collections of integral manifolds of $\mathfrak{D}$. $\mathscr{O}$ satisfies the axioms of the family of open sets and hence provides a topology

on $\mathcal{I}$. Since for an arbitrary open subset U in N we have $U \cap \mathcal{I} \in \mathcal{O}$, $\mathcal{I}$ equipped with this topology is a Hausdorff space. Let S be a connected component of $\mathcal{I}$. Then S can canonically be equipped with the structure of an r-dimensional manifold and further it is an integral manifold of $\mathfrak{D}$. Thus we have the following.

THEOREM 3.4. *Let $\mathfrak{D}$ be a distribution of dimension r on N which is not necessarily involutive and $\mathcal{I}$ be the set of points of N which are contained in integral manifolds of $\mathfrak{D}$. Then for each $p \in \mathcal{I}$, there exists a maximal integral manifold containing p, i.e., an integral manifold which is not a subset of any larger integral manifold. Any integral manifold containing p is an open submanifold of this maximal integral manifold.*

Applying Theorem 3.4, we shall show the existence of maximal totally geodesic submanifolds. H. Reckziegel in [11] proved the existence of maximally expanded isometric immersions with parallel second fundamental form. Our assertion is evidently a consequence of his result. In this paper we construct maximal totally geodesic submanifolds by another way. Let $\mathfrak{D}$ be the distribution of dimension r on the Grassmann bundle $G_r(M)$ which is defined in Theorem 3.1. We denote by $\mathcal{I}$ the set of points of $G_r(M)$ which are contained in integral manifolds of $\mathfrak{D}$. By Theorem 3.1, $\mathcal{I}$ coincides with the set of elements $\sigma = (p \, ; V)$ in $G_r(M)$ which satisfy one of the equivalent three conditions in Theorem 2.1. Then we have the following.

THEOREM 3.5. *For each $\sigma = (p \, ; V) \in \mathcal{I}$, there exists a maximal totally geodesic immersion $\phi : S \to M$ of an r-dimensional connected Riemannian manifold S into M in the following sense: Let $f : N \to M$ be any totally geodesic immersion of an r-dimensional connected Riemannian manifold N into M through p whose tangent space at p is V. Then there exists an isometric immersion $g : N \to S$ such that $\phi \circ g = f$.*

Proof. By Theorem 3.4, we take a maximal integral manifold S of $\mathfrak{D}$ containing σ. Let ϕ be the restriction to S of the projection $\pi_E : G_r(M) \to M$. Then by Theorem 3.1 ϕ is a totally geodesic immersion of S into M. Let $f : N \to M$ be a totally geodesic immersion which satisfies the assumption of our theorem. We set $g : N \to G_r(M)$ as $g(q) = (f(q) \, ; f_* T_q N)$ for $q \in N$. For each $q \in N$, we take a connected neighborhood U of q such that $f|_U : U \to M$ is imbedding. Then by Theorem 3.1 $g(U)$ is an integral manifold of $\mathfrak{D}$. Hence by Theorem 3.4, $g(U)$ is an open submanifold of some maximal integral manifold. From the connectedness of N, it follows that $g(N)$ is contained in S and that $g : N \to S$ is an immersion. Clearly we have $\phi \circ g = f$. This implies that g is an isometric immersion of N into S. $\qquad \square$

From now on we assume that M is a complete Riemannian manifold. We shall consider a necessary and sufficient condition that given a point $p \in M$ and

an r-dimensional subspace V of T_pM, there exists a complete totally geodesic submanifold of M through p whose tangent space at p is V. Our statement is an another formulation of R. Hermann's result in [7]. We will prepare the notation. Let $\sigma=(p;V)\in G_r(M)$ be given. For $v\in V$, we denote by $\gamma_v: \boldsymbol{R}\to M$ the geodesic of M such that $\gamma_v(0)=p$ and $\gamma_v'(0)=v$ and denote by $V(t)$ the subspace of $T_{\gamma_v(t)}M$ obtained by parallel translating V along γ_v to the point $\gamma_v(t)$. Evidently we have $\gamma_v'(t)\in V(t)$. We put $\sigma_v(t)=(\gamma_v(t);V(t))$. Then $\sigma_v: \boldsymbol{R}\to G_r(M)$ is a horizontal curve of $G_r(M)$ with respect to the Riemannian connection with $\sigma_v(0)=\sigma$. Moreover since $\pi_{E*}\sigma_v'(t)=\gamma_v'(t)\in V(t)$, we have $\sigma_v'(t)\in \mathfrak{D}_{\sigma_v(t)}$. Using the notation, we formulate the following.

THEOREM 3.6. *Let M be a complete Riemannian manifold. For $\sigma=(p;V)$ $\in G_r(M)$, the following two conditions are equivalent*:

(1) *There is a complete totally geodesic submanifold of M through p whose tangent space at p is V.*

(2) *For all $v\in V$ and $t\in\boldsymbol{R}$, $\sigma_v(t)\in\mathcal{J}$, where $\mathcal{J}$ denotes the subset of $G_r(M)$ defined in Theorem 3.5.*

Proof. The implication $(1)\to(2)$. Let S be a complete Riemannian manifold and $f: S\to M$ be a totally geodesic immersion with a point $o\in S$ which satisfies $f(o)=p$ and $f_*T_oS=V$. For $v\in V$, we take $u\in T_oS$ which satisfies $f_*u=v$. Let $\gamma_u: \boldsymbol{R}\to S$ be the geodesic of S through o whose tangent vector at o is u. Then we have $f(\gamma_u(t))=\gamma_v(t)$ and $f_*T_{\gamma_u(t)}S=V(t)$. Here $V(t)$ denotes the subspace of $T_{\gamma_v(t)}M$ obtained by parallel translating V along γ_v to the point $\gamma_v(t)$. Hence we see that $g(\gamma_u(t))=(f(\gamma_u(t));f_*T_{\gamma_u(t)}S)=(\gamma_v(t);V(t))=\sigma_v(t)$. For each $t\in\boldsymbol{R}$, we take a connected neighborhood U of $\gamma_u(t)$ in S such that $f|_U: U\to M$ is imbedding. Then by Theorem 3.1, $g(U)$ is an integral manifold of $\mathfrak{D}$ which contains $g(\gamma_u(t))=\sigma_v(t)$. From this it follows that $\sigma_v(t)\in\mathcal{J}$.

The implication $(2)\to(1)$. By the assumption, we have $\sigma\in\mathcal{J}$. We take a maximal integral manifold S of $\mathfrak{D}$ containing σ. Then $\phi=\pi_E|_S$ is a totally geodesic immersion of S into M. We shall prove that $\sigma_v(t)\in S$ for any $v\in V$ and $t\in\boldsymbol{R}$. By Lemma 3.2 we see that for each $t_0\in\boldsymbol{R}$ there exists a positive number ε such that $\sigma_v(t)$ for $|t-t_0|<\varepsilon$ are contained in a maximal integral manifold containing $\sigma_v(t_0)$. From the connectedness of $\boldsymbol{R}$, it follows that $\sigma_v(t)\in S$ for all $t\in\boldsymbol{R}$. Since $\phi(\sigma_v(t))=\gamma_v(t)$, σ_v is a geodesic of S. Noticing that $\phi_*(\sigma_v'(0))$ $=\gamma_v'(0)=v$ and that $\phi_{*\sigma}$ is a linear isomorphism of $T_\sigma S$ onto V, we see that the exponential mapping $\exp_\sigma$ at $\sigma\in S$ is defined on all of $T_\sigma S$. Hence S is a complete Riemannian manifold.   □

We define subsets $\mathcal{J}^{(k)}$ and $\mathcal{J}_k$ of $G_r(M)$ for $k=0, 1, \cdots$ as follows:

$$\mathcal{J}^{(k)}=\{\sigma=(p;V)\in G_r(M) \mid (\nabla^k R)(u_1, \cdots u_k; x, y, z)\in V$$

$$\text{for any } u_1, \cdots, u_k, x, y, z\in V\}$$

$$\mathcal{I}_k = \mathcal{I}^{(0)} \cap \mathcal{I}^{(1)} \cap \cdots \cap \mathcal{I}^{(k)}.$$

Then $\mathcal{I}^{(k)}$ and $\mathcal{I}_k$ are closed subsets of $G_r(M)$. Evidently we see that

$$\mathcal{I}_0 \supset \mathcal{I}_1 \supset \cdots \supset \mathcal{I}_k \supset \mathcal{I}_{k+1} \supset \cdots$$

and that $\bigcap_{k=0}^{\infty} \mathcal{I}_k \supset \mathcal{I}$. Corollary 2.2 implies that $\bigcap_{k=0}^{\infty} \mathcal{I}_k = \mathcal{I}$ for a real analytic Riemannian manifold M. In particular $\mathcal{I}$ is a closed subset of $G_r(M)$. By Theorem 2.3 we see that for a naturally reductive homogeneous space, there exists a finite integer d such that $\mathcal{I}_d = \mathcal{I}$.

Applying Theorem 3.6, we have the following.

COROLLARY 3.7. *Let M be a real analytic complete Riemannian manifold. For $\sigma = (p \; ; V) \in \mathcal{I}$, there is a complete totally geodesic submanifold of M through p whose tangent space at p is V.*

Proof. We follow the notation of Theorem 3.6. For $v \in V$ let $\sigma_v : \boldsymbol{R} \to G_r(M)$ be a curve defined in Theorem 3.6. We put $I = \{t \in \boldsymbol{R} \,|\, \sigma_v(t) \in \mathcal{I}\}$. Since $\mathcal{I}$ is a closed subset of $G_r(M)$ for a real analytic Riemannian manifold, I is closed in $\boldsymbol{R}$. On the other hand, as it has been seen in the proof of Theorem 3.6, I is open in $\boldsymbol{R}$. Since $\sigma_v(0) = \sigma \in \mathcal{I}$, I contains 0. Consequently I coincides with $\boldsymbol{R}$. From Theorem 3.6, Corollary 3.7 is thereby proved.   □

4. Characterizations of curvature-invariant subspaces

Let $\boldsymbol{R}^n$ be an n-dimensional real vector space equipped with a standard inner product $\langle , \rangle$. We denote by $\mathfrak{R}(n)$ the space of curvature tensors over $\boldsymbol{R}^n$. Given a curvature tensor $R \in \mathfrak{R}(n)$, we call an r-dimensional subspace V of $\boldsymbol{R}^n$ *curvature-invariant* with respect to R if it holds that

(4.1) $R(x, y)z \in V$ for any $x, y, z \in V$.

Here we assume that $2 \leq r \leq n-1$ when we have no reference. Viewing Theorem 2.1, we recognize that it is an important problem for investigating totally geodesic submanifolds to classify curvature-invariant subspaces with respect to a given curvature tensor. In this section we shall study characterizations of curvature-invariant subspaces.

Let $G_r(\boldsymbol{R}^n)$ be the Grassmann manifold of r-dimensional subspaces of $\boldsymbol{R}^n$. We want criteria that $V \in G_r(\boldsymbol{R}^n)$ is curvature-invariant with respect to a given curvature tensor. In the case of $r = 2$, we recall a result of R. S. Kulkarni. A curvature tensor R defines the sectional curvature

$$K(R) : G_2(\boldsymbol{R}^n) \longrightarrow \boldsymbol{R}$$

by $K(R)(V) = \langle R(e_1, e_2)e_2, e_1 \rangle$, where $\{e_1, e_2\}$ is an orthonormal basis of V. The following has been shown by R. S. Kulkarni [9] as Corollary of Proposition 5.1.

THEOREM 4.1. $V \in G_2(\boldsymbol{R}^n)$ is a curvature-invariant subspace with respect to a given curvature tensor R if and only if V is a critical point of $K(R)$.

Immediately the following two corollaries yield.

COROLLARY 4.2. For every curvature tensor R, there are always curvature-invariant 2-dimensional subspaces with respect to R.

We note that in the higher dimensional case a different phenomenon from this occurs. See Theorem 5.4 in the next section.

The following is well-known as the axiom of planes which is due to E. Cartan.

COROLLARY 4.3. If every 2-dimensional subspace is curvature-invariant with respect to R, then the curvature tensor R has constant sectional curvature.

We shall consider a generalization of Theorem 4.1 to the case of higher dimensional subspaces. First we represent the Grassmann manifold $G_r(\boldsymbol{R}^n)$ as Riemannian symmetric pair. The rotation group $SO(n)$ acts on $G_r(\boldsymbol{R}^n)$ transitively. Let $e_1, \cdots, e_n$ be a natural basis of $\boldsymbol{R}^n$ and $V_0 = \{e_1, \cdots, e_r\}_{\boldsymbol{R}}$ be the r-dimensional subspace linearly spanned by $e_1, \cdots, e_r$. We define a mapping $\pi: SO(n) \to G_r(\boldsymbol{R}^n)$ by $\pi(a) = a \cdot V_0 = \{ae_1, \cdots, ae_r\}_{\boldsymbol{R}} = \{a_1, \cdots, a_r\}_{\boldsymbol{R}}$, where $a = (a_1, \cdots, a_r, a_{r+1}, \cdots, a_n) \in SO(n)$ is written as the row of column vectors $a_1, \cdots, a_r, a_{r+1}, \cdots, a_n$. The isotropy group at $V_0 \in G_r(\boldsymbol{R}^n)$ consists of all matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in SO(n), \quad \text{where} \quad \alpha \in O(r) \text{ and } \beta \in O(n-r).$$

Denoting this subgroup by $S(O(r) \times O(n-r))$, we have $G_r(\boldsymbol{R}^n) = SO(n)/S(O(r) \times O(n-r))$. Hereafter denote simply by G and K, $SO(n)$ and $S(O(r) \times O(n-r))$, respectively. We regard G as the total space of the principal fibre bundle $G(G_r(\boldsymbol{R}^n), K)$ over $G_r(\boldsymbol{R}^n)$ with structure group K and projection π. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebra and the Lie subalgebra of $\mathfrak{g}$ corresponding to G and K, respectively. The complement $\mathfrak{p}$ of the canonical decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is given by

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & -{}^tX \\ X & 0 \end{pmatrix} \middle| X \in M_{n-r, r}(\boldsymbol{R}) \right\},$$

where $M_{n-r, r}(\boldsymbol{R})$ denotes the space of all real matrices with $n-r$ rows and r columns. The tangent space $T_{V_0} G_r(\boldsymbol{R}^n)$ is identified with $\mathfrak{p}$ by the differential π_*.

