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SOME RESULTS ON THE COMPLEX OSCILLATION FOR HIGHER ORDER HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS*

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Abstract

In this paper, we are concerned with the maximum number of linearly independent transcendental solutions with finite exponent of convergence of the zeros for a higher order homogeneous linear differential equation where its coefficients are entire functions with order less than 1/2 and one dominant. The results obtained here are the extension and deepening of J.K. Langley's.

§ 1. Introduction and results

Since 1982, various researchs have been made concerning the complex oscillation of second-order homogeneous linear differential equations. In 1991, J.K. Langley and S. Bank ([2]) discovered surprisingly that some main results obtained from second-order equations still hold for higher-order equations. They made use of asymptotic methods far different to those applied in the second-order case.

For second-order equations, S. Bank and I. Laine proved the following theorem in [1], in which etc. the notations are defined at the end of this section:

THEOREM A. *Suppose that $A(z)$ is a transcendental entire function with $\sigma(A)<1/2$. Then, the equation*

$$f''+Af=0$$

can not have two linearly independent solutions f_1, f_2 with $\max\{\lambda(f_1), \lambda(f_2)\}<+\infty$.

The assertion of Theorem A has been proved in the case where $\sigma(A)=1/2$ by Rossi ([15]) and Shen ([16]) independently.

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For higher-order equations, J. K. Langley and S. Bank proved a related conclusion in [2]:

THEOREM B. *Suppose that* $k \geqq 3$ *and that* $A_0, \cdots, A_{k-2}$ *are entire functions such that*:

(i) A_0 *is transcendental with* $\sigma(A_0)<1/2$;

(ii) *If* $\sigma(A_0)>0$, *then* $\sigma(A_j)<\sigma(A_0)$ *for* $j=1, \cdots, k-2$, *while if* $\sigma(A_0)=0$ *then* $A_1, \cdots, A_{k-2}$ *are polynomials.*

Then the equation

$$f^{(k)}+A_{k-2}f^{(k-2)}+\cdots+A_1f'+A_0f=0 \tag{1.1}$$

can not have two linearly independent solutions f_1, f_2 *with* $\max\{\lambda(f_1), \lambda(f_2)\}<+\infty$.

Afterwards, Langley ([14]) obtained a result with weaker conclusion in a more general situation than Theorem B.

THEOREM C. *Suppose that* $k \geqq 3$ *and that* $A_0, \cdots, A_{k-2}$ *are entire functions such that for some* $s \in \{1, \cdots, k-2\}$,

(i) A_s *is transcendental with* $\sigma(A_s)<1/2$;

(ii) *For* $j \neq s$, *either* A_j *is a polynomial or we have* $\sigma(A_j)<\sigma(A_s)$.

Then the equation (1.1) *can not have* k *linearly independent solutions* $f_1, \cdots, f_k$ *with* $\max\{\lambda(f_1), \cdots, \lambda(f_k)\}<+\infty$.

Recently, S.-A. Gao and J.-F. Tang ([7]) got a complement of Theorem C:

THEOREM D. *With the hypotheses of Theorem* C, *the equation* (1.1) *has at most* $k-s$ *linearly independent transcendental solutions* $f_1, \cdots, f_{k-s}$ *with* $\max\{\lambda(f_1), \cdots, \lambda(f_{k-s})\}<+\infty$.

The present paper aims to improve the upper bound $k-s$ in Theorem D, and to obtain a conclusion similar to Theorem B when A_0 will be replaced with A_s. In fact, we prove the following

THEOREM 1. *Suppose that* $k \geqq 4$ *and that* $A_0, \cdots, A_{k-2}$ *are entire functions such that for some* $s \in \{2, \cdots, k-2\}$,

(i) A_s *is transcendental with* $\sigma(A_s)<1/2$;

(ii) *For* $j \neq s$, *either* A_j *is a polynomial or we have* $\sigma(A_j)<\sigma(A_s)$.

