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HAUSDORFF DIMENSION OF SETS ARISING IN DIOPHANTINE APPROXIMATION

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Abstract

Let $g(q)$ be a nonnegative function on the set of positive integers. We studied the Hausdorff dimension of a set

$$E_g=\left\{x\in[0,1]:\left|x-\frac{p}{q}\right|<g(q)\quad\text{for infinitely many } \frac{p}{q}\right\}.$$

We prove a generalization of a result of I. Borosh and A. S. Fraenkel.

1. Introduction

It was shown by Jarník and Besicovitch that the Hausdorff dimension, denoted by $\dim_H$, of the set of real numbers for which there exist infinitely many rationals p/q satisfying

$$\left|x-\frac{p}{q}\right|<\frac{1}{q^{\beta}}$$

is $\min\{2/\beta, 1\}$. In 1972 Borosh and Fraenkel extended the above result in the following way. Let $\mathcal{L}$ be a subset of positive integers having infinitely many elements and set

$$E_{\mathcal{L}}=\left\{x\in[0,1]:\left|x-\frac{p}{q}\right|<\frac{1}{q^{\beta}}\quad\text{for infinitely many } \frac{p}{q} \text{ with } q\in\mathcal{L}\right\}.$$

THEOREM B-F ([1]). *Let ν_0 be a real number satisfying the following two conditions:*

(i) $\sum_{q\in\mathcal{L}} q^{-\nu_0}$ *is divergent,*
(ii) $\sum_{q\in\mathcal{L}} q^{-\nu_0-\varepsilon}$ *is convergent for every* $\varepsilon>0$.

Then $\dim_H E_{\mathcal{L}}=\min\{(1+\nu_0)/\beta, 1\}$.

The purpose of the present paper is to study the Hausdorff dimension of a set

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$$E_g=\left\{x\in[0, 1]:\left|x-\frac{p}{q}\right|<g(q)\ \text{ for infinitely many } \frac{p}{q}\right\},$$

where g is a nonnegative function on the set of positive integers. We set

$$C_\alpha(N)=\text{the cardinality of the set } \left\{q\leqq N: g(q)\geqq\frac{1}{q^\alpha}\right\}$$

and

$$\gamma(\alpha)=\sup\left\{\gamma:\varlimsup_{N\to\infty}\frac{C_\alpha(N)}{N^\gamma}>0\right\}.$$

Then we shall prove the following result:

THEOREM 1.1.

$$\dim_H E_g=\min\left\{\sup_{\alpha\geqq 1}\delta(\alpha), 1\right\},$$

where

$$\delta(\alpha)=\begin{cases}\dfrac{1+\gamma(\alpha)}{\alpha} & \text{if } \lim_{N\to\infty}C_\alpha(N)=\infty\\ 0 & \text{otherwise.}\end{cases}$$

Let $f(q)$ be a function on the set of positive integers with the values 0 or 1. We consider the case

$$g(q)=\frac{f(q)}{q^{\alpha_0}}.$$

In this case, $C_\alpha(N)$ is equal to the cardinality of the set $\{q\leqq N: f(q)=1\}$ if $\alpha\geqq\alpha_0$ and $C_\alpha(N)=0$ if $\alpha<\alpha_0$. Then $\delta(\alpha)=(1+\gamma(\alpha_0))/\alpha$ if $\alpha\geqq\alpha_0$ and $\delta(\alpha)=0$ if $\alpha<\alpha_0$. In this situation, we denote $E_{f/q^{\alpha_0}}$ by E_f for simplicity and set $\gamma_0=\gamma(\alpha_0)$. Then Theorem 1.1 reduces to the following

PROPOSITION 1.2.

$$\dim_H E_f=\min\left\{\frac{1+\gamma_0}{\alpha_0}, 1\right\}.$$

Set $f(q)=1$ if $q\in\mathcal{L}$ and $f(q)=0$ if $q\notin\mathcal{L}$. Then Theorem B-F can be obtained from Proposition 1.2 by proving

$$\nu_0=\gamma_0.$$

We prove Proposition 1.2 and $\nu_0=\gamma_0$ in § 2. A proof of Theorem 1.1 is given in § 3. In § 4, some examples are given.