Let F_r be the natural r-dimensional vector bundle over $G_r(\boldsymbol{R}^n)$. Namely F_r is the vector subbundle of the trivial bundle $G_r(\boldsymbol{R}^n) \times \boldsymbol{R}^n$ defined as follows:

$$F_r = \{(V ; v) \mid V \in G_r(\boldsymbol{R}^n), \ v \in V\} \subset G_r(\boldsymbol{R}^n) \times \boldsymbol{R}^n.$$

F_r can be viewed as a vector bundle with standard fibre $\boldsymbol{R}^r$ associated with the principal fibre bundle $G(G_r(\boldsymbol{R}^n), K)$ which corresponds to the Lie group homomorphism of K onto $O(r)$:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \longmapsto \alpha .$$

In fact defining a mapping of $G \times \boldsymbol{R}^r$ onto F_r by

$$(a\,;\,v) \in G \times \boldsymbol{R}^r \longrightarrow \Big(a \cdot V_0\,;\, \sum_{i=1}^{r} a_i v^i\Big) \in F_r \subset G_r(\boldsymbol{R}^n) \times \boldsymbol{R}^n,$$

we have $G \times_K \boldsymbol{R}^r = F_r$. In particular we can regard $a \in G$ as a linear isomorphism of $\boldsymbol{R}^r$ onto the fibre $(F_r)_{a \cdot V_0}$ at $a \cdot V_0 \in G_r(\boldsymbol{R}^n)$. We introduce the invariant connection on the principal fibre bundle $G(G_r(\boldsymbol{R}^n), K)$ which corresponds to the canonical decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. We denote by D the covariant differentiation in F_r defined by the connection. Let ΓF_r be the space of sections of F_r and $\exp tX$ be the 1-parameter subgroup of G generated by $X \in \mathfrak{p}$. Then under the identification of $T_{V_0} G_r(\boldsymbol{R}^n)$ with $\mathfrak{p}$, we have the following formula:

$$(4.2) \qquad D_X \varphi = I_n \Big(\frac{d}{dt}\Big|_{t=0} (\exp tX)^{-1} \varphi_{(\exp tX) \cdot V_0}\Big)$$

for $\varphi \in \Gamma F_r$ and $X \in T_{V_0} G_r(\boldsymbol{R}^n) \cong \mathfrak{p}$. Here I_n denotes the unit matrix. Moreover $\exp tX \in G$ is regarded as a linear isomorphism of $\boldsymbol{R}^r$ onto $(F_r)_{(\exp tX) \cdot V_0}$ and $(\exp tX)^{-1} \varphi_{(\exp tX) \cdot V_0}$ is understood as a curve in $\boldsymbol{R}^r$.

Let $T_3^1(F_r) = F_r^* \otimes F_r^* \otimes F_r^* \otimes F_r$ be the vector bundle of type $(1, 3)$ tensors of F_r. We shall define a mapping ϕ of $\mathfrak{R}(n)$ into the space $\Gamma T_3^1(F_r)$ of sections of $T_3^1(F_r)$. For $R \in \mathfrak{R}(n)$ and $V \in G_r(\boldsymbol{R}^n)$, we define a $(1, 3)$-type tensor $\phi(R)_V$ on $(F_r)_V \cong V$ by restricting R to V, i.e.,

$$\phi(R)_V(x, y)z = \text{the } V\text{-component of } R(x, y)z, \quad \text{for } x, y, z \in V,$$

where we take the V-component with respect to the orthogonal decomposition $\boldsymbol{R}^n = V + V^\perp$. Evidently ϕ is a linear homomorphism of $\mathfrak{R}(n)$ into $\Gamma T_3^1(F_r)$. Moreover we see that ϕ is injective. In fact, if $\phi(R)_V = 0$ for all $V \in G_r(\boldsymbol{R}^n)$, then R has constant sectional curvature 0 and hence $R \equiv 0$ (Note that $2 \leq r \leq n-1$).

Now we show the following.

THEOREM 4.4. $V \in G_r(\boldsymbol{R}^n)$ is a curvature-invariant subspace with respect to a given curvature tensor R if and only if $D\phi(R) = 0$ at $V \in G_r(\boldsymbol{R}^n)$.

Proof. We can discuss the above assertion at $V_0 = \{e_1, \cdots, e_r\}_{\boldsymbol{R}}$ without loss of generality. Let E_i^j be the $n \times n$ matrix such that the entry at the i-th column and the j-th row is 1 and other entries are all zero. We put $\mathcal{E}_k^a = E_k^a - E_a^k$ $(a = r+1, \cdots, n, \ k = 1, \cdots, r)$. Then $\{\mathcal{E}_k^a\,;\, a = r+1, \cdots, n, \ k = 1, \cdots, r\}$ forms a basis of $\mathfrak{p}$. The 1-parameter subgroup $\exp t\mathcal{E}_k^a$ of G generated by $\mathcal{E}_k^a$ is given as follows: Put

$$\exp t\mathcal{E}_k^a = (e_1(t), \cdots, e_r(t), e_{r+1}(t), \cdots, e_n(t))$$

as the row of column vectors $e_1(t), \cdots, e_r(t), e_{r+1}(t), \cdots, e_n(t)$. Then we have

$$e_i(t) = e_i \quad \text{for } i \neq k, a$$

$$e_k(t) = \cos t \; e_k + \sin t \; e_a$$

$$e_a(t) = -\sin t \; e_k + \cos t \; e_a.$$

For $1 \leq i_1, i_2, i_3, i_4 \leq r$, we see that

$$\langle (\exp t\mathcal{E}_k^a)^{-1} \phi(R)_{(\exp t\mathcal{E}_k^a) \cdot V_0}(e_{i_1}, e_{i_2})e_{i_3}, e_{i_4} \rangle$$
$$= \langle R(e_{i_1}(t), e_{i_2}(t))e_{i_3}(t), e_{i_4}(t) \rangle.$$

The formula (4.2) implies that

$$\langle (D_{\mathcal{E}_k^a}\phi(R))(e_{i_1}, e_{i_2})e_{i_3}, e_{i_4} \rangle$$

$$= \frac{d}{dt}\Big|_{t=0} \langle R(e_{i_1}(t), e_{i_2}(t))e_{i_3}(t), e_{i_4}(t) \rangle$$

$$= \delta_{i_1 k}\langle R(e_a, e_{i_2})e_{i_3}, e_{i_4} \rangle + \delta_{i_2 k}\langle R(e_{i_1}, e_a)e_{i_3}, e_{i_4} \rangle$$
$$+ \delta_{i_3 k}\langle R(e_{i_1}, e_{i_2})e_a, e_{i_4} \rangle + \delta_{i_4 k}\langle R(e_{i_1}, e_{i_2})e_{i_3}, e_a \rangle.$$

Therefore if V_0 is a curvature-invariant subspace with respect to R, we have $D\phi(R) = 0$ at V_0. Conversely suppose that $D\phi(R) = 0$ at V_0. We devide it into the three cases according to the values of r.

Case 1, $r \geq 4$. For $1 \leq i_1, i_2, i_3 \leq r$, we choose k such that $1 \leq k \leq r$ and $k \neq i_1, i_2, i_3$. For each a $(r+1 \leq a \leq n)$, we have

$$0 = \langle (D_{\mathcal{E}_k^a}\phi(R))(e_{i_1}, e_{i_2})e_{i_3}, e_{i_k} \rangle = \langle R(e_{i_1}, e_{i_2})e_{i_3}, e_a \rangle.$$

Therefore we see that V_0 is curvature-invariant with respect to R.

Case 2, $r = 3$. For each a $(4 \leq a \leq n)$, we get

$$0 = \langle (D_{\mathcal{E}_1^a}\phi(R))(e_1, e_2)e_3, e_1 \rangle$$

$$= \langle R(e_a, e_2)e_3, e_1 \rangle + \langle R(e_1, e_2)e_3, e_a \rangle$$

$$= \langle R(e_1, e_3)e_2, e_a \rangle + \langle R(e_1, e_2), e_3, e_a \rangle$$

and similarly $\langle R(e_2, e_3)e_1, e_a \rangle + \langle R(e_2, e_1)e_3, e_a \rangle = 0$. Therefore we have

$$0 = \langle R(e_1, e_2)e_3, e_a \rangle + \langle R(e_2, e_3)e_1, e_a \rangle + \langle R(e_3, e_1)e_2, e_a \rangle$$

$$= \langle R(e_1, e_2)e_3, e_a \rangle - \langle R(e_2, e_1)e_3, e_a \rangle - \langle R(e_2, e_1)e_3, e_a \rangle$$

$$= 3\langle R(e_1, e_2)e_3, e_a \rangle.$$

The others are same. Hence we see that V_0 is curvature-invariant with respect to R.

Case 3, $r=2$. For each a $(3\leq a\leq n)$, we get

$$0=\langle(D_{\mathcal{E}_1^a}\phi(R))(e_1, e_2)e_2, e_1\rangle$$

$$=\langle R(e_a, e_2)e_2, e_1\rangle+\langle R(e_1, e_2)e_2, e_a\rangle$$

$$=2\langle R(e_1, e_2)e_2, e_a\rangle.$$

Therefore we see that V_o is curvature-invariant with respect to R. $\qquad\square$

Given a curvature tensor $R\in\mathfrak{R}(n)$, we can define various functions on $G_r(\boldsymbol{R}^n)$. For example, we have the following:

$$\tau(\phi(R))(V)=\sum_{i,j=1}^{r}\langle R(\tilde{e}_i, \tilde{e}_j)\tilde{e}_j, \tilde{e}_i\rangle$$

$$\|\rho(\phi(R))\|^2(V)=\sum_{i,j,k,l=1}^{r}\langle R(\tilde{e}_i, \tilde{e}_k)\tilde{e}_k, \tilde{e}_j\rangle\langle R(\tilde{e}_i, \tilde{e}_l)\tilde{e}_l, \tilde{e}_j\rangle$$

$$\|\phi(R)\|^2(V)=\sum_{i_1,i_2,i_3,i_4=1}^{r}\langle R(\tilde{e}_{i_1}, \tilde{e}_{i_2})\tilde{e}_{i_3}, \tilde{e}_{i_4}\rangle\langle R(\tilde{e}_{i_1}, \tilde{e}_{i_2})\tilde{e}_{i_3}, \tilde{e}_{i_4}\rangle$$

$$\vdots$$

where $\{\tilde{e}_1, \cdots, \tilde{e}_r\}$ denotes an orthonormal basis of V.

Using Theorem 4.4, we can prove the following.

COROLLARY 4.5. *If $V\in G_r(\boldsymbol{R}^n)$ is a curvature-invariant subspace with respect to R, then V is a critical point of functions $\tau(\phi(R))$, $\|\rho(\phi(R))\|^2$, $\|\phi(R)\|^2$, $\cdots$.*

In the rest of this section, we shall discuss the case of $r=n-1$, i.e., hyperplanes. By simple computation, we obtain the following which will be used in section 7.

PROPOSITION 4.6. *If $V\in G_{n-1}(\boldsymbol{R}^n)$ is a curvature-invariant hyperplane with respect to a curvature tensor R, then the orthogonal complement $V^{\perp}$ is an eigenspace of its Ricci curvature ρ. When $n=3$, the converse also is true.*

Proof. We take an orthonormal basis $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ of $\boldsymbol{R}^n$ such that V is linearly spanned by $\tilde{e}_1, \cdots, \tilde{e}_{n-1}$. Since

$$\rho(\tilde{e}_n, \tilde{e}_j)=\sum_{i=1}^{n-1}\langle R(\tilde{e}_j, \tilde{e}_i)\tilde{e}_i, \tilde{e}_n\rangle \quad\text{for } j=1, \cdots, n-1,$$

the first part of the above proposition holds.

When $n=3$, let $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ be an orthonormal basis of $\boldsymbol{R}^3$ such that $\tilde{e}_3$ is an eigenvector of the Ricci tensor ρ. Since

$$0=\rho(\tilde{e}_3, \tilde{e}_1)=\langle R(\tilde{e}_1, \tilde{e}_{22})\tilde{e}, \tilde{e}_3\rangle$$

and

$$0=\rho(\tilde{e}_3, \tilde{e}_2)=\langle R(\tilde{e}_2, \tilde{e}_1)\tilde{e}_1, \tilde{e}_3\rangle,$$

$V = \{\tilde{e}_1, \tilde{e}_2\}_R$ is curvature-invariant with respect to R.   □

Let $\Lambda^2 R^n$ be the space of 2-vectors of R^n. For a given curvature tensor R, we define a symmetric endomorphism $\hat{R}$ on $\Lambda^2 R^n$, which is called a *curvature operator*. Namely put

$$\langle \hat{R}(x \wedge y), u \wedge v \rangle = \langle R(x, y)u, v \rangle \quad \text{for } x, y, u, v \in R^n$$

(cf. J. P. Bourguignon and H. Karcher [1]). The existence of curvature-invariant hyperplanes has an effect on the spectral data of the curvature operator $\hat{R}$. Namely the following holds.

PROPOSITION 4.7. *Let $V \in G_{n-1}(R^n)$ be a curvature-invariant hyperplane with respect to a curvature tensor R and ξ be a unit vector of R^n which is orthogonal to V. Then there is an orthonormal basis $\{\tilde{e}_1, \cdots, \tilde{e}_{n-1}\}$ of V such that $\xi \wedge \tilde{e}_i$ $(i=1, \cdots, n-1)$ are eigenvectors of the curvature operator $\hat{R}$. In particular the curvature operator $\hat{R}$ has at least $n-1$ decomposable eigenvectors which are linearly independent.*

Proof. Let V and ξ be the ones which satisfy the condition in Proposition 4.7. For R and ξ, we define a symmetric endomorphism $\tilde{R}_\xi$ of R^n by $\langle \tilde{R}_\xi(x), y \rangle = \langle R(\xi, x)\xi, y \rangle$ for $x, y \in R^n$. Evidently we have $\tilde{R}_\xi(V) \subset V$. We take an orthonormal basis $\{\tilde{e}_1, \cdots, \tilde{e}_{n-1}\}$ in V such that $\tilde{e}_i$ $(i=1, \cdots, n-1)$ are eigenvectors of $\tilde{R}_\xi$, i.e., $\tilde{R}_\xi(\tilde{e}_i) = \lambda_i \tilde{e}_i$. Then we see that $\xi \wedge \tilde{e}_i$ $(i=1, \cdots, n-1)$ are also eigenvectors of the curvature operator $\hat{R}$ with eigenvalues λ_i. In fact, we get

$$\langle \hat{R}(\xi \wedge \tilde{e}_i), \xi \wedge \tilde{e}_j \rangle = \langle R(\xi, \tilde{e}_i)\xi, \tilde{e}_j \rangle = \langle \tilde{R}_\xi(\tilde{e}_i), \tilde{e}_j \rangle = \lambda_i \delta_{ij}$$

and

$$\langle \hat{R}(\xi \wedge \tilde{e}_i), \tilde{e}_j \wedge \tilde{e}_k \rangle = \langle R(\xi, \tilde{e}_i)\tilde{e}_j, \tilde{e}_k \rangle = \langle R(\tilde{e}_k, \tilde{e}_j)\tilde{e}_i, \xi \rangle = 0 .   □$$

Applying Proposition 4.7, we obtain the following which is an analogous result to the axiom of planes.

THEOREM 4.8. *Let H be the connected Lie subgroup of $SO(n)$ which acts irreducibly on R^n and R be an invariant curvature tensor by the action of H. If there is a curvature-invariant hyperplane $V \in G_{n-1}(R^n)$ with respect to R, then R has constant sectional curvature.*

Proof. Let $\mathfrak{so}(n)$ be the Lie algebra which corresponds to $SO(n)$ and $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{so}(n)$ which corresponds to H. By the assumption, $\mathfrak{h}$ acts irreducibly on R^n. Let $V \in G_{n-1}(R^n)$ be a curvature invariant hyperplane with respect to R and ξ be a unit vector of R^n which is orthogonal to V. Let $V = V_1 + \cdots + V_s$ be the orthogonal decomposition of V into the eigenspaces of $\tilde{R}_\xi$ and λ_i $(i=1, \cdots, s)$ be the eigenvalues of $\tilde{R}_\xi$ on the eigenspaces V_i. It is assumed that $\lambda_1, \cdots, \lambda_s$ are mutually distinct. From Proposition 4.7, it follows

that $\hat{R}(\xi\wedge v)=\lambda_i\xi\wedge v$ for $v\in V_i$. Now we shall prepare the following lemma.