Then

(a) *An arbitrary fundamental solution set of the equation* (1.1) *includes at least* $k-s$ ($\geqq 2$) *linearly independent transcendental solutions* (*in fact, with infinite order*);

(b) *The equation* (1.1) *has at most* $m=\min\{k-s, s-1\}$ *linearly independent transcendental solutions* $f_1, \cdots, f_m$ *with* $\max\{\lambda(f_1), \cdots, \lambda(f_m)\}<+\infty$;

(c) *If* $s=2$, *or if* $k-1$ *and* $k-s$ *are relatively prime, then the equation* (1.1) *can not have two linearly independent transcendental solutions* f_1, f_2 *with* $\max\{\lambda(f_1), \lambda(f_2)\}<+\infty$.

Theorem 1(b) is an improvement of Theorem D. Under the assumptions above, Theorem 1(c) is the expected result similar to Theorem B. But the case $s=1$ is exceptional and it remains open here whether we have similar improvements in this case.

In this paper, we use the standard notations of Nevanlinna theory, e.g. see [11]. Secondly, we denote by $\sigma(g)$, $\mu(g)$ and $\lambda(g)$, respectively, the order, the lower order and the exponent of convergence of the zeros for a meromorphic function $g(z)$. Thirdly, for a set $E\subset(1, +\infty)$, denote $m_1(E)=\int_1^\infty \chi_E(t)dt/t$, $\underline{\text{log dens}}\,(E)=\varliminf_{r\to\infty}\{m_1(E\cap(1, r])/\log r\}$, $\overline{\text{log dens}}\,(E)=\varlimsup_{r\to\infty}\{m_1(E\cap(1, r])/\log r\}$, where $\chi_E(t)$ denotes the characteristic function of the set E.

§2. Lemmas needed for the proof of Theorem 1

LEMMA 1 ([10]). *Let* w *be a transcendental meromorphic function with finite order* $\sigma(w)=\rho<+\infty$, *let* $\Gamma=\{(k_1, j_1), \cdots, (k_m, j_m)\}$ *denote a finite set of distinct pairs of integers that satisfy* $k_i>j_i\geqq 0$ *for* $i=1, \cdots, m$, *and let* $\varepsilon>0$ *be a given constant. Then there exists a set* $E_1\subset(1, +\infty)$ *with* $m_1(E_1)<+\infty$, *such that for all* z *satisfying* $|z|\notin E_1\cup[0, 1]$ *and* $(k, j)\in\Gamma$, *we have*

$$|w^{(k)}(z)/w^{(j)}(z)|\leqq|z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.1}$$

LEMMA 2 ([3]). *Let* $f(z)$ *be entire of order* $\sigma(f)=\rho<1/2$. *Denote* $A(r)=\inf_{|z|=r}\log|f(z)|$, $B(r)=\sup_{|z|=r}\log|f(z)|$. *If* $\rho<\alpha<1$, *then*

$$\underline{\text{log dens}}\ \{r: A(r)>(\cos\pi\alpha)B(r)\}\geqq 1-\rho/\alpha. \tag{2.2}$$

LEMMA 3 (Theorem 5 of [4]). *Let* $f(z)$ *be entire with* $\mu(f)=\mu<1/2$ *and* $\mu<\rho=\sigma(f)$. *If* $\mu\leqq\delta<\min(\rho, 1/2)$ *and* $\delta<\alpha<1/2$, *then* $\overline{\text{log dens}}\ \{r: A(r)>(\cos\pi\alpha)B(r)>r^\delta\}>C(\rho, \delta, \alpha)$, *where* $C(\rho, \delta, \alpha)$ *is a positive constant depending only on* ρ, δ *and* α.

LEMMA 4 ([6], p. 205). *If* $f(z)$ *is a non-trivial solution of* (1.1), *where* $A_0, \cdots, A_{k-2}$ *are entire functions of finite order, then there is a positive constant* N *such that*

$$\log|f(z)|\leqq\exp(|z|^N). \tag{2.3}$$

LEMMA 5. *Suppose that* $k \geqq 3$ *and that* $A_0, \cdots, A_{k-2}$ *are entire functions of order less than* $1/2$ *such that for some* $s \in \{1, \cdots, k-2\}$,

(i) A_s *is transcendental*;
(ii) *For* $j \neq s$, *either* A_j *is a polynomial or we have* $\sigma(A_j) < \sigma(A_s)$.