2. Proof of Proposition 1.2

In this section, we give the proof of the Proposition 1.2. We first show the inequality $\dim_H(E_f) \leqq (1+\gamma_0)/\alpha_0$. For each positive integer q, we set

$$F_q = \left\{ x \in [0, 1] : \left| x - \frac{p}{q} \right| < \frac{f(q)}{q^{\alpha_0}} \quad \text{for some integer } p \right\}.$$

Then F_q consists of $q-1$ intervals of length $2f(q)/q^{\alpha_0}$ and two end intervals of length $f(q)/q^{\alpha_0}$. Clearly, $E_f \subset \bigcup_{q=k}^{\infty} F_q$ for each positive integer k, so taking the intervals of F_q for $q \geqq k$ as a cover of E_f gives that

$$\mathcal{H}_\delta^s(E_f) \leqq \sum_{q=k}^{\infty} (q+1) \left(\frac{2f(q)}{q^{\alpha_0}} \right)^s$$

If $2/k^{\alpha_0} \leqq \delta$, where $\mathcal{H}_\delta^s(E)$ is the infimum of $\sum_{i=1}^{\infty} |U_i|^s$ over all countable δ-covers $\{U_i\}$ of E. The right hand of the above inequality is smaller than

$$\sum_{q=k}^{\infty} 2q \left(\frac{2f(q)}{q^{\alpha_0}} \right)^s = 2^{s+1} \sum_{q=k}^{\infty} \frac{f(q)}{q^{s\alpha_0 - 1}}.$$

Hence if the series $\sum_{q=k}^{\infty} f(q)/q^{s\alpha_0-1}$ converges, then the Hausdorff s-dimensional measure $\mathcal{H}^s(E_f) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(E_f) = 0$.

Fix a positive real number γ with $\gamma > \gamma_0$. Let $C_f(N)$ be the cardinality of the set $\{q : q \leqq N \text{ and } f(q)=1\}$. Note that $C_f(N) = C_{\alpha_0}(N)$. Then, for $\lim_{N\to\infty} C_f(N^{1/\gamma})/N = \lim_{N\to\infty} C_f(N)/N^\gamma = 0$, there exists an integer N_0 such that if $N \geqq N_0$ then $C_f(N^{1/\gamma}) \leqq N$. We define M_k by $C_f(M_k) = kN_0$ $(k=1, \cdots)$. Since $C_f((kN_0)^{1/\gamma}) < kN_0 = C_f(M_k)$, we have

$$(kN_0)^{1/\gamma} < M_k.$$

Set

$$\mathcal{M}_k = \{q : f(q)=1 \quad \text{and} \quad M_{k-1} < q \leqq M_k\}$$

then $\#\mathcal{M}_k = N_0$ from the definition of M_k. Now we have

$$\sum_{q=1}^{\infty} \frac{f(q)}{q^{s\alpha_0-1}} = \sum_{k=0}^{\infty} \sum_{q \in \mathcal{M}_{k+1}} \frac{1}{q^{s\alpha_0-1}} < \sum_{q \in \mathcal{M}_1} \frac{1}{q^\beta} + \sum_{k=1}^{\infty} \frac{N_0}{((kN_0)^{1/\gamma})^{s\alpha_0-1}}$$

$$< N_0 + \frac{1}{N_0^{(s\alpha_0-1)/\gamma}} \sum_{k=1}^{\infty} \frac{1}{k^{(s\alpha_0-1)/\gamma}}.$$

If $s > (1+\gamma)/\alpha_0$, then $\sum_{k=1}^{\infty} 1/k^{(s\alpha_0-1)/\gamma}$ is convergent. Hence if $s > (1+\gamma)/\alpha_0$ we have $\mathcal{H}^s(E_f)=0$ and therefore $\dim_H(E_f) \leqq (1+\gamma)/\alpha_0$ for any $\gamma > \gamma_0$. This implies that $\dim_H(E_f) \leqq (1+\gamma_0)/\alpha_0$.

We need some lemmas to prove converse inequality. Let $C_f(N, M)$ be the cardinality of the set $\{q : N \leqq q \leqq M \text{ and } f(q)=1\}$. We set

$$\gamma_1 = \sup\left\{ \gamma : \varlimsup_{N\to\infty} \frac{C_f(N, 2N)}{N^\gamma} > 0 \right\}.$$

LEMMA 2.1.

$$\gamma_1 = \gamma_0.$$

Proof. It is clear that $0 \leqq \gamma_1 \leqq \gamma_0$. Hence it is sufficient to show that $\gamma_1 \geqq \gamma_0$. We can assume that $\gamma_0 > 0$. Let ε be a positive number with $\gamma_0 - \varepsilon > 0$ and $\{n_j\}$ be a sequence of positive integers such that

$$\frac{C_f(n_j)}{(\log_2 n_j) n_j^{\gamma_0}} \cdot n_j^{\varepsilon} \to \infty \quad (j \to \infty).$$

It is possible to choose such a sequence as above by definition of γ_0. We divide the interval $[1, n_j]$ into k-small intervals

$$\left[1, \frac{n_j}{2^k}\right), \left[\frac{n_j}{2^k}, \frac{n_j}{2^{k-1}}\right), \cdots, \left[\frac{n_j}{2}, n_j\right],$$

where k is the greatest integer satisfying $k < \log_2 n_j$. Let m be the number such that $C_f(n_j/2^{m+1}, n_j/2^m)$ is the greatest among $C_f(n_j/2^{l+1}, n_j/2^l)$, $l = 0, 1, \cdots, k$.