LEMMA. *For each $X\in\mathfrak{h}$ and $v\in V_i$, the following formulas hold*:
(1) $\hat{R}(\xi\wedge Xv)=\lambda_i\xi\wedge Xv,$
(2) $\hat{R}(X\xi\wedge v)=\lambda_i X\xi\wedge v.$

Proof of Lemma. Let $\exp tX$ be the 1-parameter subgroup of H generated by $X\in\mathfrak{h}$. Since the curvature tensor R is invariant by the action of H, we have $\hat{R}\circ\exp tX=\exp tX\circ\hat{R}$ on $\Lambda^2\boldsymbol{R}^n$. Hence

$$\hat{R}(\exp tX(\xi\wedge v))=\exp tX(\hat{R}(\xi\wedge v))=\lambda_i\exp tX(\xi\wedge v).$$

Differentiating this equation at $t=0$, we have

$$\hat{R}(X\xi\wedge v+\xi\wedge Xv)=\lambda_i(X\xi\wedge v+\xi\wedge Xv).$$

We note that $\hat{R}(\Lambda^2 V)\subset\Lambda^2 V$ and $\hat{R}(\xi\wedge V)\subset\xi\wedge V$, since V is curvature-invariant with respect to R. Since $X\xi\in V$, we obtain $\hat{R}(\xi\wedge Xv)=\lambda_i\xi\wedge Xv$ and $\hat{R}(X\xi\wedge v)=\lambda_i X\xi\wedge v$. Thus Lemma has been proved.

By Lemma (1), we see that for each eigenspace V_i and any $X\in\mathfrak{h}$, $X(V_i)\subset V_i+\boldsymbol{R}\cdot\xi$. If $X\xi=0$ for any $X\in\mathfrak{h}$, it contradicts to the irreducibility of $\mathfrak{h}$. Therefore there exists some $X_0\in\mathfrak{h}$ such that $X_0\xi\neq0$. Accordingly, for some $i\in\{1,\cdots,s\}$ and $v_0\in V_i$ we have $\langle X_0\xi, v_0\rangle\neq0$. For this i, we shall show that $X\xi\in V_i$ for any $X\in\mathfrak{h}$. We take an arbitrary vector $v'\in V_j$ for $j\neq i$. By Lemma (2), $X_0\xi\wedge v'$ and $X\xi\wedge v_0$ are eigenvectors of $\hat{R}$ with distinct eigenvalues λ_j and λ_i, respectively. Therefore they are mutually orthogonal. Hence

$$0=\langle X_0\xi\wedge v', X\xi\wedge v_0\rangle$$
$$=\langle X_0\xi, X\xi\rangle\langle v', v_0\rangle-\langle X_0\xi, v_0\rangle\langle X\xi, v'\rangle$$
$$=-\langle X_0\xi, v_0\rangle\langle X\xi, v'\rangle.$$

Since $\langle X_0\xi, v_0\rangle\neq0$, we have $\langle X\xi, v'\rangle=0$. From this it follows that $X\xi\in V_i$. This together with the preceding fact implies for this i $V_i+\boldsymbol{R}\cdot\xi$ is an invariant subspace by the action of $\mathfrak{h}$. By the irreducibility of $\mathfrak{h}$, we have $V_i+\boldsymbol{R}\cdot\xi=\boldsymbol{R}^n$. This means that the whole V is an eigenspace of $\tilde{R}_\xi$, whose eigenvalue will be denoted by λ. In particular it holds that

$$R(\xi, x)y=\lambda\{\langle\xi, y\rangle x-\langle x, y\rangle\xi\}\quad\text{for any }x, y\in\boldsymbol{R}^n.$$

Now we define a curvature tensor R_0 of $\boldsymbol{R}^n$ by $R_0(x, y)z=\langle y, z\rangle x-\langle x, z\rangle y$ for $x, y, z\in\boldsymbol{R}^n$. For the preceding eigenvalue λ of $\tilde{R}_\xi$, put a subspace $\mathfrak{N}$ of $\boldsymbol{R}^n$ as follows:

$$\mathfrak{N}=\{x\in\boldsymbol{R}^n\mid i(x)(R+\lambda R_0)=0\}.$$

Here $i(x)(R+\lambda R_0)$ denotes the $(1, 2)$-tensor on $\boldsymbol{R}^n$ given by

$$i(x)(R+\lambda R_o)(u, v)=(R+\lambda R_o)(x, u)v \quad \text{for } u, v \in \mathbf{R}^n.$$

Since R and R_o are invariant by the action of H, $\mathfrak{N}$ is an invariant subspace of $\mathbf{R}^n$ by the action of H. By the preceding discussion, we have $\xi \in \mathfrak{N}$. From the irreducibility of H, it follows that $\mathfrak{N}=\mathbf{R}^n$. Therefore it holds that $R(x, y)z=-\lambda\{\langle y, z\rangle x-\langle x, z\rangle y\}$ for $x, y, z \in \mathbf{R}^n$. Thus our assertion has been proved. □

As an application of Theorem 4.8, we obtain the following, which is a slight generalization of Theorem 1 in B. Y. Chen [2].

COROLLARY 4.9. *Let $M=G/H$ be a Riemannian homogeneous space such that the identity component H_o of H acts irreducibly on the tangent space* (dim $M\geq3$). *If M admits a totally umbilical hypersurface S, then M has constant sectional curvature.*

Remark. B. Y. Chen in [2] investigated the case when M is an irreducible Riemannian symmetric space and showed the same conclusion as above. J. A. Wolf ([15]) investigated and classified Riemannian homogeneous spaces which satisfy the assumption stated in Corollary 4.9.

Proof of Corollary 4.9. We can prove this corollary following the same process as the proof of Theorem 1 in [2]. The coset H is denoted by o. We can assume that the totally umbilical hypersurface S is through o. M is Einsteinian and hence from the equation of Codazzi, it follows that the mean curvature of S is constant. Using the equation of Codazzi once more, we see that the tangent space T_oS is a curvature-invariant hyperplane of T_oM. The curvature tensor R at o is invariant by the linear isotropy action of H_o. By Theorem 4.8 we obtain our conclusion.   □

5. Curvature-invariant subspaces—consideration of examples

In this section, we shall classify the curvature-invariant subspaces for curvature tensors of special form. Let A and B be symmetric endomorphisms of $\mathbf{R}^n$ with respect to the standard inner product $\langle,\rangle$. We define a (1, 3)-tensor $A\wedge B$ by

(5.1) $A\wedge B(x, y)z=\langle By, z\rangle Ax-\langle Bx, z\rangle Ay+\langle Ay, z\rangle Bx-\langle Ax, z\rangle By$

(cf. J. P. Bourguignon and H. Karcher [1]). Then $A\wedge B$ is a curvature tensor on $\mathbf{R}^n$. We try to classify curvature-invariant subspaces with respect to $A\wedge B$. First we fix notations. Given a subspace V in $\mathbf{R}^n$, we have the orthogonal decomposition $\mathbf{R}^n=V+V^\perp$. For $v\in\mathbf{R}^n$, we denote by v_a and v_b the V-component and the $V^\perp$-component of v, respectively. For a symmetric endomorphism A of $\mathbf{R}^n$, we define a mapping $A^a: V\to V$ and $A^b: V\to V^\perp$ by $A^ax=(Ax)_a$ and $A^bx=(Ax)_b$ for $x\in V$, respectively.

The following is easily shown.

PROPOSITION 5.1. *Let $R = A \wedge I$ be a curvature tensor given by a symmetric endomorphism A and an identity transformation I of $\mathbf{R}^n$ and V be a subspace of $\mathbf{R}^n$ with $\dim V \geq 2$. Then V is curvature-invariant with respect to R if and only if V is an invariant subspace of the symmetric endomorphism A.*

Remark. If a Riemannian manifold M is conformally flat, its curvature tensor R has the form $A \wedge I$ for some symmetric tensor field A.

PROPOSITION 5.2. *Let $R = A \wedge A$ be a curvature tensor given by a symmetric endomorphism A of $\mathbf{R}^n$ and V be a subspace of $\mathbf{R}^n$ with $\dim V \geq 2$. Then V is curvature-invariant with respect to R if and only if one of the following holds:*
  (1) *V is an invariant subspace of the symmetric endomorphism A.*
  (2) *V is a null-subspace with respect to A, i.e.,*

$$\langle Ax, y \rangle = 0 \quad \text{for any } x, y \in V$$

  (3) *The image $A(V)$ of V by A is of 1-dimensional.*

Proof. Let V be a curvature-invariant subspace with respect to R. Then we have

$$\langle Ay, z \rangle A^b x - \langle Ax, z \rangle A^b y = 0 \quad \text{for any } x, y, z \in V.$$

We discuss dividing into the following three cases:

$$\text{Case 1,} \quad \ker A^b = V$$

$$\text{Case 2,} \quad \ker A^a = V$$

$$\text{Case 3,} \quad \ker A^a \neq V \text{ and } \ker A^b \neq V,$$

where $\ker A^a$ and $\ker A^b$ denote the kernel of A^a and that of A^b, respectively.

Case 1 means that V is an invariant subspace of A and Case 2 means that V is a null-subspace with respect to A. Therefore we consider Case 3.

ASSERTION 1. $\ker A^a = \ker A^b$. *In particular they coincide with* $\ker A \cap V$.

Proof of Assertion 1. First we shall prove $\ker A^b \subset \ker A^a$. We take a non-zero vector y which belongs to the orthogonal complement of $\ker A^b$ in V. Then $A^b y$ is not zero. For $x \in \ker A^b$, we have $-\langle Ax, z \rangle A^b y = 0$ and hence $\langle Ax, z \rangle = 0$ for any $z \in V$. This implies that $x \in \ker A^a$.

Next we shall prove $\ker A^a \subset \ker A^b$. We take a non-zero vector y which belongs to the orthogonal complement of $\ker A^a$ in V. Then $A^a y$ is not zero. For $x \in \ker A^a$, we have $\langle Ay, z \rangle A^b x = 0$. Putting $z = A^a y$, we obtain $A^b x = 0$.

We put $V' = \ker A^a = \ker A^b = \ker A \cap V$ and denote by V'' the orthogonal complement of V' in V.

ASSERTION 2. $\dim V''=1$.

Proof of Assertion 2.   We fix non-zero $x \in V''$. Then $A^a x$ and $A^b x$ are both not zero. For arbitrary $y \in V''$, $\langle Ay, A^a x \rangle A^b x - \langle Ax, A^a x \rangle A^b y = 0$ and hence $A^b y = (\langle A^a x, A^a y \rangle / \|A^a x\|^2) A^b x$. Therefore we have $\dim A^b(V'')=1$. Since A^b is injective in V'', $\dim V''=1$.

From these, it follows that in Case 3, $\dim A(V)=1$.

Conversely, it is evident that if one of the three conditions in Proposition 5.2 holds for V, then V is curvature-invariant with respect to $R=A \wedge A$.   □

Remark 1.   Let M be a hypersurface immersed in a real space form $\widetilde{M}(\tilde{c})$ of constant sectional curvature $\tilde{c}$ and A be its shape operator for a unit normal vector field. Then by the equation of Gauss, the curvature tensor R of M has the form $R=1/2\{\tilde{c}I \wedge I + A \wedge A\}$. Therefore we can apply Proposition 5.2 to the classification of its curvature-invariant subspaces. In fact, this Proposition will be used in section 6.

Remark 2.   If A is non-singular, the case (3) in Proposition 5.2 does not occur. Moreover if A is positive-definite, the case (1) only may occur.

PROPOSITION 5.3.   *Let* $R=A \wedge B$ *be a curvature tensor given by the positive-definite symmetric endomorphisms A, B of $\mathbf{R}^n$ and V be a subspace of $\mathbf{R}^n$ with* $\dim V \geqq 3$. *Then V is curvature-invariant with respect to $R=A \wedge B$ if and only if one of the following holds*:
   (1)   *V is an invariant subspace by both A and B.*
   (2)   *There exists a positive number λ such that*

$$B^a = \lambda A^a \quad and \quad B^b = -\lambda A^b \quad on \ V.$$

   (3)   *There exist a positive number λ and a subspace V' of codimension 1 in V such that the following holds*:

   (i) $A(V') \subset V$, $B(V') \subset V$ *and* $Bx = \lambda Ax$ *for* $x \in V'$

   (ii) $B^b y = -\lambda A^b y$ *for* $y \in V''$,

where V'' denotes the orthogonal complement of V' in V.

Proof.   Let V be a curvature-invariant subspace of $\mathbf{R}^n$ with respect to $R=A \wedge B$. We assume that $\dim V \geqq 3$.

ASSERTION 1.   *For each $x \in V$, there exists a positive number λ (which may depend on x) such that $B^b x = -\lambda A^b x$. In particular we have $\ker A^b = \ker B^b$ and $A^b(V)=B^b(V)$.*

Proof of Assertion 1.   Since $\dim V \geqq 3$, there exists a non-zero vector $z \in V$ such that $\langle Ax, z \rangle = \langle Bx, z \rangle = 0$. From (5.1), it follows that $\langle By, z \rangle Ax +$

$\langle Ay, z \rangle Bx \in V$. Hence $\langle By, z \rangle A^b x + \langle Ay, z \rangle B^b x = 0$. We put $y = z$. Since A and B are both positive-definite, $\langle Az, z \rangle > 0$ and $\langle Bz, z \rangle > 0$. Putting $\lambda = \langle Bz, z \rangle / \langle Az, z \rangle$, we have $B^b x = -\lambda A^b x$.

Owing to Assertion 1, we define a subspace V' of V by $V' = \ker A^b = \ker B^b$ and denote by V'' the orthogonal complement of V' in V.

ASSERTION 2. *If* $\dim V'' \geq 2$, *there exists a positive number* λ *such that* $B^a x = \lambda A^a x$ *and* $B^b x = -\lambda A^b x$ *for any* $x \in V$.

Proof of Assertion 2. Let x and y be linearly independent vectors in V''. By Assertion 1, we have $B^b x = -\lambda(x) A^b x$ and $B^b y = -\lambda(y) A^b y$, where $\lambda(x)$ and $\lambda(y)$ denote positive constants depending on x and y, respectively. We shall prove that $\lambda(x) = \lambda(y)$. We have

$$B^b(x+y) = -\lambda(x+y) A^b(x+y) = -\lambda(x+y) A^b x - \lambda(x+y) A^b y .$$

On the other hand,

$$B^b(x+y) = B^b x + B^b y = -\lambda(x) A^b x - \lambda(y) A^b y .$$

Hence

$$(\lambda(x) - \lambda(x+y)) A^b x + (\lambda(y) - \lambda(x+y)) A^b y = 0 .$$

Since $A^b x$ and $A^b y$ are linearly independent, we see that $\lambda(x) = \lambda(x+y) = \lambda(y)$. From this, it follows that the constant λ does not depend on the choice of vectors in V''.

Again, let x and y be linearly independent vectors in V''. By (5.1), it holds that

$$\langle By, z \rangle A^b x - \langle Bx, z \rangle A^b y + \langle Ay, z \rangle B^b x - \langle Ax, z \rangle B^b y = 0$$

for any $z \in V$. Hence $\langle By - \lambda Ay, z \rangle A^b x - \langle Bx - \lambda Ax, z \rangle A^b y = 0$. Since $A^b x$ and $A^b y$ are linearly independent, we have $\langle Bx - \lambda Ax, z \rangle = 0$. Since z is an arbitrary vector in V, we see that $B^a x = \lambda A^a x$.