If the equation (1.1) *has a solution* g *of finite order, then* g *is a polynomial.*

Proof. If $\sigma(A_s) > 0$ then we take σ, τ such that for $j \neq s$,

$$\sigma(A_j) < \sigma < \tau < \sigma(A_s). \tag{2.4}$$

By Lemma 2 (if $\mu(A_s) = \sigma(A_s)$) or by Lemma 3 (if $\mu(A_s) < \sigma(A_s)$) we know that there exists a set $H \subset (1, +\infty)$ with $m_1(H) = +\infty$, such that for all z satisfying $|z| = r \in H$, we have

$$\log |A_s(z)| > r^{\tau}. \tag{2.5}$$

If $\sigma(A_s) = 0$ (thus $\mu(A_s) = \sigma(A_s) = 0$), by Lemma 2 there also exists a set $H \subset (1, +\infty)$ with $m_1(H) = +\infty$, for $r \in H$

$$\frac{\min\{\log |A_s(z)| : |z| = r\}}{\log r} \to +\infty \tag{2.6}$$

as $r \to +\infty$. For convenience later on, we define σ to be zero if $\sigma(A_s) = 0$, so that we will always have, if $j \neq s$ and z is large,

$$|A_j(z)| = O(r^{M_1} \exp(r^{\sigma})), \quad M_1 > 0 \tag{2.7}$$

(Denote some fixed positive constants by $M_1, M_2, \cdots$ henceforth). If g is a solution of finite order of (1.1) and g is transcendental, by Lemma 1, there exists $H_1 \subset (1, +\infty)$ with $m_1(H_1) < +\infty$, for all z satisfying $|z| = r \notin H_1$,

$$|g^{(j)}(z)/g(z)| \leqq |z|^{M_2}, \tag{2.8}$$

for $j = 1, \cdots, k$. By (1.1) we obtain

$$g^{(k)}/g + A_{k-2} g^{(k-2)}/g + \cdots + A_s g^{(s)}/g + \cdots + A_0 = 0. \tag{2.9}$$

By (2.7), (2.8) and (2.9) we can get when $|z| = r \notin H_1$,

$$|A_s g^{(s)}/g| = O(r^{M_3} \exp(r^{\sigma})). \tag{2.10}$$

And by (2.5), (2.6) and (2.10), for all z satisfying $|z| = r \in H \backslash H_1$,

$$|z^s g^{(s)}/g| = o(1). \tag{2.11}$$

On the other hand, by Wiman-Valiron theory (e.g. see [12], [13]), there exists $H_2 \subset (1, +\infty)$ with $m_1(H_2) < +\infty$, such that if z satisfies $|z| = r \notin H_2$ and $|g(z)| = M(r, g)$, then $g^{(s)}(z)/g(z) \sim (\nu(r)/z)^s$, where $\nu(r)$ is the central index of g. From (2.11) we get $\nu(r) = o(1)$ when $r \in H \backslash (H_1 \cup H_2)$. Since g is transcendental, this is a contradiction. So g must be a polynomial.

The following Lemma 6 (see [13, p. 244]) is an improvement of a result of M. Frei ([5], [6]):

LEMMA 6. *Suppose that the coefficients of the linear differential equation*

$$a_n w^{(n)} + a_{n-1} w^{(n-1)} + \cdots + a_0 w = 0 \tag{2.12}$$

are entire functions, at least one of a_j $(o \leqq j \leqq n-1)$ *is transcendental, and* a_p $(0 \leqq p \leqq n-1)$ *is the first one in order of* $a_0, a_1, \cdots, a_{n-1}$ *such that*

$$\varliminf_{\substack{r\to\infty \\ r\notin D}} \left\{ \left(\sum_{j=p+1}^{n} m(r, a_j) \right) \Big/ m(r, a_p) \right\} < 1, \tag{2.13}$$

where D is some set in $(0, +\infty)$ *with finite measure. Then the equation* (2.12) *has at most p linearly independent meromorphic solutions with the property*

$$\varlimsup_{\substack{r\to\infty \\ r\notin D}} \{\log T(r, w)/m(r, a_p)\} = 0. \tag{2.14}$$

§ 3. Proof of Theorem 1

We first prove part (a) of Theorem 1.