Let $M_j = n_j/2^m$. Then $M_j \to \infty$ if $j \to \infty$. Because if $M_j < K$ for some constant K, then we have

$$\begin{aligned} C_f(K) &\geqq C_f(M_j) \\ &\geqq \frac{C_f(n_j)}{\log_2 n_j} \\ &= \frac{C_f(n_j)}{(\log_2 n_j) n_j^{\gamma_0}} \cdot n_j^{\varepsilon} \cdot n_j^{\gamma_0 - \varepsilon} \to \infty \quad (j \to \infty). \end{aligned}$$

This is a contradiction.

Now for $\gamma_0 - \varepsilon > 0$,

$$\frac{C_f(n_j)}{n_j^{\gamma_0 - \varepsilon}} \leqq \frac{(\log_2 n_j) C_f(M_j/2, M_j)}{M_j^{\gamma_0 - \varepsilon}}.$$

Since $C_f(n_j)/(\log_2 n_j) n_j^{\gamma_0 - \varepsilon} \to \infty$ $(j \to \infty)$ we have

$$\frac{C_f(M_j/2, M_j)}{M_j^{\gamma_0 - \varepsilon}} \to \infty \quad (j \to \infty).$$

Hence $\gamma_1 \geqq \gamma_0 - \varepsilon$. Since ε can be taken arbitrary small, it follows that $\gamma_1 \geqq \gamma_0$. ∎

Let $P(N) = N^{\gamma_0}/C_f(N, 2N)$. Then by Lemma 2.1

$$\varlimsup_{N \to \infty} N^{\varepsilon} P(N) = +\infty \quad \text{and} \quad \varliminf_{N \to \infty} N^{-\varepsilon} P(N) = 0$$

for any $\varepsilon > 0$. So, for each $\varepsilon > 0$, we choose a sequence of positive integers $\{N_j\}$ such that $N_j > 2N_{j-1}$ and

$$\lim_{j \to \infty} N_j^{\varepsilon} P(N_j) = +\infty \quad \text{and} \quad \lim_{j \to \infty} N_j^{-\varepsilon} P(N_j) = 0.$$

First we consider the case $\alpha_0 > 1 + \gamma_0$. Set $L_j = \{N_j \leqq q \leqq 2N_j : f(q) = 1\}$ and $l_j = \# L_j = C_f(N_j, 2N_j)$. Then $l_j P(N_j)/N_j^{\gamma_0} = 1$. Let $L_j = \{q_1, \cdots, q_{l_j}\}$ with $N_j \leqq q_1 < q_2 < \cdots < q_{l_j} \leqq 2N_j$. Let $\tilde{G}_{q_k}$ be the set of reduced fractions p/q_k in the interval [0, 1] whose numerators p are prime numbers. Let $G_{q_1}=\tilde{G}_{q_1}$ and let $G_{q_2}\subset\tilde{G}_{q_2}$ be the set of reduced fractions t_2/q_2 in the interval [0, 1], whose numerators are prime numbers, satisfying

$$(1)\qquad \left|\frac{t_2}{q_2}-\frac{t_1}{q_1}\right|>\frac{P(N_j)}{N_j^{1+\gamma_0}(\log N_j)^2}$$

for any element t_1/q_1 of G_{q_1}.

LEMMA 2.2.

$$\#(\tilde{G}_{q_2}-G_{q_2})\leqq\frac{8N_j^{1-\gamma_0}P(N_j)}{(\log N_j)^2}.$$

Proof. If t_2/q_2 satisfies the inequality

$$|t_2q_1-t_1q_2|>\frac{4N_j^{1-\gamma_0}P(N_j)}{(\log N_j)^2}$$

for any $t_1/q_1\in G_{q_1}$, it satisfies (1) for any $t_1/q_1\in G_{q_1}$. So we count up the number of fractions t_2/q_2 which satisfies

$$(2)\qquad |t_2q_1-t_1q_2|\leqq\frac{4N_j^{1-\gamma_0}P(N_j)}{(\log N_j)^2}$$

for some $t_1/q_1\in G_{q_1}$. Since the number of solutions (t_1, t_2) of the equation

$$|t_2q_1-t_1q_2|=k \quad (k \text{ is a positive integer})$$

in the range $0\leqq t_1\leqq q_1$, $0\leqq t_2\leqq q_2$ is at most two, the number of reduced fractions t_2/q_2 which satisfies (2) for some $t_1/q_1\in G_{q_1}$ is at most $8N_j^{1-\gamma_0}P(N_j)/(\log N_j)^2$. Hence we have the lemma. ∎