Finally, let x and y be a vector in V' and a non-zero vector in V'', respectively. Then we have $-\langle Bx, z \rangle A^b y - \langle Ax, z \rangle B^b y = 0$ and hence $\langle Bx - \lambda Ax, z \rangle A^b y = 0$. Since $A^b y \neq 0$, $Bx = \lambda Ax$. Therefore it has been shown that there exists a positive number λ such that $B^a x = \lambda A^a x$ and $B^b x = -\lambda A^b x$ for any $x \in V$.

ASSERTION 3. *If* $\dim V'' = 1$, *there exists a positive number* λ *such that* $Bx = \lambda Ax$ *for* $x \in V'$ *and* $B^b y = -\lambda A^b y$ *for* $y \in V''$.

Proof of Assertion 3. From Assertion 1, it follows that $B^b y = -\lambda A^b y$ for a non-zero vector $y \in V''$. Then we have $Bx = \lambda Ax$ for $x \in V'$. It is shown by the same way as Assertion 2.

The case when dim $V''=0$ means that $A(V)\subset V$ and $B(V)\subset V$. This, together with Assertions 2 and 3 implies that if V is curvature-invariant, one of three conditions in Proposition 5.3 holds.

Conversely, it is easily shown that if one of three conditions in Proposition 5.3 holds for V, V is curvature-invariant with respect to $R=A\wedge B$.   □

Applying Propositions 5.2 and 5.3, we shall construct examples of curvature tensors which do not all have curvature-invariant subspaces of dimension not less than 3. We denote by $\{e_1, \cdots, e_n\}$ the natural basis of $\mathbf{R}^n$. Let A be a positive definite symmetric endomorphism of $\mathbf{R}^n$ such that each e_ι $(i=1, \cdots, n)$ is an eigenvector of A with mutually distinct eigenvalue a_ι $(i=1, \cdots, n)$. For such an A, we choose a symmetric endomorphism B which satisfies the following two conditions:

(5.2)-(i)   Let $(b_{ij})_{i,j=1,\cdots,n}$ be the $n\times n$-matrix which represents B with respect to the natural basis. Then each entry b_{ij} is not zero.

(5.2)-(ii) Each eigenspace of $A^{-1}B$ is of dimension 1.

Here we shall make a remark about the endomorphism $A^{-1}B$ of $\mathbf{R}^n$. Denote by g a new inner product on $\mathbf{R}^n$ which is defined by $g(x, y)=\langle Ax, y\rangle$ for $x, y\in\mathbf{R}^n$. Then $A^{-1}B$ is a symmetric endomorphism with respect to this inner product g. In fact we have

$$g(A^{-1}Bx, y)=\langle AA^{-1}Bx, y\rangle=\langle Bx, y\rangle=\langle x, By\rangle$$

$$=\langle x, AA^{-1}By\rangle=\langle Ax, A^{-1}By\rangle=g(x, A^{-1}By).$$

Thus $A^{-1}B$ has real eigenvalues and mutually orthogonal eigenspaces with respect to g.

THEOREM 5.4.   *Let A and B be symmetric endomorphisms of $\mathbf{R}^n$ taken as above. We define a curvature tensor R_ε by $R_\varepsilon=A\wedge(A+\varepsilon B)$ for $\varepsilon\in\mathbf{R}$. Then for sufficiently small ε $(\neq 0)$ the curvature tensor R_ε does not at all have curvature-invariant subspaces of dimension not less than 3.*

Proof.   We fix an integer r with $3\leq r\leq n-1$. First we shall classify r-dimensional curvature-invariant subspaces with respect to $R_0=A\wedge A$. By Remark 2 of Proposition 5.2, each r-dimensional curvature-invariant subspace with respect to R_0 is invariant by A. Since each eigenspace of A is of dimension 1, an r-dimensional invariant subspace by A is linearly spanned by $e_{\iota_1}, \cdots, e_{\iota_r}$, $\iota_1<\cdots<\iota_r$, where $\{e_1, \cdots, e_n\}$ denotes the natural basis of $\mathbf{R}^n$. Consequently each r-dimensional curvature-invariant subspace is given by $\{e_{\iota_1}, \cdots, e_{\iota_r}\}_R$ for some $i_1<\cdots<i_r$. We note that for sufficiently small ε, curvature-invariant subspaces with respect to $R_\varepsilon=A\wedge(A+\varepsilon B)$ lie near those with respect to $R_0=A\wedge A$ if they exist. Applying Theorem 4.4, we can easily show this fact.

Since $A+\varepsilon B$ is a positive definite symmetric endomorphism for sufficiently small ε, we can apply Proposition 5.3 to the curvature tensor R_ε. We shall show that if non-zero ε is sufficiently small, neither of three cases in Proposition 5.3 holds for any r-dimensional subspace V.

ASSERTION 1. *The case* (1) *in Proposition* 5.3 *does not hold for any r-dimensional subspace V.*

Proof of Assertion 1. If V is an invariant subspace by both A and $A+\varepsilon B$, then V is invariant by B. An r-dimensional invariant subspace V by A is given by $\{e_{i_1}, \cdots, e_{i_r}\}_R$ for some $i_1 < \cdots < i_r$. On the other hand, since $Be_{i_1} = \sum_{j=1}^n b_{ji_1}e_j$ and $b_{ji_1} \neq 0$ for $j \neq i_1, \cdots, i_r$ owing to (5.2)-(i), Be_{i_1} is not contained in V. Therefore the case (1) does not hold.

ASSERTION 2. *The case* (3) *in Proposition* 5.3 *does not hold for any r-dimensional subspace V.*

Proof of Assertion 2. Suppose that there exist a positive number λ and $r-1$-dimensional subspace V' in V such that $(A+\varepsilon B)x = \lambda Ax$ for any $x \in V'$. Accordingly, we have $Bx = ((\lambda-1)/\varepsilon)Ax$ and hence $A^{-1}Bx = ((\lambda-1)/\varepsilon)x$ for $x \in V'$. Therefore V' is a subspace of the eigenspace of $A^{-1}B$ with eigenvalue $(\lambda-1)/\varepsilon$. Since $\dim V' \geq 2$, it is contrary to (5.2)-(ii). Therefore the case (3) does not hold.

ASSERTION 3. *The case* (2) *in Proposition* 5.3 *does not hold for any r-dimensional subspace V if ε is sufficiently small.*

Proof of Assertion 3. First recalling section 4, we prepare some notations. Let $\mathrm{End}(F_r) = F_r^* \otimes F_r$ be the vector bundle consisting of endomorphisms of the natural vector bundle F_r over the Grassmann manifold $G_r(\mathbf{R}^n)$ and $\Gamma\mathrm{End}(F_r)$ be the space of sections of $\mathrm{End}(F_r)$. For $C \in \mathrm{End}(\mathbf{R}^n)$ and $V \in G_r(\mathbf{R}^n)$, we define an endomorphism $\phi(C)_V$ of $(F_r)_V \cong V$ by restricting C to V, i.e.,

$$\phi(C)_V(x) = \text{the } V\text{-component of } Cx \text{ for } x \in V,$$

where we take the V-component with respect to the orthogonal decomposition $\mathbf{R}^n = V + V^\perp$. Thus we obtain a linear map $\phi : \mathrm{End}(\mathbf{R}^n) \to \Gamma\mathrm{End}(F_r)$. For symmetric endomorphisms A and B in Theorem 5.4, we define a subset $C(A\,;B)$ of $G_r(\mathbf{R}^n)$ as follows:

$$C(A\,;B) = \{V \in G_r(\mathbf{R}^n) \mid \text{there exists a number } \mu \text{ such that } \phi(B)_V = \mu\phi(A)_V\}.$$

It is easily seen that $C(A\,;B)$ is a closed subset in $G_r(\mathbf{R}^n)$.

Suppose that for $V \in G_r(\mathbf{R}^n)$, (2) in Proposition 5.3 holds with respect to $R_\varepsilon = A \wedge (A+\varepsilon B)$. That is, there exists a positive number λ such that $(A+\varepsilon B)^a = \lambda A^a$ and $(A+\varepsilon B)^b = -\lambda A^b$ on V. Accordingly, it holds that $B^a = ((\lambda-1)/\varepsilon)A^a$

on V, i.e., $\phi(B)_V = ((\lambda-1)/\varepsilon)\phi(A)_V$. Therefore V belongs to $C(A;B)$. On the other hand, we shall show that if ε is sufficiently small, curvature-invariant subspaces with respect to R_ε must belong to $G_r(\boldsymbol{R}^n) - C(A;B)$. Let $W \in G_r(\boldsymbol{R}^n)$ be a curvature-invariant subspace with respect to $R_0 = A \wedge A$. Then W does not belong to $C(A;B)$. In fact W is given by $\{e_{i_1}, \cdots, e_{i_r}\}_R$ for some $i_1 < \cdots < i_r$. From our assumptions of A and B it follows that $\phi(A)_W e_{i_1} = a_{i_1} e_{i_1}$ and $\phi(B)_W e_{i_1}$ $= b_{i_1 i_1} e_{i_1} + b_{i_2 i_1} e_{i_2} + \cdots + b_{i_r i_1} e_{i_r}$. Since $b_{i_2 i_1} \neq 0$, there does not exist μ such that $\phi(B)_W = \mu\phi(A)_W$. We recall that for sufficiently small ε, curvature-invariant subspaces with respect to R_ε must lie near those with respect to R_0 if they exist. Noticing that $G_r(\boldsymbol{R}^n) - C(A;B)$ is an open subset in $G_r(\boldsymbol{R}^n)$, we obtain our claim.   □

6. Totally geodesic submanifolds of hypersurfaces in a Euclidean space

Let $\widetilde{M}^{n+1}(\tilde{c})$ be an $n+1$-dimensional real space form, i.e., a simply connected complete Riemannian manifold with constant sectional curvature $\tilde{c}$. Let M be a hypersurface in $\widetilde{M}^{n+1}(\tilde{c})$. Here we assume that M is a regular submanifold of $\widetilde{M}^{n+1}(\tilde{c})$. In this section we shall study totally geodesic submanifolds of M applying Proposition 5.2. In particular, we shall classify totally geodesic submanifolds of hypersurfaces in $\boldsymbol{R}^{n+1}$ defined by homogeneous polynomials of degrees 2 and 3.

We fix some notation. We denote $\widetilde{\nabla}$ and ∇ the Riemannian connections of $\widetilde{M}^{n+1}(\tilde{c})$ and M, respectively. We denote by ξ, α and A a unit normal vector field, the second fundamental form, and the shape operator of M, respectively.

PROPOSITION 6.1. *Let M be a hypersurface in $\widetilde{M}^{n+1}(\tilde{c})$ and S be an r-dimensional totally geodesic connected submanifold of M. We assume that $\dim A(T_x S)$ ≥ 2 at any point $x \in S$. Then one of the following holds:*

(1) There exists an $r+1$-dimensional totally geodesic submanifold $\widetilde{M}^{r+1}(\tilde{c})$ of $\widetilde{M}^{n+1}(\tilde{c})$ which contains S such that an open subset in $\widetilde{M}^{r+1}(\tilde{c}) \cap M$ containing S is an r-dimensional regular submanifold of M and S is its open submanifold.

(2) There exists an r-dimensional totally geodesic submanifold $\widetilde{M}^r(\tilde{c})$ of $\widetilde{M}^{n+1}(\tilde{c})$ in which S is an open submanifold.

Proof. From the equation of Gauss, it follows that at $x \in M$ the curvature tensor R of M is given by

$$R(X, Y)Z = \tilde{c}\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\} + \{\langle AY, Z\rangle AX - \langle AX, Z\rangle AY\}$$

$X, Y, Z \in T_x M$. Since S is a totally geodesic submanifold of M, the tangent space $T_x S$ is a curvature-invariant subspace of $T_x M$ at any point $x \in S$. Since $\dim A(T_x S) \geq 2$, by Proposition 5.2 one of the following holds for $T_x S$:
  (1) $T_x S$ is invariant by A, i.e., $A(T_x S) \subset T_x S$.
  (2) $T_x S$ is a null-subspace with respect to A, i.e., $\langle AX, Y\rangle = 0$ for any $X, Y \in T_x S$.

We define subsets S_1 and S_2 in S as follows:

$$S_1 = \{x \in S \mid T_x S \text{ is invariant by } A\}$$

$$S_2 = \{x \in S \mid T_x S \text{ is a null-subspace with respect to } A\}.$$

Then we have $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$ (empty). Clearly, S_1 and S_2 are both closed subsets in S and hence open. Since S is connected, we have $S = S_1$ or $S = S_2$.

Let $T\widetilde{M}|_S$ be the vector bundle over S induced by the tangent bundle $T\widetilde{M}$ of $\widetilde{M}$. Then we have the orthogonal decomposition:

$$T\widetilde{M}|_S = TS + T^\perp S + \boldsymbol{R} \cdot \xi,$$

where $T^\perp S$ denotes the orthogonal complement of TS in $TM|_S$. We have the following formulas: for $X, Y \in \Gamma TS$

$$(6.1) \qquad \widetilde{\nabla}_X Y = \nabla_X Y + \langle \alpha(X, Y), \xi \rangle \xi$$

$$(6.2) \qquad \widetilde{\nabla}_X \xi = -AX.$$

We note that $\nabla_X Y \in \Gamma TS$, since S is a totally geodesic submanifold of M.

We consider the case $S = S_1$. TS is an invariant subbundle of TM by A. This, together with (6.2), implies that $TS + \boldsymbol{R} \cdot \xi$ is a parallel subbundle of $T\widetilde{M}|_S$ with respect to $\widetilde{\nabla}$. Therefore by the well-known reduction theorem (J. Erbacher [6]), it follows that there exists an $r+1$-dimensional totally geodesic submanifold $\widetilde{M}^{r+1}(\tilde{c})$ of $\widetilde{M}^{n+1}(\tilde{c})$ such that S is a submanifold of $\widetilde{M}^{r+1}(\tilde{c})$. In particular, we have $S \subset \widetilde{M}^{r+1}(\tilde{c}) \cap M$. Since $T_x \widetilde{M}^{r+1} = T_x S + \boldsymbol{R} \cdot \xi_x$ at an arbitrary point $x \in S$, $\widetilde{M}^{r+1} \cap M$ is an r-dimensional regular submanifold of $\widetilde{M}^{r+1}$ in a neighborhood of S and S is its open submanifold. Since both M and $\widetilde{M}^{r+1}(\tilde{c})$ are regular submanifolds of $\widetilde{M}^{n+1}(\tilde{c})$, an open subset in $\widetilde{M}^{r+1} \cap M$ containing S is also a regular submanifold of M.

If $S = S_2$, by (6.1), S is a totally geodesic submanifold of $\widetilde{M}^{n+1}(\tilde{c})$. Hence we obtain our assertion (2) in Proposition 6.1. $\qquad\square$

Now we shall consider the converse of Proposition 6.1. Let S be an r-dimensional submanifold of M. If S is a totally geodesic submanifold of $\widetilde{M}^{n+1}(\tilde{c})$, evidently it is a totally geodesic submanifold of M. Corresponding to Proposition 6.1 (1), the following is easily seen.