It is well-known that if $A_0, \cdots, A_{k-1}$ are entire functions, then all solutions of the equation (1.1) are entire. As the proof in Lemma 5, by (2.4), (2.5), (2.6) and (2.7), it is easy to check that for an arbitrary set D of r with finite measure we have

$$\varliminf_{\substack{r\to\infty \\ r\notin D}} \left\{ \left(\sum_{j=s+1}^{k} m(r, A_j) \right) \Big/ m(r, A_s) \right\} \leqq \varliminf_{\substack{r\to\infty \\ r\in H\setminus D}} \left\{ \left(\sum_{j=s+1}^{k} m(r, A_j) \right) \Big/ m(r, A_s) \right\} < 1,$$

i.e. (2.13) holds for the equation (1.1). Additionally, since A_s is transcendental, it is easy to see that for an arbitrary polynomial solution $g(z)$, we have

$$\lim_{r\to\infty} (\log m(r, g)/m(r, A_s)) = 0,$$

i.e. (2.14) holds. Therefore, it follows that from Lemma 6 the equation (1.1) has at most s linearly independent polynomial solutions. This implies assertion (a) holds from Lemma 5.

Now, we start to prove part (b) and part (c).

Suppose that the equation (1.1) has q linearly independent transcendental solutions $f_1, \cdots, f_q$ each with $\lambda(f_j) < +\infty$. Thus, f_j can be written in the form $f_j = w_j e^{h_j}$ with $\sigma(w_j) < +\infty$, $j = 1, \cdots, q$. By Lemma 5, $\sigma(f_j) = +\infty$, so h_j must be a transcendental entire function. But by Lemma 4, $\sigma(h_j) < +\infty$. Therefore, by Lemma 1, there exists a set $E_2 \subset (1, +\infty)$ with $m_1(E_2) < +\infty$, for $|z| \notin E_2 \cup [0, 1]$, $m = 1, \cdots, k$ and $j = 1, \cdots, q$,

$$|w_j^{(m)}/w_j| + |h_j^{(m)}/h_j'| + \sum_{i \neq j} |(w_j/w_i)^{(m)}/(w_j/w_i)'| \leqq |z|^{M_4}. \tag{3.1}$$

If $\sigma(A_s)>0$, for $j\neq s$ we take σ, τ such that

$$\sigma(A_j)<\sigma<\tau<\sigma(A_s). \tag{3.2}$$

By the same reasoning as in the proof of Lemma 5, there exists a sequence $\{r_n\}$, $r_n\to\infty$, such that $r_n\notin E_2$ and such that on $|z|=r_n$ we have

$$\begin{cases} \log|A_s(z)|>r_n^\tau, & \text{if } \sigma(A_s)>0\,; \\ \dfrac{\min\{\log|A_s(z)| : |z|=r_n\}}{\log r_n}\to+\infty, & \text{if } \sigma(A_s)=0, \end{cases} \tag{3.3}$$

and moreover, for $j\neq s$, by defining $\sigma=0$ when $\sigma(A_s)=0$, we have always

$$|A_j(z)|=O(|z|^{M_5}\exp(|z|^\sigma)). \tag{3.4}$$

Now we estimate h_j' on $|z|=r_n$. By (3.1), there exists $N>0$ (N can be taken large enough) such that if z on $|z|=r_n$ and $|h_j'(z)|\geqq|z|^N$ then we have

$$f_j^{(p)}(z)/f_j(z)=(h_j')^p(1+O(|z|^{-M_6})), \tag{3.5}$$

where $p=1,\cdots,k$. Substituting $f_j=w_je^{h_j}$ into (1.1) and dividing through by f_j, at a point z on $|z|=r_n$ satisfying $|h_j'(z)|\geqq|z|^N$, by (3.4) and (3.5) we have