Now we inductively define $G_{q_k}\subset\tilde{G}_{q_k}$. Let G_{q_k} be a set of reduced fractions t_k/q_k with prime numerators which satisfies

$$\left|\frac{t_k}{q_k}-\frac{t}{q}\right|>\frac{P(N_j)}{N_j^{1+\gamma_0}(\log N_j)^2}$$

for all $t/q\in\bigcup_{i=1}^{k-1}G_{q_i}$. Then by the similar argument as the proof of Lemma 2.2, we have

$$\#(\tilde{G}_{q_k}-G_{q_k})\leqq\frac{8(k-1)N_j^{1-\gamma_0}P(N_j)}{(\log N_j)^2}.$$

Set $H_j=\bigcup_{k=1}^{l_j}G_{q_k}$ and $\tilde{H}_j=\bigcup_{k=1}^{l_j}\tilde{G}_{q_k}$. Then we get

LEMMA 2.3.

$$\#(\tilde{H}_j-H_j)<\frac{4l_jN_j}{(\log N_j)^2}.$$

Proof. By the preceeding discussion, we have

$$\#(\tilde{H}_j-H_j)=\sum_{k=1}^{l_j}\#(\tilde{G}_{q_k}-G_{q_k})\leqq\sum_{k=1}^{l_j}\frac{8(k-1)N_j^{1-\gamma_0}P(N_j)}{(\log N_j)^2}.$$

Therefore we have

$$\begin{aligned}\#(\tilde{H}_j-H_j)&\leqq\frac{4l_j(l_j-1)N_j^{1-\gamma_0}P(N_j)}{(\log N_j)^2}\\&<\frac{4l_jN_j}{(\log N_j)^2}\cdot\frac{l_jP(N_j)}{N_j^{\gamma_0}}\\&=\frac{4l_jN_j}{(\log N_j)^2},\end{aligned}$$

since $l_jP(N_j)/N_j^{\gamma_0}=1$. ■

Now we complete the proof of Proposition 1.2. For any two different elements p_1/r_1 and p_2/r_2 of H_j, we have

$$\left|\frac{p_1}{r_1}-\frac{p_2}{r_2}\right|>\frac{P(N_j)}{N_j^{1+\gamma_0}(\log N_j)^2}.$$

Set

$$I_{p/q}\left\{x\in[0,1]:\left|x-\frac{p}{q}\right|<\frac{f(q)}{q^{\alpha_0}}\right\}$$

and

$$K_j=\bigcup_{p/q\in H_j}I_{p/q}.$$

Then the distance between two different intervals $I_{p/q}$ and $I_{p'/q'}$ is at least

$$\frac{P(N_j)}{N_j^{1+\gamma_0}(\log N_j)^2}-\frac{2}{N_j^{\alpha_0}}>\frac{P(N_j)}{2N_j^{1+\gamma_0}(\log N_j)^2}$$

for sufficiently large N_j, since $1+\gamma_0<\alpha_0$.

Let $E_0=[0, 1]$ and E_j the set of intervals of K_j which are completely contained in some of E_{j-1}. Then the intervals of E_j are separated by gaps of at least

$$\varepsilon_j=\frac{P(N_j)}{2N_j^{1+\gamma_0}(\log N_j)^2}=\frac{1}{2N_jl_j(\log N_j)^2}.$$

Let $I=[a, b]\subset[0, 1]$ be an interval with $|I|>3/N_j$. We count the number of intervals of K_j in I.

LEMMA 2.4. *The number of intervals of K_j contained in I is at least*

$$\frac{(b-a)N_jl_j}{16\log N_j},$$

for sufficiently large N_j.