LEMMA 6.2. *Let $\widetilde{M}^{r+1}(\tilde{c})$ be an $r+1$-dimensional totally geodesic submanifold of $\widetilde{M}^{n+1}(\tilde{c})$. Suppose that $S = \widetilde{M}^{r+1} \cap M$ is an r-dimensional submanifold of $\widetilde{M}^{n+1}(\tilde{c})$ (and hence it is also a submanifold of both $\widetilde{M}^{r+1}$ and M). Let ξ be a unit normal vector field of M. If $\xi_x \in T_x \widetilde{M}^{r+1}$ at an arbitrary point $x \in S$, then S is a totally geodesic submanifold of M.*

Let $\boldsymbol{R}^{n+1}$ be an $n+1$-dimensional real vector space with standard inner product $\langle\,,\,\rangle$. Let F be a symmetric multilinear form of degree m on $\boldsymbol{R}^{n+1}$ and f be a homogeneous polynomial defined by $f(x)=F(x,\cdots,x)$ for $x\in\boldsymbol{R}^{n+1}$. We set a subset M of $\boldsymbol{R}^{n+1}$ by $M=\{x\in\boldsymbol{R}^{n+1}\,|\,f(x)=1\}$. As usual, we identify the tangent space $T_x\boldsymbol{R}^{n+1}$ of $\boldsymbol{R}^{n+1}$ at x with $\boldsymbol{R}^{n+1}$. Under this identification, we have $df_x(v)=mF(x,\cdots,x,v)$ for $v\in T_x\boldsymbol{R}^{n+1}\cong\boldsymbol{R}^{n+1}$. Since $df_x(x)=m\neq0$ at $x\in M$, M is a smooth hypersurface of $\boldsymbol{R}^{n+1}$ if M is not empty. Easily we see that M is a real analytic and complete Riemannian manifold. We try to classify totally geodesic submanifolds of such hypersurfaces.

We consider two types of totally geodesic submanifolds:

Type I. We take an $r+1$-dimensional linear subspace W of $\boldsymbol{R}^{n+1}$ and a vector $C\in W^{\perp}$, where $W^{\perp}$ denotes the orthogonal complement of W in $\boldsymbol{R}^{n+1}$. We denote by $C+W$ the affine subspace defined by $C+w$, $w\in W$. Let S be one of connected components of $(C+W)\cap M$. Now we assume that

$$(6.3) \qquad\qquad F(x,\cdots,x,W^{\perp})=0 \quad\text{at any point } x\in S.$$

Type II. We take a point $x\in M$ and an r-dimensional linear subspace W. Now we assume that

$$(6.4)\qquad\qquad F(\overbrace{x,\cdots,x}^{k},\overbrace{W,\cdots,W}^{m-k})=0 \quad\text{for } 0\leq k\leq m-1.$$

PROPOSITION 6.3.   (1) *Let S be a subset of M given in Type* I *above. Then S is an r-dimensional totally geodesic submanifold of M.*

(2) *Let x and W be a point of M and an r-dimensional subspace given in Type* II, *respectively. Then the affine subspace $x+W$ is contained in M and hence it is an r-dimensional totally geodesic submanifold of M.*

Totally geodesic submanifolds in both cases are maximal in the sense of Theorem 3.5.

Proof. (1) First we shall show that S is a smooth hypersurface in an $r+1$-dimensional affine subspace $C+W$. Let x be an arbitrary point of S. We can identify the tangent space $T_x(C+W)$ with W. If $df_x(w)=mF(x,\cdots,x,w)$ $=0$ for all $w\in T_x(C+W)\cong W$, then the condition (6.3) implies that $F(x,\cdots x,v)$ $=0$ for all $v\in\boldsymbol{R}^{n+1}$. It contradicts to $F(x,\cdots,x)=1$. Therefore there exists a vector $w\in T_x(C+W)\cong W$ such that $df_x(w)\neq0$. Hence S is a smooth hypersurface in $C+W$.

We note that the tangent space T_xM is given by

$$T_xM=\{v\in T_x\boldsymbol{R}^{n+1}\cong\boldsymbol{R}^{n+1}\,|\,df_x(v)=mF(x,\cdots,x,v)=0\}.$$

From (6.3), it follows that $W^{\perp}\subset T_xM$ at every point $x\in S$. Let ξ be the unit normal vector field of M in $\boldsymbol{R}^{n+1}$. Then at every point $x\in S$, ξ_x is orthogonal to $W^{\perp}$. Consequently we have $\xi_x\in W\cong T_x(C+W)$. By Lemma 6.2, S is a totally geodesic submanifold of M.

(2) For an arbitrary vector $w \in W$, the condition (6.4) implies

$$F(x+w, \cdots, x+w) = \sum_{k=0}^{m-1} \binom{m}{k} F(\overbrace{x, \cdots, x}^{k}, \overbrace{w, \cdots, w}^{m-k}) + F(x, \cdots, x) = 1.$$

Therefore the affine subspace $x+W$ is contained in M.

Since totally geodesic submanifolds given above are complete Riemannian manifold and imbedded in M, they are maximal in the sense of Theorem 3.5. $\square$

Remark. Let ϕ be a linear transformation of $\boldsymbol{R}^{n+1}$. We define a multilinear form F' by $F' = \phi \cdot F$, i.e., $F'(v_1, \cdots, v_m) = F(\phi^{-1} v_1, \cdots, \phi^{-1} v_m)$ and denote by M' the hypersurface of $\boldsymbol{R}^{n+1}$ defined by F'. Then we have $\phi(M) = M'$. Moreover if an affine subspace $x+W$ is contained in M, then the affine subspace $\phi(x+W)$ is contained in M' and vice versa. The existence of totally geodesic submanifolds of Type II is a problem which belongs to affine differential geometry.

We add one more assumption to F. We say that a symmetric multilinear form F is *non-singular* when $F(x, \cdots, x, v) = 0$ for $v \in \boldsymbol{R}^{n+1}$ only if $x=0$.

THEOREM 6.4. *Let F be a non-singular symmetric multilinear form of degree m on $\boldsymbol{R}^{n+1}$ and M be a hypersurface defined by F. Let S be an r-dimensional $(2 \le r \le n-1)$ connected Riemannian manifold and φ be a totally geodesic immersion of S into M. Then there exists an r-dimensional totally geodesic submanifold $\tilde{S}$ obtained by Proposition 6.3 such that $\varphi(S) \subset \tilde{S}$. That is, maximal totally geodesic submanifolds are exhausted by ones given in Proposition 6.3. In particular, a maximal totally geodesic submanifold of M is imbedded and closed in M.*

Proof. We note that S is a real analytic Riemannian manifold and φ is a real analytic immersion. Let $\varphi^{-1}TM$ be an induced bundle by φ from the tangent bundle of M. Then we have an orthogonal decomposition: $\varphi^{-1}TM = TS + T^{\perp}S$. We denote by A the shape operator of M in $\boldsymbol{R}^{n+1}$. We view it as a section of the endomorphism bundle $\mathrm{End}(\varphi^{-1}TM)$. We put $k = \max_{x \in S} \dim A(T_x S)$. Then the following three cases may occur:

Case 1. $k=0$.
Case 2. $k=1$.
Case 3. $k \ge 2$.

ASSERTION 1. *Case 1 does not occur.*

Proof of Assertion 1. Since our discussion below is local, we don't distinguish S and $\varphi(S)$ and may assume that S is an imbedded submanifold of M. We shall show that at a fixed point $x \in S$, the following holds:

(6.5) $F(u, \cdots, u, v) = 0$ for any $u \in T_x S$ and $v \in \boldsymbol{R}^{n+1}$.

Therefore it contradicts that F is non-singular. Now we shall prove (6.5). In our case S is a totally geodesic submanifold of $\boldsymbol{R}^{n+1}$. For $x \in S$ and $u \in T_x S$,

we put $\gamma(t)=x+tu$. Then there exists a positive number ε such that $\gamma(t)\in S$ for $|t|<\varepsilon$. Since $A(TS)=0$ on S, the tangent bundle TM is a parallel subbundle of TR^{n+1} along γ with respect to $\tilde{\nabla}$. Under a usual identification, we see that if $v\in T_xM$, then $v\in T_{\gamma(t)}M$ for $|t|<\varepsilon$. Therefore we have

$$0=\frac{1}{m}df_{\gamma(t)}(v)=F(\gamma(t),\ \cdots,\ \gamma(t),\ v)$$

$$=F(x,\ \cdots,\ x,\ v)+(m-1)tF(x,\ \cdots,\ x,\ u,\ v)+\ \cdots\ +t^{m-1}F(u,\ \cdots,\ u,\ v).$$

Hence it follows that $F(u,\ \cdots,\ u,\ v)=0$ for any $v\in T_xM$. Since $\gamma(t)\in S\subset M$ for $|t|<\varepsilon$, we have

$$1=F(\gamma(t),\ \cdots,\ \gamma(t))$$

$$=1+mtF(x,\ \cdots,\ x,\ u)+\ \cdots\ +mt^{m-1}F(u,\ \cdots,\ u,\ x)+t^mF(u,\ \cdots,\ u).$$

Hence it follows that $F(u,\ \cdots,\ u,\ x)=0$. Consequently we obtain the formula (6.5).

ASSERTION 2. *In Case 2 at any point $x\in S$, T_xS is a null-subspace with respect to A i.e., $\langle AX,\ Y\rangle=0$ for any $X,\ Y\in T_xS$. Therefore S is a totally geodesic submanifold of R^{n+1}.*

Proof of Assertion 2. Suppose that at a point $x\in S$, T_xS is not a null-subspace with respect to A. Then there exists a neighborhood S' of x in S such that at each $y\in S'$, T_yS' is not a null-subspace with respect to A. We define a subspace $\mathfrak{D}_y$ of T_yS' at $y\in S'$ by $\mathfrak{D}_y=\{X\in T_yS'\,|\,AX=0\}$. Then $\mathfrak{D}$ is a distribution in S' with codimension 1. $\mathfrak{D}$ is exactly a relative nullity distribution when we view S' as a submanifold of R^{n+1}. Therefore $\mathfrak{D}$ is involutive and each leaf of $\mathfrak{D}$ is a totally geodesic submanifold of S'. We denote by N one of leaves of $\mathfrak{D}$. Then N is a totally geodesic submanifold of M which satisfies $A(T_yN)=0$ at any point $y\in N$. By the proof of Assertion 1, we see that Assertion 1 is also true when S is a 1-dimensional totally geodesic submanifold of M, i.e., a geodesic. Therefore such a submanifold N does not exist. It's a contradiction. Hence our assertion has been proved.

Now we consider Case 3. There exists an open subset S' in S such that $\dim A(T_xS)\geq 2$ at each point $x\in S'$. If there exists an r-dimensional totally geodesic submanifold $\tilde{S}$ obtained by Proposition 6.3 such that $\varphi(S')\subset\tilde{S}$, then by the real analyticity of φ it holds that $\varphi(S)\subset\tilde{S}$. Therefore we may discuss, assuming that S is an imbedded submanifold of M and $\dim A(T_xS)\geq 2$ at each point $x\in S$. By Proposition 6.1, one of the following holds:

Case 3-i. There exists an $r+1$-dimensional affine subspace $\tilde{M}^{r+1}$ which contains S such that an open subset in $\tilde{M}^{r+1}\cap M$ containing S is an r-dimensional regular submanifold of M and S is its open submanifold.

Case 3-ii. There exists an r-dimensional affine subspace $\tilde{M}^r$ in which S is an open submanifold.

Now we consider Case 3-i. We note that by the proof of Proposition 6.1 $\xi_x \in T_x \tilde{M}^{r+1}$ at any point $x \in S$. Let $\tilde{S}$ be a connected component of $\tilde{M}^{r+1} \cap M$ which contains S.

ASSERTION 3. $\tilde{S}$ is an r-dimensional regular submanifold of M and moreover $\tilde{S}$ is a complete totally geodesic submanifold of M.

Proof of Assertion 3. We fix a point $p \in S$. Applying Corollary 3.7 to $T_p S$, we see that there is a complete totally geodesic submanifold $\bar{S}$ immersed in M which contains S as an open submanifold. We denote by ϕ the totally geodesic immersion of $\bar{S}$ into M. By the real analyticity of ϕ, we have $\phi(\bar{S}) \subset \tilde{M}^{r+1}$ and hence $\phi(\bar{S}) \subset \tilde{S}$. In the same manner, by the real analyticity of ξ, it holds that $\xi_{\phi(q)} \in T_{\phi(q)} \tilde{M}^{r+1}$ at any point $q \in \bar{S}$. We put $\bar{\bar{S}} = \phi(\bar{S})$. We shall prove that $\bar{\bar{S}}$ is an r-dimensional regular submanifold of M. We denote by $\tilde{f}$ the polynomial function on $\tilde{M}^{r+1}$ obtained by restricting $f(x) = F(x, \cdots, x)$ to $\tilde{M}^{r+1}$. Then we have $\tilde{M}^{r+1} \cap M = \{x \in \tilde{M}^{r+1} \mid \tilde{f}(x) = 1\}$. Since $\xi_{\phi(q)} \in T_{\phi(q)} \tilde{M}^{r+1}$ at an arbitrary point $q \in \bar{S}$, $d\tilde{f}_{\phi(q)} \neq 0$ in the cotangent space $T^*_{\phi(q)} \tilde{M}^{r+1}$. Therefore there exists a neighborhood U of $\phi(q)$ in $\tilde{M}^{r+1}$ such that $M \cap U = \{x \in U \mid \tilde{f}(x) = 1\}$ is an r-dimensional regular submanifold of $\tilde{M}^{r+1}$. Hence we can take a neighborhood $\bar{U}$ of q in $\bar{S}$ such that ϕ is a diffeomorphism of $\bar{U}$ onto $M \cap V$, where V is an adequate neighborhood of $\phi(q)$ in $\tilde{M}^{r+1}$. From this, it follows that $\bar{\bar{S}} \cap V = M \cap V$. Consequently it has been proved that $\bar{\bar{S}}$ is an r-dimensional regular submanifold of $\tilde{M}^{r+1}$ and hence is also a regular submanifold of M. Moreover ϕ is a Riemannian covering map of $\bar{S}$ onto $\bar{\bar{S}}$. Since $\bar{S}$ is a complete Riemannian manifold, so is $\bar{\bar{S}}$. In our proof above, it has been also shown that $\bar{\bar{S}}$ is an open subset of $\tilde{S}$.

Finally we shall show that $\bar{\bar{S}}$ is a closed subset of $\tilde{S}$. If this is shown, we have $\bar{\bar{S}} = \tilde{S}$ and obtain our assertion. Let $\{p_j\}$ and p be a sequence of points in $\bar{\bar{S}}$ and a point of $\tilde{S}$ which satisfy $\lim_{j \to \infty} p_j = p$ in $\tilde{S}$. Since $\xi_{p_j} \in T_{p_j} \tilde{M}^{r+1}$ for any j, then we have $\xi_p \in T_p \tilde{M}^{r+1}$. Therefore there exists a neighborhood V of p in $\tilde{M}^{r+1}$ such that $M \cap V$ is an r-dimensional regular submanifold of $\tilde{M}^{r+1}$. We choose p_j which belongs to a normal coordinate neighborhood $\mathcal{V}$ of p in $M \cap V$. Let γ be a geodesic from p_j to p in $\mathcal{V}$. We parametrize γ such that $\gamma(0) = p_j$ and $\gamma(1) = p$. Since $\bar{\bar{S}} \cap V$ is an open submanifold of $M \cap V$, there exists an open interval $I \subset [0, 1]$ containing 0 such that $\gamma(I)$ is contained in $\bar{\bar{S}}$ and $\gamma|_I$ is a geodesic of $\bar{\bar{S}}$. Since $\bar{\bar{S}}$ is a complete Riemannian manifold, we have $I = [0, 1]$ and hence $p = \gamma(1)$ belongs to $\bar{\bar{S}}$. Consequently it has been proved that $\bar{\bar{S}}$ is a closed subset of $\tilde{S}$.