$$\begin{aligned} &(h_j')^k(1+o(1))+A_s(h_j')^s(1+o(1))+\sum_{\substack{m=0\\(m\neq s)}}^{k-2}O(r_n^{M_5}\exp(r_n^\sigma))(h_j')^m=0, \text{ or} \\ &((h_j')^{k-s}/A_s)(1+o(1))+1+\sum_{\substack{m=0\\(m\neq s)}}^{k-2}O(r_n^{M_5}\exp(r_n^\sigma))(h_j')^{m-k}(h_j')^{k-s}/A_s=0, \text{ or} \\ &((h_j')^{k-s}/A_s)(1+o(1))+1+((h_j')^{k-s}/A_s)O(r_n^{M_5}\exp(r_n^\sigma))/(h_j')^2=0, \text{ or} \\ &((h_j')^{k-s}/A_s)(1+O(r_n^{M_5}\exp(r_n^\sigma))/(h_j')^2)=-1. \end{aligned} \tag{3.6}$$

If there are infinitely many n, say n_k, such that at a point z on $|z|=r_{n_k}$ satisfying $|h_j'(z)|\geqq|z|^N$, we have $|h_j'(z)|\leqq\exp(r_{n_k}^\sigma)$, then by (3.3), (3.6) can not hold as n_k is large enough (because the left side of (3.6) tends to zero as $n_k\to\infty$). So if n is large enough, at a point z on $|z|=r_n$ satisfying $|h_j'(z)|\geqq|z|^N$, $|h_j'(z)|>\exp(r_n^\sigma)$ must hold. Thus, at a point z on $|z|=r_n$ satisfying $|h_j'(z)|\geqq|z|^N$, from (3.6) we can get

$$(h_j')^{k-s}/A_s=-1+o(1), \tag{3.7}$$

and

$$A_s/(h_j')^{k-s}=-(1+O(r_n^{M_5})/h_j'). \tag{3.8}$$

Now set $G_1=\{z: |h_j'(z)|\geqq|z|^N,\ |z|=r_n\}$, $G_2=\{z: |h_j'(z)|<|z|^N,\ |z|=r_n\}$, $G=\{z: |z|=r_n\}$, then $G=G_1\cup G_2$, $G_1\cap G_2=\emptyset$. By (3.3) and (3.7), it is not difficult to see that $G_1=\{z: |h_j'(z)|>|z|^N,\ |z|=r_n\}$ as r_n is large enough. Since the linearly connected open set G can be separated into the union of two open sets and $G_1 \cap G_2 = \emptyset$, one of the sets G_1 and G_2 must be empty. Because $h_j'(z)$ is a transcendental entire function, so $G_1 \neq \emptyset$ as r_n is large enough, and this is to say $G_2 = \emptyset$. Therefore, (3.5), (3.6), (3.7) and (3.8) always hold on $|z| = r_n$ as r_n is large enough.

Using (3.7), from (3.8) we obtain

$$\begin{aligned} A_s^{1/(k-s)}/h_j' &= (-1+O(r_n^{M_5})/h_j')^{1/(k-s)} \\ &= (-1)^{1/(k-s)}(1+O(r_n^{M_5})/h_j'). \end{aligned}$$

So on $|z| = r_n$,

$$A_s^{1/(k-s)} - (-1)^{1/(k-s)} h_j' = O(r_n^{M_5}).$$

Hence, on $|z| = r_n$, there exists $c_{j,n}$ such that $c_{j,n}^{k-s} = -1$ and

$$|h_j'(z) - c_{j,n} A_s(z)^{1/(k-s)}| \leqq r_n^{M_7}. \tag{3.9}$$

Especially,

$$|h_1'(z) - c_{1,n} A_s(z)^{1/(k-s)}| \leqq r_n^{M_7}.$$

Therefore, there exists $a_{j,n}$ such that $a_{j,n}^{k-s} = 1$ and such that on $|z| = r_n$