Proof. The number of rationals in I whose denominators are q_k and numerators are prime is equal to the number of primes in the interval $[aq_k, bq_k]$. By prime number theorem there are at least

$$\frac{1}{2}\left(\frac{bq_k}{\log bq_k}-\frac{aq_k}{\log aq_k}\right)$$

primes in $[aq_k, bq_k]$. Then we have

$$\frac{1}{2}\left(\frac{bq_k}{\log bq_k}-\frac{aq_k}{\log aq_k}\right)>\frac{1}{4}\cdot\frac{(b-a)q_k}{\log q_k}=\frac{1}{4}\cdot\frac{q_k|I|}{\log q_k}$$

$$>\frac{1}{8}\cdot\frac{N_j|I|}{\log N_j}.$$

Then by Lemma 2.3,

$$\begin{aligned}\#(I\cap H_j)&=\#(I\cap\widetilde{H}_j)-\#(I\cap(\widetilde{H}_j-H_j))\\&\geqq\frac{1}{8}\cdot\frac{N_jl_j|I|}{\log N_j}-\frac{4l_jN_j}{(\log N_j)^2}\\&=l_j\left(\frac{N_j|I|}{8\log N_j}-\frac{4N_j}{(\log N_j)^2}\right)\\&>\frac{N_jl_j|I|}{16\log N_j}.\end{aligned}$$

Hence at least $N_jl_j|I|/16\log N_j$ intervals of K_j are contained in I for sufficiently large N_j. ∎

We can take the sequence $\{N_j\}$ to satisfy the inequality

$$\frac{2}{(2N_{j-1})^{\alpha_0}}>\frac{64}{\log N_j}.$$

As the length of each interval in E_{j-1} is at least $2/(2N_{j-1})^{\alpha_0}$, by Lemma 2.4, E_{j-1} contains at least

$$m_j=\frac{N_jl_j(2N_{j-1})^{-\alpha_0}}{16\log N_j}=\frac{cN_jl_jN_{j-1}^{-\alpha_0}}{\log N_j}$$

intervals of K_j, where $c=16^{-1}\cdot 2^{-\alpha_0}$ and we set $m_1=1$. By choosing a subsequence of N_j, we can assume that $\log N_j>j\log N_{j-1}$. Then clearly $\varepsilon_j>\varepsilon_{j+1}$ for sufficiently large j. By Example 4.6 (p. 58) in [2], we have

$$\begin{aligned}\dim_H\left(\bigcap_{j=1}^{\infty}E_j\right)&\geqq\varliminf_{j\to\infty}\frac{\log(m_1\cdots m_{j-1})}{-\log(m_j\varepsilon_j)}\\&=\varliminf_{j\to\infty}\frac{\log[c^{j-2}N_1^{-\alpha_0}(N_2\cdots N_{j-2})^{1-\alpha_0}(l_2\cdots l_{j-2})(\log N_2\cdots\log N_{j-2})^{-1}l_{j-1}N_{j-1}]}{-\log[cN_{j-1}^{-\alpha_0}(\log N_{j-1})^{-3}]}.\end{aligned}$$

The numerator can be rewritten as following:

$$(j-2)\log c-\alpha_0\log N_1+(1+\gamma_0-\alpha_0)(\log N_2+\cdots+\log N_{j-2})$$
$$-(\log\log N_2+\cdots+\log\log N_{j-2})$$
$$-(\log P(N_2)+\cdots+\log P(N_{j-2}))+(1+\gamma_0)\log N_{j-1}-\log P(N_{j-1}).$$

Since $\log N_j>j\log N_{j-1}$ we have

$$\frac{\log N_2+\cdots+\log N_{j-2}}{\log N_{j-1}}<\frac{2}{j-1}.$$

We may assume that $N_j^{\varepsilon}P(N_j)>1$ and $N_j^{-\varepsilon}P(N_j)<1$ for all j. Then $N_j^{-\varepsilon}<P(N_j)<N_j^{\varepsilon}$ and hence

$$\left|\frac{\log P(N_j)}{\log N_j}\right|<\varepsilon.$$

Hence we have

$$\left|\frac{\log P(N_2)+\cdots+\log P(N_{j-2})}{\log N_{j-1}}\right|\leqq\varepsilon\frac{(\log N_2+\cdots+\log N_{j-2})}{\log N_{j-1}}$$
$$<\frac{2\varepsilon}{j-1}.$$

Thus the principal term of the numerator is $(1+\gamma_0)\log N_{j-1}-\log P(N_{j-1})>(1+\gamma_0-\varepsilon)\log N_{j-1}$, and that of denominator is $\alpha_0\log N_{j-1}$. Hence we get

$$\dim_H\left(\bigcap_{j=1}^{\infty}E_j\right)\geqq\frac{1+\gamma_0-\varepsilon}{\alpha_0}.$$

If $x\in E_j$ for all j, then x lies in infinitely many of the F_q and so $x\in E_f$. Therefore

$$\dim_H(E_f)\geqq\dim_H\left(\bigcap_{j=1}^{\infty}E_j\right)\geqq\frac{1+\gamma_0-\varepsilon}{\alpha_0}.$$

Since $\varepsilon>0$ is arbitrary, we have $\dim_H(E_f)\geqq(1+\gamma_0)/\alpha_0$.