We continue our discussion on Case 3-i. We denote by W an $r+1$-dimensional linear subspace of $\mathbf{R}^{n+1}$ which corresponds to the tangent space $T_x \tilde{M}^{r+1}$ at $x \in \tilde{M}^{r+1}$ under the usual identification $T_x \mathbf{R}^{n+1} \cong \mathbf{R}^{n+1}$ and denote by $W^\perp$ the

orthogonal complement of W in R^{n+1}. Let C be a vector of $W^{\perp}$ which is the image of $\tilde{M}^{r+1}$ by the orthogonal projection of R^{n+1} onto $W^{\perp}$. Then the affine subspace $\tilde{M}^{r+1}$ is given by $C+W$. As has been shown in Assertion 3, for an arbitrary point $x \in \tilde{S}$, ξ_x belongs to $T_x\tilde{M}^{r+1}$ and is orthogonal to $W^{\perp}$. Therefore we have $F(x, \cdots, x, W^{\perp})=(1/m)df_x(W^{\perp})=0$ and obtain the formula (6.3).

Finally we consider Case 2 and Case 3-ii. In each case, there exists an r-dimensional affine subspace $\tilde{M}^r$ in which S is an open submanifold. We denote by W an r-dimensional linear subspace of R^{n+1} which corresponds to $T_xS=T_x\tilde{M}^r$. For an arbitrary vector $w \in W$, we take a straight line $\gamma(t)=x+tw$. For sufficiently small t, $\gamma(t)$ belong to S and hence to M. Therefore we have

$$1=F(\gamma(t), \cdots, \gamma(t))$$
$$=F(x, \cdots, x)+ \cdots +\binom{m}{k}t^{m-k}F(\overbrace{x, \cdots, x}^{k}, \overbrace{w, \cdots, w}^{m-k})+ \cdots t^mF(w, \cdots, w).$$

Hence we have $F(\overbrace{x, \cdots, x}^{k}, \overbrace{w, \cdots, w}^{m-k})=0$ for $0 \leq k \leq m-1$. Thus $\gamma(t)$ for all $t \in R$ belong to M and hence the affine subspace $x+W=\tilde{M}^r$ is contained in M. $\square$

In the rest of this section we shall classify totally geodesic submanifolds of hypersurfaces defined by symmetric bilinear forms and trilinear forms on R^{n+1}. Let F be a non-singular symmetric bilinear form on R^{n+1}. We use the same symbol F for the symmetric endomorphism of R^{n+1} which corresponds to the symmetric bilinear form by the standard inner product $\langle , \rangle$. Let M be a hypersurface defined by F. Then the following holds.

THEOREM 6.5. *If S is an r-dimensional $(2 \leq r \leq n-1)$ maximal totally geodesic submanifold of M, then S is one of the following:*

(1) *a connected component of $W \cap M$, where W is an $r+1$-dimensional invariant subspace by F,*

(2) *an r-dimensional affine subspace of R^{n+1} which is contained in M.*

Proof. It is sufficient to consider Case 3-i in the proof of Theorem 6.4. Namely there exist an $r+1$-dimensional linear subspace W and a vector $C \in W^{\perp}$ and S is a connected component of $(C+W) \cap M$ which satisfies $F(x, v)=0$ for any $x \in S$ and any $v \in W^{\perp}$. Let v be an arbitrary vector of $W^{\perp}$. We view F and v as parallel tensor fields with respect to the Riemannian connection $\tilde{\nabla}$ of R^{n+1}. For vector fields U_1 and U_2 over S, we have

$$0=(\tilde{\nabla}_{U_1}F)(x, v)=U_1F(x, v)-F(U_1, v)$$
$$=-F(U_1, v)$$

and

$$0=(\tilde{\nabla}_{U_2}F)(U_1, v)=U_2F(U_1, v)-F(\tilde{\nabla}_{U_2}U_1, v)$$
$$=-F(\alpha(U_2, U_1), v).$$

From the above computations, it follows that at point $x \in S$

$$F(x, v) = 0,$$

$$F(u, v) = 0 \quad \text{for any} \quad u \in T_x S,$$

$$F(\alpha(u_1, u_2), v) = 0 \quad \text{for any} \quad u_1, u_2 \in T_x S.$$

We may assume that α restricted to $T_x S$ is not zero. Therefore there are u_1 and u_2 in $T_x S$ such that $\alpha(u_1, u_2) \neq 0$. The above last equation implies that $F(\xi_x, v) = 0$. Since $W = T_x S + \mathbf{R} \xi_x$, we obtain $F(W, W^{\perp}) = 0$. Hence W is an invariant subspace by F. Since $F(x, W^{\perp}) = 0$, we have $F(C, W^{\perp}) = 0$. On the other hand, since $C \in W^{\perp}$, we have $F(C, W) = 0$. From the non-singularity of F, it follows that $C = 0$. Consequently S is a connected component of $W \cap M$. $\square$

By fundamental properties of symmetric bilinear forms we can show the following.

PROPOSITION 6.6. *In Theorem* 6.5, *we assume that F has the index ν. Then the dimension of an affine subspace contained in M is not greater than the minimum of ν and $n-\nu$. Let $\tilde{S}_1$ and $\tilde{S}_2$ be affine subspaces of the same dimension which are contained in M. Then there exists a linear transformation ϕ of $\mathbf{R}^{n+1}$ which preserves F such that $\phi(\tilde{S}_1) = \tilde{S}_2$.*

Examples. Let F be a symmetric bilinear form of $\mathbf{R}^{n+1}$ given by $F = \sum_{i=1}^{n+1} a_i e_i^* \otimes e_i^*$, where $\{e_1^*, \cdots, e_{n+1}^*\}$ denotes the dual basis of the natural basis $\{e_1, \cdots, e_{n+1}\}$ of $\mathbf{R}^{n+1}$. If every a_i is positive, the hypersurface M defined by F does not contain any affine subspace of positive dimension. In addition suppose that $a_1, \cdots, a_{n+1}$ are mutually distinct. Then the number of F-invariant $r+1$-dimensional subspaces is $\binom{n+1}{r+1}$. We put $M' = \{x = {}^t(x^1, \cdots, x^{n+1}) \in M \mid x^i \neq 0$ for any $i\}$. Then M' is an open submanifold of M. The Riemannian manifold M' does not have any $r(2 \leq r \leq n-1)$-dimensional totally geodesic submanifolds. In fact, the F-invariant subspace which contains a point $x \in M'$ is the whole space $\mathbf{R}^{n+1}$.

Next we consider the following symmetric trilinear form F on $\mathbf{R}^{n+1}$: $F = \sum_{i=1}^{n+1} e_i^* \otimes e_i^* \otimes e_i^*$. Then F is non-singular in our sense. In the rest of this section, we denote by Q^n the hypersurface of $\mathbf{R}^{n+1}$ defined by F, i.e.,

$$Q^n = \{x \in \mathbf{R}^{n+1} \mid F(x, x, x) = 1\}$$

$$= \left\{x = {}^t(x^1, \cdots, x^{n+1}) \in \mathbf{R}^{n+1} \,\middle|\, \sum_{i=1}^{n+1} (x^i)^3 = 1\right\}.$$

Let S_{n+1} be the symmetric group consisting of permutations of $\{1, 2, \cdots, n+1\}$. For $\sigma \in S_{n+1}$, we define a linear transformation T_σ of $\mathbf{R}^{n+1}$ as follows: $T_\sigma(e_i) = e_{\sigma(i)}$ $(i = 1, \cdots, n+1)$. Then T_σ is an orthogonal transformation which preserves

the above trilinear form F. Hence T_σ yields an isometry of Riemannian manifold Q^n.

We shall classify totally geodesic submanifolds of Q^n. First we present examples of totally geodesic submanifolds. For $2 \leq m \leq n-1$, Q^m can be viewed as a submanifold of Q^n, naturally, i.e.,

$$Q^m = \{x = {}^t(x^1, \cdots, x^{n+1}) \in Q^n \mid x^{m+2} = \cdots = x^{n+1} = 0\}.$$

Let W be an $m+1$-dimensional linear subspace of R^{n+1} spanned by $e_1, \cdots, e_{m+1}$. Its orthogonal complement $W^\perp$ is spanned by $e_{m+2}, \cdots, e_{n+1}$. Then evidently it holds that $Q^m = W \cap Q^n$ and $F(x, y, v) = 0$ for $x, y \in W$, $v \in W^\perp$. By Proposition 6.3 (1), Q^m is totally geodesic in Q^n. Another example is given as follows: Let $k_1, \cdots, k_l$ be integers which satisfy $2 \leq k_1 \leq \cdots \leq k_l$ and $k_1 + \cdots + k_l \leq n+1$. We put

$$S = \{x = {}^t(x^1, \cdots, x^{n+1}) \in Q^n \mid x^1 = \cdots = x^{k_1}, \ x^{k_1+1} = \cdots = x^{k_1+k_2},$$

$$\cdots, \ x^{k_1+\cdots+k_{l-1}+1} = \cdots = x^{k_1+\cdots+k_{l-1}+k_l}\}.$$

Then it is easily seen that S is connected and is a submanifold of Q^n with codimension $k_1 + \cdots + k_l - l$. We define cyclic permutations $\sigma_1, \cdots, \sigma_l$ by $\sigma_1 = (1, \cdots, k_1), \cdots, \sigma_l = (k_1 + \cdots + k_{l-1}+1, \cdots, k_1 + \cdots + k_{l-1}+k_l)$ and a permutation σ by $\sigma = \sigma_1 \cdots \sigma_l$. Then S is a fixed point set of Q^n by T_σ. Therefore S is totally geodesic in Q^n. For convenience, we denote this submanifold by $S_{k_1, \cdots, k_l}$ in this section. We shall show that these submanifolds and affine subspaces exhaust totally geodesic submanifolds of Q^n.

THEOREM 6.7. *If S is an r-dimensional maximal totally geodesic submanifold of Q^n, then S is congruent to one of the following totally geodesic submanifolds by an isometry T_σ, $\sigma \in S_{n+1}$:*

(1) $S_{k_1, \cdots, k_l}$ *in Q^m for some $m \leq n$, where $r = m + l - (k_1 + \cdots + k_l)$,*

(2) *an r-dimensional affine subspace of R^{n+1} which is contained in Q^n.*

Proof. Owing to Theorem 6.4, we consider Case 3-i in the proof of Theorem 6.4. Namely, there exist an $r+1$-dimensional linear subspace W and a vector $C \in W^\perp$ and S is a connected component of $(C+W) \cap Q^n$ which satisfies $F(x, x, v) = 0$ for any $x \in S$ and $v \in W^\perp$. Let ξ, A, α, and $\nabla \alpha$ denote the unit normal vector field, the shape operator, the second fundamental form and its covariant derivative of Q^n in R^{n+1}, respectively. We note that at generic points $x \in S$, α restricted to $T_x S$ do not vanish as it is known by the argument in the proof of Theorem 6.4.

To prove our Theorem, we prepare some lemmas.

LEMMA 1. *For $x \in S$, $v \in W^\perp$, and $u, u_1, u_2, u_3, u_4 \in T_x S$, the following formulas hold:*

(1)　　$F(x, u, v)=0$,

(2)　　$F(u_1, u_2, v)+F(x, \alpha(u_1, u_2), v)=0$,

(3)　　$F(\alpha(u_1, u_2), u_3, v)+F(\alpha(u_2, u_3), u_1, v)+F(\alpha(u_3, u_1), u_2, v)$

　　　　$+F(x, (\bar{\nabla}_{u_3}\alpha)(u_1, u_2), v)=0$,

(4)　　$F(\alpha(u_1, u_2), \alpha(u_3, u_4), v)+F(\alpha(u_2, u_3), \alpha(u_1, u_4), v)+F(\alpha(u_3, u_1), \alpha(u_2, u_4), v)$

　　　　$+F((\bar{\nabla}_{u_1}\alpha)(u_2, u_3), u_4, v)+F((\bar{\nabla}_{u_2}\alpha)(u_3, u_4), u_1, v)+F((\bar{\nabla}_{u_3}\alpha)(u_4, u_1), u_2, v)$

　　　　$+F((\bar{\nabla}_{u_4}\alpha)(u_1, u_2), u_3, v)-F(A_{\alpha(u_3, u_1)}u_4, u_2, v)-F(A_{\alpha(u_1, u_2)}u_4, u_3, v)$

　　　　$-F(A_{\alpha(u_2, u_3)}u_4, u_1, v)+F(x, (\bar{\nabla}_{u_4}\bar{\nabla}_{u_3}\alpha)(u_1, u_2), v)=0$.

Proof of Lemma 1. Differentiating $F(x, x, v)=0$, we obtain the first formula (1). Succeedingly differentiating one formula, we obtain next one.

We fix a point $x \in S$ at which α restricted to T_xS does not vanish. We define a subspace W' in $W^{\perp}$ as follows:

$$W'=\{v \in W^{\perp} \mid F(x, \xi_x, v)=0\}.$$

Then we have next lemma.

LEMMA 2. *The following formula holds*:

(6.6)　　　　$F(w_1, w_2, v)=0$ *for any* $v \in W'$ *and* $w_1, w_2 \in \mathbf{R}C+W$.

Proof of Lemma 2. For $u, u_1, u_2 \in T_xS$ and $v \in W'$, we have

　　(i)　$F(x, x, v)=0$,　　(ii)　$F(x, \xi_x, v)=0$,
　　(iii)　$F(x, u, v)=0$,　　(iv)　$F(u_1, u_2, v)=0$,
　　(v)　$F(u, \xi_x, v)=0$,　　(vi)　$F(\xi_x, \xi_x, v)=0$.

The formula (iii) is exactly Lemma 1 (1) and (iv) is obtained by Lemma 1 (2) and (ii). Since α restricted to T_xS does not vanish, by Lemma 1 (3) and (ii) we have (v). Similarly, by Lemma 1 (4) and (ii), (iv) and (v), the formula (vi) holds. Since $W=T_xS+\mathbf{R}\xi_x$, by (iv), (v), and (vi), we have $F(w_1, w_2, v)=0$ for $w_1, w_2 \in W$. Moreover by (ii) and (iii), we have $F(x, w, v)=0$ for $w \in W$. Since $x \in S$ is written as $x=C+w$ for some $w \in W$, Lemma 2 holds.