$$|h_j'(z) - a_{j,n} h_1'(z)| \leqq r_n^{M_8}. \tag{3.10}$$

Since $a_{j,n}$ is a $(k-s)$-th root of unity, there exists a fixed one, say a_j with $a_j^{k-s} = 1$, which makes (3.10) to hold for infinitely many n. Thus, $h_j'(z) - a_j h_1'(z)$ is a polynomial and so is $h_j(z) - a_j h_1(z)$. Set $h_j(z) - a_j h_1(z) = P_j(z)$, then $f_j = w_j e^{h_j} = w_j e^{P_j} \cdot e^{a_j h_1}$. Since the term e^{P_j} with P_j a polynomial may be incorporated into w_j, we conclude that without loss of generality $h_j(z) = a_j h_1(z)$, $j = 1, \cdots, q$. Then f_j can be written in the form

$$f_j = w_j e^{a_j h_1}, \quad j = 1, \cdots, q. \tag{3.11}$$

As we have known, a_j is some $(k-s)$-th root of unity, but we will further show that a_j is also some $(s-1)$-th root of unity. Using (3.7) and (3.3), from (3.6) we can get an improving form of (3.8): On $|z| = r_n$,

$$A_s/(h_j')^{k-s} = -(1+O(r_n^{M_5})/(h_j')^{1+\varepsilon}), \tag{3.8$'$}$$

where $0 < \varepsilon < 1$ is a given constant. In addition, by [11], Lemma 3.5, by (3.1) and by a straightforward inductive argument applied to $f_j'/f_j = h_j' + w_j'/w_j$ on $|z| = r_n$, we get

$$f_j^{(p)}/f_j = O(r_n^{M_4})(h_j')^{p-2} + \binom{p}{1}(h_j')^{p-1} w_j'/w_j + \binom{p}{2}(h_j')^{p-2} h_j'' + (h_j')^p \tag{3.12}$$

for $p = 2, \cdots, k$. Moreover, by (3.1) and (3.8)$'$, we may write on $|z| = r_n$

$$(3.13)\qquad A_s f_j^{(s)}/f_j = A_s\Big[O(r_n^{M_4})(h_j')^{s-2}+\binom{s}{1}(h_j')^{s-1}w_j'/w_j+\binom{s}{2}(h_j')^{s-2}h_j''+(h_j')^s\Big]$$
$$=O(r_n^{M_9})(h_j')^{k-1-\varepsilon}-\binom{s}{1}(h_j')^{k-1}w_j'/w_j-\binom{s}{2}(h_j')^{k-2}h_j''-(h_j')^k.$$

Since f_1 and f_j are solutions of (1.1), we have

$$f_1^{(k)}/f_1+\cdots+A_1f_1'/f_1=f_j^{(k)}/f_j+\cdots+A_1f_j'/f_j.$$

From it and by (3.4), (3.5), on $|z|=r_n$, we can get

$$f_1^{(k)}/f_1+A_sf_1^{(s)}/f_1+\sum_{\substack{m=1\\(m\neq s)}}^{k-2}O(r_n^{M_5}\exp(r_n^\sigma))(h_1')^m$$
$$=f_j^{(k)}/f_j+A_sf_j^{(s)}/f_j+\sum_{\substack{m=1\\(m\neq s)}}^{k-2}O(r_n^{M_5}\exp(r_n^\sigma))(h_j')^m.$$

By (3.12), (3,13), (3.11), (3.7), (3.3) and (3.2), dividing through by $(h_1')^{k-1}$, the above formula can be changed into the form on $|z|=r_n$

$$(3.14)\qquad 2(k-s)w_1'/w_1+[k(k-1)-s(s-1)]h_1''/h_1'$$
$$=a_j^{k-1}\{2(k-s)w_j'/w_j+[k(k-1)-s(s-1)]h_1''/h_1'\}+O(r_n^{M_{10}})(h_1')^{-\varepsilon}.$$

Denote by F_1 and F_j respectively the entire functions $w_1^{2(k-s)}(h_1')^{k(k-1)-s(s-1)}$ and $w_j^{2(k-s)}(h_1')^{k(k-1)-s(s-1)}$. By (3.9), h_1' has lower order less than 1/2, so that F_1 and F_j have infinitely many zeros. But the argument principle and (3.14) give

$$n(r_n,1/F_1)=a_j^{k-1}n(r_n,1/F_j)+O(r_n^{M_{10}})M(r_n,(h_1')^{-\varepsilon}).$$

Since, by (3.7) and (3.3), $O(r_n^{M_{10}})M(r_n,(h_1')^{-\varepsilon})\to 0$ as $n\to\infty$ on $|z|=r_n$, the above formula implies that $a_j^{k-1}=1$, i.e. a_j is also a $(k-1)$-th root of unity. And from $1=a_j^{k-1}=a_j^{k-s+s-1}=a_j^{s-1}$, we see that a_j is also a $(s-1)$-th root of unity.