Finally we consider the case $\alpha_0\leqq 1+\gamma_0$. It is clear that

$$E_f^{1+\gamma_0+\varepsilon}\subset E_f^{\alpha_0}$$

for any positive number ε. Then we have from our preceeding results

$$\frac{1+\gamma_0}{1+\gamma_0+\varepsilon}\leqq\dim_H E_f^{\alpha_0}\leqq 1$$

for any $\varepsilon>0$. Hence we get

$$\dim_H E_f^{\alpha_0}=1.$$

We now show a lemma to obtain the Theorem B-F from Proposition 1.2.

LEMMA 2.5. *The critical exponent* ν_0 *of*

$$\sum_{q\in\mathcal{L}}\frac{1}{q^{\nu}}$$

is equal to $\gamma_0=\sup\{\gamma:\overline{\lim}_{N\to\infty}C_{\mathcal{L}}(N)/N^{\gamma}>0\}$, *where* $C_{\mathcal{L}}(N)=\#\{q\leqq N: q\in\mathcal{L}\}$.

Proof. Let $\nu<\gamma_0$. We can choose a sequence $\{N_i\}$ with $N_j\to\infty$ $(i\to\infty)$ such that $C_{\mathcal{L}}(N_i)>2N_i^{\nu}$. We also choose a subsequence $\{L_j\}$ of $\{N_i\}$ such that

$$L_j>C_{\mathcal{L}}(L_{j-1})^{1/\nu}L_{j-1}.$$

Then we have

$$C_{\mathcal{L}}(L_j)-C_{\mathcal{L}}(L_{j-1})>L_j^{\nu}.$$

We set $\mathcal{L}_i=\{q\in\mathcal{L}: L_{j-1}<q\leqq L_j\}$. Then

$$\sum_{q\in\mathcal{L}}\frac{1}{q^{\nu}}=\sum_{j=1}^{\infty}\sum_{q\in\mathcal{L}_j}\frac{1}{q^{\nu}}>\sum_{j=1}^{\infty}\frac{C_{\mathcal{L}}(L_j)-C_{\mathcal{L}}(L_{j-1})}{L_j^{\nu}}>\sum_{j=1}^{\infty}1=\infty.$$

When $\nu>\gamma_0$, we can show that the series $\sum_{q\in\mathcal{L}}1/q^{\nu}$ is convergent by the same way used in the proof of the inequality $\dim_H E_f\leqq(1+\gamma_0)/\alpha_0$. ■

3. Proof of Theorem 1.1

In this section, we prove the Theorem 1.1. Let g be a nonnegative function defined on the set of all natural numbers. Let

$$E_g=\left\{x\in[0,1]:\left|x-\frac{p}{q}\right|<g(q)\ \text{ for infinitely many } q\right\}.$$

Set

$$\gamma(\alpha)=\sup\left\{\gamma:\overline{\lim_{N\to\infty}}\frac{C_{\alpha}(N)}{N^{\gamma}}>0\right\}$$

for $\alpha\geqq1$, where $C_{\alpha}(N)$ is the cardinality of the set $\{q\leqq N: g(q)\geqq1/q^{\alpha}\}$. We define a function $\delta(\alpha)$ as follows

$$\delta(\alpha)=\begin{cases}\dfrac{1+\gamma(\alpha)}{\alpha} & \text{if } \lim_{N\to\infty}C_{\alpha}(N)=\infty\\ 0 & \text{otherwise.}\end{cases}$$

Then we have

THEOREM 1.1.

$$\dim_H E_g=\min\left\{\sup_{\alpha\geqq1}\delta(\alpha),\ 1\right\}.$$

Proof. Set

$$E=\left\{x\in[0, 1]: \left|x-\frac{p}{q}\right|<g(q) \text{ for infinitely many } q\right\},$$

$$E_\alpha=\left\{x\in[0, 1]: \left|x-\frac{p}{q}\right|<g(q) \text{ and } g(q)\geqq\frac{1}{q^\alpha} \text{ for infinitely many } q\right\}$$

and

$$F_\alpha=\left\{x\in[0, 1]: \left|x-\frac{p}{q}\right|<g(q) \text{ and } g(q)<\frac{1}{q^\alpha} \text{ for infinitely many } q\right\}.$$

Set $F_\infty=\bigcap_{\alpha\geqq 1}F_\alpha$ then it is clear that

$$E-\bigcup_{\alpha\geqq 1}E_\alpha\subset F_\infty.$$

We can show that $\dim_H F_\infty=0$ as follows. Let

$$G_\alpha=\left\{x\in[0, 1]: \left|x-\frac{p}{q}\right|<\frac{1}{q^\alpha} \text{ for infinitely many } q\right\}.$$

Then we have

$$F_\infty\subset F_\alpha\subset G_\alpha$$

for $\alpha\geqq 1$. Hence $\dim_H F_\infty\leqq\dim G_\alpha=2/\alpha$. Since α is arbitrary large, $\dim_H F_\infty=0$.