Our important step is to prove that the vector C is zero. We suppose that $C \neq 0$ and will show a contradiction. Moreover we assume that $\dim W'= \dim W^{\perp}-1$, where W' denotes the subspace of $W^{\perp}$ defined in Lemma 2 (however even if $W'=W^{\perp}$, arguments below will hold). To present $W^{\perp}$, we will choose an appropriate basis of $W^{\perp}$. Let $\{v_1, \cdots, v_q\}$ be a basis of $W^{\perp}$ ($\dim W^{\perp}=q$) such that $\{v_1, \cdots, v_{q-1}\}$ is a basis of W'. Placing column vectors $v_1, \cdots, v_q$ of $\mathbf{R}^{n+1}$, we obtain an $(n+1, q)$-matrix $P=(v_1 \cdots v_q)$. We assume that first $(q-1, q-1)$-matrix and first (q, q)-matrix in P are both non-singular. If not so, we take a suitable

permutation $\sigma \in S_{n+1}$ and transform W, C etc. by a linear isometry T_σ. Then our assumption will be satisfied. By change of bases, we may assume that the matrix $P=(v_1 \cdots v_q)$ has the form:

$$(6.7) \qquad \begin{pmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ v_1^q & \cdots & v_{q-1}^q & 1 \\ \cdot & & \cdot & v_q^{q+1} \\ \vdots & & \vdots & \vdots \\ v_1^{n+1} & \cdots & v_{q-1}^{n+1} & v_q^{n+1} \end{pmatrix}.$$

LEMMA 3. *In* (6.7), *for each* j $(j=1, \cdots, q-1)$, $v_j^q, v_j^{q+1}, \cdots, v_j^{n+1}$ *are zero except at most one and if* $v_j^i \neq 0$ $(i=q, \cdots, n+1)$, *then* $v_j^i=-1$. *Moreover* W' *is orthogonal to* $RC+W$.

Proof of Lemma 3. We prepare some notation. For a vector $v={}^t(v^1, v^2, \cdots, v^{n+1}) \in \mathbf{R}^{n+1}$, we denote by $m_+(v)$ and $m_-(v)$, the number of positive components v^i and the number of negative components v^i, respectively. We denote by F_v a symmetric bilinear form on $\mathbf{R}^{n+1}$ defined by $F_v(x, y)=F(x, y, v)$, i.e., $F_v=\sum_{i=1}^{n+1} v^i e_i^* \otimes e_i^*$. We note that the dimension of a null subspace with respect to F_v is not greater than $n+1-\max\{m_+(v), m_-(v)\}$.

ASSERTION 1. *For each* j $(j=1, \cdots, q-1)$, $v_j^q, v_j^{q+1}, \cdots, v_j^{n+1}$ *are zero except at most one and if* $v_j^i \neq 0$ $(i=q, \cdots, n+1)$, *then* v_j^i *is negative.*

Proof of Assertion 1. By Lemma 2, $RC+W$ is a null subspace with respect to F_v for $v \in W'$. Since $C \neq 0$, we have dim $RC+W=r+2=n+1-(q-1)$. We shall prove our assertion for $j=1$. The other cases are same. Suppose that there exists a positive number in $v_1^q, v_1^{q+1}, \cdots, v_1^{n+1}$. We assume that $v_1^i>0$. We choose sufficiently small positive numbers $\lambda_2, \cdots, \lambda_{q-1}$ such that $v_1^i+\lambda_2 v_2^i+ \cdots + \lambda_{q-1}v_{q-1}^i>0$. Put $v=v_1+\lambda_2 v_2+ \cdots +\lambda_{q-1}v_{q-1}$. Then we have $m_+(v) \geq q$. Therefore the dimension of a null subspace with respect to F_v is not greater than $n+1-q$. It is a contradiction. Next we suppose that there exist two negative numbers in $v_1^q, v_1^{q+1}, \cdots, v_1^{n+1}$. We assume that they are v_1^i and v_1^j $(q \leq i<j \leq n+1)$. Choosing sufficiently small positive numbers $\lambda_2, \cdots, \lambda_{q-1}$, we have $v_1^i-\lambda_2 v_2^i- \cdots -\lambda_{q-1}v_{q-1}^i<0$ and $v_1^j-\lambda_2 v_2^j- \cdots -\lambda_{q-1}v_{q-1}^j<0$. Put $v=v_1-\lambda_2 v_2- \cdots -\lambda_{q-1}v_{q-1}$. Then we have $m_-(v) \geq q$. Again it yields a contradiction. Hence our first assertion has been shown.

Now we take vectors w_i $(i=1, \cdots, r+1)$ as follows:

$$w_i = \begin{pmatrix} v_1^q v_q^{q+i} - v_1^{q+i} \\ v_2^q v_q^{q+i} - v_2^{q+i} \\ \vdots \\ v_{q-1}^q v_q^{q+i} - v_{q-1}^{q+i} \\ -v_q^{q+i} \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longleftarrow (q+i)\text{-th row.}$$

Then $\{w_1, \cdots, w_{r+1}\}$ is a basis of W. We take a vector $C' \in C+W$ which has the form $C' = {}^t(c'^1, \cdots, c'^q, 0, \cdots, 0)$. Since $C \neq 0$, we have $C' \neq 0$. Evidently $C'+W=C+W$.

ASSERTION 2. *There exist non-zero ones in* $v_q^{q+1}, \cdots, v_q^{n+1}$.

Proof of Assertion 2. Suppose that $v_q^{q+1} = \cdots = v_q^{n+1} = 0$. For $x \in S$, we have $0 = F(x, x, v_q) = (x^q)^2$ and hence $x^q = 0$. Since S is contained in $C'+W$, $0 = \langle v_q, x-C' \rangle = -c'^q$ and hence $c'^q = 0$. Since for j $(1 \leq j \leq q-1)$, $0 = F(C', C', v_j) = (c'^j)^2$, it holds that $c'^j = 0$. Consequently we have $C' = 0$. It is contrary to $C' \neq 0$. Therefore there exist non-zero ones in $v_q^{q+1}, \cdots, v_q^{n+1}$.

ASSERTION 3. *If* $v_j^i \neq 0$ *for* $j=1, \cdots, q-1$ *and* $i=q, \cdots, n+1$, *then* $v_j^i = -1$.

Proof of Assertion 3. For $i=1, \cdots, r+1$, we have

$$0 = F(w_i, w_i, v_j) = (v_j^q v_q^{q+i} - v_j^{q+i})^2 + v_j^q(-v_q^{q+i})^2 + v_j^{q+i}.$$

Suppose that $v_j^{q+i} \neq 0$ for some i $(i=1, \cdots, r+1)$. Then by Assertion 1, we have $v_j^q = 0$. This, with the formula above, implies that $v_j^{q+i}(v_j^{q+i}+1) = 0$ and hence $v_j^{q+i} = -1$. Suppose that $v_j^{q+i} = 0$ for all $i = 1, \cdots, r+1$. By Assertion 2, $v_q^{q+i} \neq 0$ for some i $(i=1, \cdots, r+1)$. From the formula above, it follows that $(v_q^{q+i})^2 v_j^q(v_j^q+1) = 0$. Therefore we have $v_j^q = 0$ or $v_j^q = -1$.

ASSERTION 4. *For* $j=1, \cdots, q-1$, *if* $v_j^q = 0$, *then* $c'^j = 0$ *and if* $v_j^q = -1$, *then* $c'^j = c'^q$. *In particular* W' *is orthogonal to* $RC'+W=RC+W$.

Proof of Assertion 4. From Lemma 2, it follows that $0 = F(C', C', v_j) = (c'^j)^2 + (c'^q)^2 v_j^q$. Therefore if $v_j^q = 0$, then $c'^j = 0$. Next suppose that $v_j^q = -1$. We

choose an integer i such that $v_q^{q+i} \neq 0$. Since $0 = F(C', w_i, v_j) = v_q^{q+i}(c'^q - c'^j)$, we obtain $c'^j = c'^q$.

Hence Lemma 3 has been completely proved.

We continue our proof of Theorem 6.7. We use the same notation as Lemma 3 and its proof. We put vectors w_i $(i=0, \cdots, r+1)$ as follows: $w_i = {}^t(-v_1^{q+i}, \cdots, -v_{q-1}^{q+i}, 0, \cdots, 0, 1, 0, \cdots, 0)$, where v_j^{q+i} $(j=1, \cdots, q-1)$ denote the components of vectors v_j given in Lemma 3 and $(q+i)$-th row is 1. Then $\{w_0, w_1, \cdots, w_{r+1}\}$ is a basis of $RC+W$. In fact, for each i, w_i is orthogonal to $v_1, \cdots, v_{q-1}$ and hence by Lemma 3, w_i belongs to $RC+W$. For $y \in RC+W$, we write $y = \sum_{i=0}^{r+1} y^i w_i$. Then S is a connected component of

$$(C+W) \cap Q^n = \left\{ y \in RC+W \mid y^0 + \sum_{i=1}^{r+1} v_q^{q+i} y^i = c'^q, \ \sum_{i=0}^{r+1} m_i (y^i)^3 = 1 \right\}.$$

Here m_i $(i=0, \cdots, r+1)$ denote positive integers defined by $m_i = 1 + $ the number of $\{j \mid v_j^{q+i} = -1\}$. Moreover it holds that $F(y, y, v_q) = 0$ for all $y \in S$, i.e., $(y^0)^2 + \sum_{i=1}^{r+1} v_q^{q+i}(y^i)^2 = 0$. From now on, we simply write v^i $(i=1, \cdots, r+1)$ and c for v_q^{q+i} and c'^q, respectively. We note that $c = c'^q$ is not zero. In fact, if $c'^q = 0$, then by Assertion 4 in the proof of Lemma 3, we have $C' = 0$ and it is contrary to our assumption. We define functions f, g, and h on $RC+W$ by $f(y) = \sum_{i=0}^{r+1} m_i (y^i)^3$, $g(y) = y^0 + \sum_{i=1}^{r+1} v^i y^i$, and $h(y) = (y^0)^2 + \sum_{i=1}^{r+1} v^i (y^i)^2$, respectively.

As is shown in Assertion 2, there exist non-zero ones in $v^1, \cdots, v^{r+1}$. We shall show that there exist negative ones in them. Suppose that $v^1, \cdots, v^{r+1}$ are all non-negative. Since $h(y) = 0$ for $y \in S$, we have $g(y) = 0$. It is contrary to $g(y) = c \neq 0$ for $y \in S$. Therefore there exist negative ones. So we assume that v^1 is negative. We define a projection π of $RC+W$ onto the subspace V linearly spanned by $w_2, \cdots, w_{r+1}$ as follows: $\pi(y) = \pi(\sum_{i=0}^{r+1} y^i w_i) = \sum_{i=2}^{r+1} y^i w_i$ for $y \in RC+W$. It is easily seen that π restricted to $(C+W) \cap Q^n$ is a homeomorphism onto V. Therefore $(C+W) \cap Q^n$ is connected and hence coincides with S. Next we shall show that there exist non-zero ones in $v^2, \cdots, v^{r+1}$. Suppose that $v^2 = \cdots = v^{r+1} = 0$. We put $y = c w_0 + y^2 w_2$, where $y^2 = \sqrt[3]{(1-m_0 c^3)/m_2}$. Then y belongs to S and $h(y) = c^2 \neq 0$. It is a contradiction. Therefore there exist non-zero ones in $v^2, \cdots, v^{r+1}$. So we assume that $v^2 \neq 0$.

We take y^0 and y^1 which satisfy $y^0 + v^1 y^1 = c$ and $m_0(y^0)^3 + m_1(y^1)^3 = 1$ and put $y = y^0 w_0 + y^1 w_1$. Then y belongs to S. Since $h \equiv 0$ on S, it holds that $dh = \lambda df + \mu dg$ in $T_y^*(RC+W)$ for some $\lambda, \mu \in R$. At $y = y^0 w_0 + y^1 w_1$, we have $df = 3m_0(y^0)^2 dy^0 + 3m_1(y^1)^2 dy^1$, $dg = dy^0 + \sum_{i=1}^{r+1} v^i dy^i$ and $dh = 2y^0 dy^0 + 2v^1 y^1 dy^1$. Comparing coefficients of dy^2 in both sides, we obtain $\mu = 0$. Since $dh = \lambda df$, it follows that $2y^0 = 3\lambda m_0(y^0)^2$ and $2v^1 y^1 = 3\lambda m_1(y^1)^2$. Since $h(y) = 0$, we have $0 = 2\{(y^0)^2 + v^1(y^1)^2\} = 3\lambda \{m_0(y^0)^3 + m_1(y^1)^3\} = 3\lambda$ and hence $\lambda = 0$. From this it follows that $dh = 0$ and hence $y^0 = y^1 = 0$. It is a contradiction. Consequently it has been proved that the vector C is zero.

By the argument above, it follows that S is a connected component of $W \cap Q^n$, where W is an $(r+1)$-dimensional linear subspace of $\boldsymbol{R}^{n+1}$. Since $F(x, x, v)=0$ for any $x \in S$ and $v \in W^{\perp}$, we have $F(w_1, w_2, v)=0$ for any $w_1, w_2 \in W$ and $v \in W^{\perp}$. By the same consideration as Lemma 3, we see that there exists a basis $\{v_1, \cdots, v_q\}$ of $W^{\perp}$ which has the following form (up to a linear isometry T_σ, $\sigma \in S_{n+1}$):

$$(v_1 \cdots v_q) = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ 0 & & 1 & \\ v_1^{q+1} & \cdots & & v_q^{q+1} \\ \vdots & & & \vdots \\ v_1^{n+1} & \cdots & & v_q^{n+1} \end{pmatrix}.$$

Here for each j $(j=1, \cdots, q)$, $v_j^{q+1}, \cdots, v_j^{n+1}$ are zero except at most one and if $v_j^i \neq 0$ $(i=q+1, \cdots, n+1)$, then $v_j^i=-1$. Therefore it has been proved that S is congruent to $S_{k_1, \cdots, k_l}$ in Q^m for some $m \leq n$. $\qquad \square$

On case (2) in Theorem 6.7 the following holds:

PROPOSITION 6.8. *The dimension of an affine subspace contained in Q^n is not greater than $n/2$. When $n=2q$, a q-dimensional affine subspace contained in Q^n is congruent to the following affine subspace S by a linear transformation T_σ, $\sigma \in S_{n+1}$:*

$$S = \{x = {}^t(x^1, \cdots, x^{n+1}) \in \boldsymbol{R}^{n+1} \mid x^i + x^{q+i}=0 \ (i=1, \cdots, q), \ x^{n+1}=1\}$$

Proof. Let S be an r-dimensional affine subspace of $\boldsymbol{R}^{n+1}$ which is contained in Q^n and W be an r-dimensional subspace which corresponds to the tangent space $T_x S$ at $x \in S$. Then the following formulas hold:

(i) $F(x, x, x)=1$,   (ii) $F(x, x, w_1)=0$,
(iii) $F(x, w_2, w_1)=0$,   (iv) $F(w_3, w_2, w_1)=0$

for $w_1, w_2, w_3 \in W$.

We use the same notation as Lemma 3 in Theorem 6.7. The formulas (ii), (iii), and (iv) imply that an $r+1$-dimensional linear subspace $\boldsymbol{R}x+W$ is a null subspace with respect to F_w for any $w \in W$. We may take a basis $\{v_1, \cdots, v_r\}$ of W which has the following form (up to a linear transformation T_σ, $\sigma \in S_{n+1}$):

(6.8)
$$(v_1 \cdots v_r) = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ 0 & & 1 & \\ v_1^{r+1} & \cdots & & v_r^{r+1} \\ \vdots & & & \vdots \\ v_1^{n+1} & \cdots & & v_r^{n+1} \end{pmatrix}.$$

We put $v=v_1+\cdots+v_r$. Since $m_+(v)\geqq r$, the dimension of a null subspace with respect to F_v is not greater than $n+1-r$. Therefore we have $r+1\leqq n+1-r$ and hence $r\leqq n/2$. Thus the first part of Proposition 6.8 has been proved.