Since a_j is of above properties, it follows that: 1. In part (b), if $q>\min\{k-s, s-1\}$, then there must exist i and p, $i\neq p$, such that $a_i=a_p$. Thus, by (3.11) we have $h_i=h_p$. 2. In part (c), assume $q=2$. Since $s=2$ or $k-1$ and $k-s$ are relatively prime, $a_j=1$ must hold for $j=1, 2$. Therefore, in both cases, there are two transcendental solutions, which may be denoted by f_1 and f_2 in Case 1 without loss of generality, of the form $f_1=w_1e^h$, $f_2=w_2e^h$, where we set $h=h_1=h_2$. We proceed to show that this is impossible, which proves part (b) and part (c).

Set $f=uf_1$, $v=u'$, and substitute them into (1.1), we obtain

$$(3.15)\qquad v^{(k-1)}+d_{k-2}v^{(k-2)}+\cdots+d_0v=0,$$

where

$$(3.16)\qquad \begin{cases} d_{k-2}=kf_1'/f_1, \\ d_{k-3}=A_{k-2}+\binom{k}{2}f_1''/f_1, \\ \quad\cdots\cdots \\ d_j=A_{j+1}+\binom{j+2}{1}A_{j+2}f_1'/f_1+\cdots+\binom{j+i}{i-1}A_{j+i}f_1^{(i-1)}/f_1+\cdots \\ \qquad +\binom{s}{s-j-1}A_sf_1^{(s-j-1)}/f_1+\cdots+\binom{k}{k-j-1}f_1^{(k-j-1)}/f_1, \\ \quad\cdots\cdots \\ d_0=A_1+\binom{2}{1}A_2f_1'/f_1+\cdots+\binom{s}{s-1}A_sf_1^{(s-1)}/f_1+\cdots+kf_1^{(k-1)}/f_1. \end{cases}$$

Since $f_1^{(j)}/f_1=(h')^j(1+o(1))$ $(1\leqq j\leqq k)$ on $|z|=r_n$, by (3.3), (3.4), (3.7) and (3.16), we obtain

$$(3.17)\qquad \begin{cases} d_0=(k-s)(h')^{k-1}(1+o(1)), \\ d_1=\left(\binom{k}{k-2}-\binom{s}{s-2}\right)(h')^{k-2}(1+o(1)), \\ \quad\cdots\cdots \\ d_{s-1}=\left(\binom{k}{k-s}-\binom{s}{0}\right)(h')^{k-s}(1+o(1)), \\ \quad\cdots\cdots \\ d_{k-2}=\binom{k}{1}h'(1+o(1)) \end{cases}$$

on $|z|=r_n$. Obviously, $v=(f_2/f_1)'=(w_2/w_1)'$ is a solution of (3.15). By (3.1), when $|z|\notin E_2\cup[0, 1]$,

$$(3.18)\qquad |v^{(j)}(z)/v(z)|\leqq|z|^{M_4}.$$

for $j=1, \cdots, k-1$. Dividing through (3.15) by v, we obtain

$$(3.19)\qquad v^{(k-1)}/v+d_{k-2}v^{(k-2)}/v+\cdots+d_1v'/v+d_0=0.$$

Dividing through (3.19) by $(h')^{k-1}$ again, by (3.3), (3.7), (3.17) and (3.18) on $|z|=r_n$, we obtain $k-s=0$ as $r_n\to\infty$. This is a contradiction, since $s\in\{2, \cdots, k-2\}$.

The proof of Theorem 1 has been completed.

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