Therefore $\dim_H(E-\bigcup_{\alpha\geqq 1}E_\alpha)=0$ and hence $\dim_H E=\dim\bigcup_{\alpha\geqq 1}E_\alpha$.

First we show the inequality $\dim_H E\geqq\min\{\sup_{\alpha\geqq 1}\delta(\alpha), 1\}$. Set

$$H_\alpha=\left\{x\in[0, 1]: \left|x-\frac{p}{q}\right|<\frac{\chi_{A_\alpha}(q)}{q^\alpha} \text{ for infinitely many } q\right\},$$

where $A_\alpha=\{q\in\boldsymbol{N}: g(q)\geqq 1/q^\alpha\}$ and χ_{A_α} is the characteristic function of A_α. Then by the Proposition 1.2, we have

$$\dim_H H_\alpha=\min\left\{\frac{1+\gamma(\alpha)}{\alpha}, 1\right\}$$

if A_α is an infinite set and $\dim_H H_\alpha=0$ if A_α finite. Hence $\dim_H H_\alpha=\min\{\delta(\alpha), 1\}$. Since $E_\alpha\supset H_\alpha$,

$$\dim_H E_\alpha\geqq\dim_H H_\alpha=\min\{\delta(\alpha), 1\}.$$

Therefore

$$\begin{aligned}\dim_H E=\dim_H\bigcup_{\alpha\geqq 1}E_\alpha&\geqq\sup_{\alpha\geqq 1}\dim_H E_\alpha\\&\geqq\sup_{\alpha\geqq 1}\min\{\delta(\alpha), 1\}\\&=\min\left\{\sup_{\alpha\geqq 1}\delta(\alpha), 1\right\}.\end{aligned}$$

Next, we show the converse inequality $\dim_H E\leqq\min\{\sup_{\alpha\geqq 1}\delta(\alpha), 1\}$. For positive real numbers α, β such that $\alpha>\beta$, we set

$$E_\alpha^\beta=\left\{x\in[0,\,1]:\left|x-\frac{p}{q}\right|<g(q)\ \text{ and }\ \frac{1}{q^\beta}>g(q)>\frac{1}{q^\alpha}\ \text{ for infinitely many } q\right\}$$

and

$$H_\alpha^\beta=\left\{x\in[0,\,1]:\left|x-\frac{p}{q}\right|<\frac{\chi_{A_\alpha}(q)}{q^\beta}\ \text{ for infinitely many } q\right\}.$$

It is clear that $E_\alpha^\beta\subset H_\alpha^\beta$. So we have, by using Proposition 1.2 again,

$$\begin{aligned}\dim_H E_\alpha^\beta&\leqq\dim_H H_\alpha^\beta\\&=\min\left\{\frac{1+\gamma(\alpha)}{\beta},\,1\right\}\\&=\min\left\{\frac{1+\gamma(\alpha)}{\alpha}\cdot\frac{\alpha}{\beta},\,1\right\},\end{aligned}$$

if A_α is infinite and $\dim_H E_\alpha^\beta=0$ if A_α finite. Hence

$$\dim_H E_\alpha^\beta\leqq\min\left\{\delta(\alpha)\cdot\frac{\alpha}{\beta},\,1\right\}.$$

Fix an $\varepsilon>0$ arbitrary. Define the sequence $\{\alpha_n\}_{n=0}^\infty$ by setting $\alpha=1+n\varepsilon$. Then it is easily seen that

$$\bigcup_{\alpha\geqq1}E_\alpha=\left(\bigcup_{n=1}^\infty E_{\alpha_n}^{\alpha_{n-1}}\right)\cup E_1.$$