From now on we assume that $n=2q$ and that S is a q-dimensional affine subspace of $\boldsymbol{R}^{n+1}$ which is contained in Q^n. Let $\{v_1, \cdots, v_q\}$ be a basis of W which has the same form as (6.8). By the same argument as Lemma 3 in Theorem 6.7, we see that for each j ($j=1, \cdots, q$), $v_j^{q+1}, \cdots, v_j^{n+1}$ are zero except at most one. Since $0=F(v_j, v_j, v_j)=1+\sum_{i=1}^{q+1}(v_j^{q+i})^3$, there exists only one v_j^{q+i} such that $v_j^{q+i}=-1$. For $k\neq j$, we have $0=F(v_k, v_k, v_j)=v_j^{q+i}(v_k^{q+i})^2=-(v_k^{q+i})^2$ and hence $v_k^{q+i}=0$. Therefore transforming by T_σ, $\sigma\in S_{n+1}$, if necessary, we can take a basis $\{v_1, \cdots, v_q\}$ of W which has the following form:

$$(v_1 \cdots v_q)=\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ -1 & & 0 \\ & \ddots & \\ 0 & & -1 \\ 0 & \cdots & 0 \end{pmatrix}.$$

Since $F(x, v_j, v_j)=0$, we have $x^j+x^{q+j}=0$ ($j=1, \cdots, q$). Therefore the vector ${}^t(0, \cdots, 0, 1)$ belongs to $S=x+W$. Consequently the second part of Proposition 6.8 has been proved.   □

7. Totally geodesic surfaces of 3-dimensional Lie groups with left invariant Riemannian metric

Let G be a Lie group with left invariant Riemannian metric and let $\mathfrak{g}$ be the corresponding Lie algebra with an inner product $\langle , \rangle$. We want to clarify various properties on totally geodesic submanifolds in G. In this paper we will study the simplest case, i.e., 2-dimensional submanifolds in 3-dimensional Lie groups. From now on we denote by G a 3-dimensional Lie group. We reduce the classification problem of totally geodesic surfaces of G to the following problem: Classify 2-dimensional subspaces in T_eG which are tangent spaces of totally geodesic surfaces in G. Here e denotes the identity of G. As usual, we identify T_eG with $\mathfrak{g}$. In this paper we specialize to the unimodular case and state our results.

First we recall the classification of 3-dimensional unimodular Lie groups following J. Milnor [10]. Let $\mathfrak{g}$ be a 3-dimensional unimodular Lie algebra with an inner product $\langle , \rangle$. Then there exists an orthonormal basis e_1, e_2, e_3 for which the bracket operation is given as follows:

(7.1) $[e_2, e_3]=\lambda_1 e_1, \quad [e_3, e_1]=\lambda_2 e_2, \quad [e_1, e_2]=\lambda_3 e_3.$

According to signs of λ_1, λ_2, λ_3, six kinds of Lie algebras are obtained. The following table is due to J. Milnor [10]:

Table 1.

Signs of λ_1, λ_2, λ_3	Associated Lie group
+, +, +	$SU(2)$ or $SO(3)$
+, +, −	$SL(2, \boldsymbol{R})$ or $O(1, 2)$
+, +, 0	$E(2)$
+, −, 0	$E(1, 1)$
+, 0, 0	Heisenberg group
0, 0, 0	$\boldsymbol{R}^3$

Now we give the formulas of the Riemannian connection ∇ and the curvature properties, which are due to [10]. For convenience, we define numbers μ_i by
$$\mu_i = 1/2(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i \quad (i = 1, 2, 3).$$

LEMMA 7.1. *Let e_1, e_2, e_3 be an orthonormal basis of $(\mathfrak{g}, \langle , \rangle)$ chosen as (7.1).*
(1) *The Riemannian connection ∇ is given with respect to e_1, e_2, e_3 as follows:*

$$\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0$$

$$\nabla_{e_1} e_2 = \mu_1 e_3, \qquad \nabla_{e_1} e_3 = -\mu_1 e_2$$

$$\nabla_{e_2} e_1 = -\mu_2 e_3, \qquad \nabla_{e_2} e_3 = \mu_2 e_1$$

$$\nabla_{e_3} e_1 = \mu_3 e_2, \qquad \nabla_{e_3} e_2 = -\mu_3 e_1.$$

(2) *The orthonormal basis e_1, e_2, e_3 diagonalizes the Ricci curvature tensor and the principal Ricci curvatures are given by*

$$\rho(e_1, e_1) = 2\mu_2 \mu_3, \quad \rho(e_2, e_2) = 2\mu_1 \mu_3, \quad \rho(e_3, e_3) = 2\mu_1 \mu_2.$$

(3) *The covariant derivative of the curvature tensor R is given as follows:*

$$(\nabla_{e_1} R)(e_1, e_2)e_1 = 2\mu_1^2(\mu_2 - \mu_3)e_3, \qquad (\nabla_{e_1} R)(e_1, e_3)e_1 = 2\mu_1^2(\mu_2 - \mu_3)e_2$$

$$(\nabla_{e_1} R)(e_1, e_2)e_3 = -2\mu_1^2(\mu_2 - \mu_3)e_1, \qquad (\nabla_{e_1} R)(e_1, e_3)e_2 = -2\mu_1^2(\mu_2 - \mu_3)e_1$$

$$(\nabla_{e_2} R)(e_2, e_1)e_2 = 2\mu_2^2(\mu_3 - \mu_1)e_3, \qquad (\nabla_{e_2} R)(e_2, e_3)e_2 = 2\mu_2^2(\mu_3 - \mu_1)e_1$$

$$(\nabla_{e_2} R)(e_2, e_1)e_3 = -2\mu_2^2(\mu_3 - \mu_1)e_2, \qquad (\nabla_{e_2} R)(e_2, e_3)e_1 = -2\mu_2^2(\mu_3 - \mu_1)e_2$$

$$(\nabla_{e_3} R)(e_3, e_1)e_3 = 2\mu_3^2(\mu_1 - \mu_2)e_2, \qquad (\nabla_{e_3} R)(e_3, e_2)e_3 = 2\mu_3^2(\mu_1 - \mu_2)e_1$$

$$(\nabla_{e_3} R)(e_3, e_1)e_2 = -2\mu_3^2(\mu_1 - \mu_2)e_3, \qquad (\nabla_{e_3} R)(e_3, e_2)e_1 = -2\mu_3^2(\mu_1 - \mu_2)e_3$$

the others $(\nabla_{e_i} R)(e_j, e_k)e_l = 0$.

Proof. (1) and (2) are found in [10] and (3) is obtained by the straight-forward computation.   □

Let Aut($\mathfrak{g}$) and $O(\mathfrak{g} ; \langle , \rangle)$ be the group consisting of automorphisms of the Lie algebra $\mathfrak{g}$ and that of orthogonal transformations of $(\mathfrak{g}, \langle , \rangle)$, respectively. For $\Phi \in \mathrm{Aut}(\mathfrak{g})$, we take the (local) automorphism φ of the Lie group G such that $\varphi_{*e} = \Phi$. Moreover if $\Phi \in O(\mathfrak{g} ; \langle , \rangle)$, then φ is an (local) isometry of G with the left invariant Riemannian metric. Therefore a 2-dimensional subspace V in $T_e G \cong \mathfrak{g}$ is the tangent space of a totally geodesic surface in G if and only if so is $\Phi(V)$. We will classify such subspaces up to the action of $\mathrm{Aut}(\mathfrak{g}) \cap O(\mathfrak{g} ; \langle , \rangle)$.

THEOREM 7.2. *Let G be a 3-dimensional unimodular Lie group with left invariant Riemannian metric and let e_1, e_2, e_3 be an orthonormal basis of $(\mathfrak{g} ; \langle , \rangle)$ chosen as (7.1). Then totally geodesic surfaces of G are classified as the following table: to describe totally geodesic surfaces, we list up 2-dimensional subspaces in $T_e G \cong \mathfrak{g}$ which are the tangent spaces of totally geodesic surfaces. We remark that the signs of λ_1, λ_2, λ_3 in the table below are chosen as Table 1.*

G	metric	t.g. surfaces	remark
$SU(2)$	$\lambda_1 = \lambda_2 = \lambda_3$	*all 2-dim subspaces*	(1)
	the others	*none*	
$SL(2, \boldsymbol{R})$	$\lambda_1 > \lambda_2 - \lambda_3$	*none*	
	$\lambda_1 = \lambda_2 - \lambda_3$	$\{\cos \theta e_1 + \sin \theta e_3, \ e_2\}_R$ $\left(\tan^2 \theta = -\dfrac{\lambda_3}{\lambda_1}\right)$	(2)
	$\lambda_1 < \lambda_2 - \lambda_3$	*none*	
$E(2)$	$\lambda_1 > \lambda_2$	*none*	
	$\lambda_1 = \lambda_2$	*all 2-dim subspaces*	(3)
$E(1, 1)$	$\lambda_1 \neq -\lambda_2$	*none*	
	$\lambda_1 = -\lambda_2$	$\left\{\dfrac{1}{\sqrt{2}}(e_1 + e_2), \ e_3\right\}_R$ *and* $\left\{\dfrac{1}{\sqrt{2}}(e_1 - e_2), \ e_3\right\}_R$	(4)
Heisenberg gr.		*none*	
$\boldsymbol{R}^3$		*all 2-dim subspaces*	(5)

Remarks:

(1) *This metric is of positive constant sectional curvature.*

(2) *These subspaces are equivalent under the action of $\mathrm{Aut}(\mathfrak{g}) \cap O(\mathfrak{g} ; \langle , \rangle)$. They are Lie subalgebras of $\mathfrak{g}$.*

(3) *This metric is flat.*

(4) *These two subspaces are equivalent under the action of* $\mathrm{Aut}(\mathfrak{g}) \cap O(\mathfrak{g}\,;\langle\,,\rangle)$. *They are Lie subalgebras of* $\mathfrak{g}$.

(5) *This metric is flat.*

Proof. We shall prove our computation results when $G = SL(2, \boldsymbol{R})$. The other cases are similar. We may assume that $\lambda_1 \geq \lambda_2$ without loss of generality. Since $\lambda_3 < 0$, we have $\mu_1 = 1/2(-\lambda_1 + \lambda_2 + \lambda_3) < 0$, $\mu_3 = 1/2(\lambda_1 + \lambda_2 - \lambda_3) > 0$, and $\mu_2 < \mu_3$. The sign of μ_2 is uncertain. We consider the following three cases:

Case I. $\mu_2 > 0$.

Case II. $\mu_2 = 0$.

Case III. $\mu_2 < 0$.

Case I. By Lemma 7.1 (2), the eigenvalues of Ricci curvature tensor are mutually distinct. Therefore by Proposition 4.6 curvature-invariant subspaces are $\{e_1, e_2\}_R$, $\{e_2, e_3\}_R$, and $\{e_1, e_3\}_R$. Since $(\nabla_{e_1} R)(e_1, e_2)e_1 = 2\mu_1^2(\mu_2 - \mu_3)e_3$, there does not exist a totally geodesic surface whose tangent space at e is $\{e_1, e_2\}_R$. For subspaces $\{e_2, e_3\}_R$ and $\{e_1, e_3\}_R$, the same holds. Hence in this case there isn't a totally geodesic surface.

Case II. In this case, we have $\rho(e_1, e_1) = \rho(e_3, e_3) = 0$ and $\rho(e_2, e_2) < 0$. Therefore curvature-invariant subspaces are $\{e_1, e_3\}_R$ and $\{\cos\theta e_1 + \sin\theta e_3, e_2\}_R$ for $\theta \in \boldsymbol{R}$.

Case 1, $\{e_1, e_3\}_R$. Since $(\nabla_{e_1} R)(e_1, e_3)e_1 = -2\mu_1^2\mu_3 e_2$, there does not exist a totally geodesic surface whose tangent space at e is $\{e_1, e_3\}_R$.

Case 2, $\{\cos\theta e_1 + \sin\theta e_3, e_2\}_R$. We put $f = \cos\theta e_1 + \sin\theta e_3$. Making use of Lemma 7.1 (3), we have

$$(\nabla_f R)(f, e_2)f = 2\mu_1\mu_3(\cos^2\theta\,\mu_1 + \sin^2\theta\,\mu_3)(\sin\theta e_1 - \cos\theta e_3).$$

Therefore if $\cos^2\theta\,\mu_1 + \sin^2\theta\,\mu_3 \neq 0$, there does not exist a totally geodesic surface whose tangent space at e is $\{f, e_2\}_R$. From now on we choose θ which satisfies $\tan^2\theta = -\mu_1/\mu_3 = -\lambda_3/\lambda_1$. In this case, we put $\mathfrak{h} = \{f, e_2\}_R$. Then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. In fact we have

$$[f, e_2] = -\lambda_1 \sin\theta e_1 + \lambda_3 \cos\theta e_3 = -\lambda_1 \tan\theta f.$$

Let H be a connected Lie subgroup of G which corresponds to $\mathfrak{h}$. Then H is totally geodesic in G. In fact we have $\nabla_f f = \sin\theta\cos\theta(-\mu_1 + \mu_3)e_2$, $\nabla_{e_2} f = 0$, and $\nabla_{e_2} e_2 = 0$.

Case III. We devide this case into the following two subcases:

Case 1. $\lambda_1 > \lambda_2$.

Case 2. $\lambda_1 = \lambda_2$.

Case 1. Since $\mu_1 - \mu_2 = \lambda_2 - \lambda_1 < 0$, the eigenvalues of Ricci curvature tensor are mutually distinct. Similarly to Case I, we see that there does not exist a totally geodesic surface.

Case 2. In this case, we have $\rho(e_1, e_1)=\rho(e_2, e_2)<0$ and $\rho(e_3, e_3)>0$. Therefore curvature-invariant subspaces are $\{e_1, e_2\}_R$ and $\{\cos\theta e_1+\sin\theta e_2, e_3\}_R$ for $\theta\in R$. Moreover it is easily seen that

$$\left\{\begin{pmatrix}\cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1\end{pmatrix} \mid \alpha\in R\right\} \subset \mathrm{Aut}(\mathfrak{g})\cap O(\mathfrak{g}\,;\,\langle\,,\,\rangle).$$

Therefore it is sufficient to consider the subspaces $\{e_1, e_2\}_R$ and $\{e_1, e_3\}_R$. Since $(\nabla_{e_1}R)(e_1, e_2)e_1=2\mu_1^2(\mu_2-\mu_3)e_3$ and $(\nabla_{e_1}R)(e_1, e_3)e_1=2\mu_1^2(\mu_2-\mu_3)e_2$, there does not exist a totally geodesic surface whose tangent space at e is $\{e_1, e_2\}_R$ or $\{e_1, e_3\}_R$. □

Observations of Theorem 7.2. (1) There do not exist totally geodesic surfaces in almost all 3-dimensional Lie groups with left invariant Riemannian metrics.

(2) Except the cases of constant sectional curvature, totally geodesic surfaces are Lie subgroups. In particular, the equivariance of such immersions holds.

(3) After we classified totally geodesic surfaces, we see that given a 2-dimensional subspace V in T_eG there is a totally geodesic surface whose tangent space at e is V if and only if $R(x, y)z\in V$ and $(\nabla_uR)(x, y)z\in V$ for $u, x, y, z\in V$. That is, we can conclude that there exists or not a totally geodesic submanifold by the data of R and ∇R.

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