If E_1 is not empty, then the set $\{q:g(q)\geqq1/q\}$ is infinite. Hence $\delta(1)=(1+\gamma(1)/1\geqq1$. Therefore $\min\{\sup_{\alpha\geqq1}\delta(\alpha),\,1\}=1$ and hence the inequality is obvious. So we may assume that E_1 is empty. For a countable family of sets $\{X_n\}_{n=1}^\infty$, the Hausdorff dimension has the following property:

$$\dim_H\bigcup_{n=1}^\infty X_n=\sup_{n\geqq1}\dim_H X_n.$$

So we have

$$\begin{aligned}\dim_H E&=\dim_H\bigcup_{\alpha\geqq1}E_\alpha\\&=\dim_H\bigcup_{n=1}^\infty E_{\alpha_n}^{\alpha_{n-1}}\\&=\sup_{n\geqq1}\dim_H E_{\alpha_n}^{\alpha_{n-1}}\\&\leqq\sup_{n\geqq1}\min\left\{\delta(\alpha_n)\cdot\frac{\alpha_n}{\alpha_{n-1}},\,1\right\}\\&=\min\left\{\sup_{n\geqq1}\delta(1+n\varepsilon)\cdot\frac{1+n\varepsilon}{1+n\varepsilon-\varepsilon},\,1\right\}\\&\leqq\min\left\{\sup_{\alpha\geqq1}\delta(\alpha)\cdot\frac{\alpha}{\alpha-\varepsilon},\,1\right\}\end{aligned}$$

$$\leqq \min\left\{\frac{1}{1-\varepsilon}\sup_{\alpha\geqq 1}\delta(\alpha),\ 1\right\}.$$

Since $\varepsilon>0$ is arbitrary, we have the desired conclusion. ■

4. Examples

For a function $g(q)$ defined on $\boldsymbol{N}$, we set

$$a_q=\begin{cases} -\dfrac{\log g(q)}{\log q} & \text{if } g(q)>0 \\ \infty & \text{if } g(q)\leqq 0. \end{cases}$$

Then we see

$$\{q\in \boldsymbol{N}:\ a_q<\alpha\}=\left\{q\in \boldsymbol{N}:\ g(q)>\frac{1}{q^{\alpha}}\right\}$$

for $\alpha\in \boldsymbol{R}$. Hence the Hausdorff dimension of E is determined by the distribution of the sequence $\{a_q\}$.

PROPOSITION 4.1. *If the cardinality of the set* $\{q\in \boldsymbol{N}:\ a_q<\alpha\}$ *is finite for any* $\alpha\in \boldsymbol{R}$, *then* $\dim_H E=0$.

Proof. By the assumption we have $\lim_{N\to\infty} C_\alpha(N)=\#\{q\in \boldsymbol{N}:\ a_q<\alpha\}<\infty$. Hence from Theorem 4.1, $\dim_H E=0$. ■

We may apply Proposition 4.1 for following functions

(1) $$g(q)=\frac{1}{q^q},\ \frac{1}{(\log q)^q},\ \frac{1}{q^{\log q}},\ \frac{1}{(\log q)^{\log q}},\ \text{etc.}$$

(2) $$g(q)=\frac{1}{q^{\varphi(q)}},$$

where $\varphi(q)$ is the Euler function.

(3) $$g(q)=\frac{1}{a^q},$$

where a is a constant with $a>1$.

(4) $$g(q)=\frac{1}{q!}.$$

By Theorem 1.1, we have

PROPOSITION 4.2. *If the sequence* $\{a_q\}$ *is distributed in an interval* $[s, t]$ $(2\leqq s\leqq t)$ *in such a way that the limit*

$$\lim_{N\to\infty}\frac{\#\{q\leqq N: a_q\in[s, \alpha]\}}{N^{\gamma}}$$

exists and its value is positive for any $s<\alpha$, *where* γ *is a constant* $0<\gamma\leqq 1$, *then*

$$\dim_H E=\frac{1+\gamma}{s}.$$

We can apply Proposition 4.2 for following cases:

(1) $\{a_q\}$ is uniformly distributed in $[s, t]$. Then $\gamma=1$ and $\dim_H E=2/s$.
(2) $a_q=s+|\sin q|$.

The sequence $\{q/\pi\}$ is uniformely distributed in $[0, 1]$ mod 1. Hence $\{a_q\}$ satisfies the condition of Proposition 4.2 for $\gamma=1$ and $\dim_H E=2/s$.

Example 4.3. We set $a_q=s+k$ if $q\equiv k \pmod n$, where n is a fixed natural number and s ($\geqq 2$) is also fixed. Then if $s+k<\alpha\leqq s+k+1$,

$$\begin{aligned} C_\alpha(N)&=\#\left\{q\leqq N: \frac{1}{q^{a_q}}>\frac{1}{q^{\alpha}}\right\} \\ &=\#\{q\leqq N: q\equiv 0, 1, \cdots, k-1 \pmod n\} \\ &\sim\frac{k}{n}N. \end{aligned}$$

Hence $\gamma(\alpha)=1$ and we have

$$\dim_H E=\sup_{s<\alpha}\frac{2}{\alpha}=\frac{2}{s}.$$

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