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LINEAR ISOMETRIC OPERATORS ON THE $C_0^{(n)}(X)$ TYPE SPACES

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Abstract

In this paper, we try to investigate the representations of isometries, isometry groups and the space classifications of the $C_0^{(n)}(X)$ type spaces $(X \subset \boldsymbol{R}^m, m, n \geqq 1)$.

§ 0. Introduction

Let $\boldsymbol{Z}_+$ be the set of non-negative integers. We make the following notations:

$$\boldsymbol{x}=(x_1, x_2, \cdots, x_m) \in \boldsymbol{R}^m \quad \boldsymbol{r}=(r_1, r_2, \cdots, r_m) \in \boldsymbol{Z}_+^m$$

$$\boldsymbol{r}!=r_1!r_2! \cdots r_m! \qquad |\boldsymbol{r}|=r_1+r_2+\cdots+r_m$$

$$f^{(\boldsymbol{r})}(\boldsymbol{x})=\frac{\partial^{r_1+r_2+\cdots+r_m} f(\boldsymbol{x})}{\partial x_1^{r_1} \partial x_2^{r_2} \cdots \partial x_m^{r_m}}.$$

If Ω is a locally compact Hausdorff space, $C_0(\Omega)$ denotes the Banach space consisting of continuous function f on Ω vanishing at infinity (i.e., $\{\omega \in \Omega : |f(\omega)| \geqq \varepsilon\}$ is compact for all $\varepsilon>0$), with the norm $\|f\|=\sup\{|f(\omega)| : \omega \in \Omega\}$. For any integers m, $n \geqq 1$, set $\Gamma=\{\boldsymbol{r}=(r_1, \cdots, r_m) \in \boldsymbol{Z}_+^m : r_1+\cdots+r_m \leqq n\}$. A subset X of $\boldsymbol{R}^m$ is called to be NIP: if for any line L parallel to one of the axes of $\boldsymbol{R}^m$ the set $L \cap X$ contains no isolated points. If X is a locally compact and NIP subset of $\boldsymbol{R}^m$, we use $C_0^{(n)}(X)$ to denote the normed space consisting of all function f on X which satisfies: $f^{(\boldsymbol{r})} \in C_0(X)$ for all $\boldsymbol{r} \in \Gamma$, with the norm $\|f\| =\sup_{\boldsymbol{x} \in X} \sum_{\boldsymbol{r} \in \Gamma} |f^{(\boldsymbol{r})}(\boldsymbol{x})|/\boldsymbol{r}!$. We set $C_0^{(0)}(X)=C_0(X)$ and use $S_{n,X}$ to denote the unit sphere of $C_0^{(n)}(X)$.

For the case $n=m=1$ and $X, Y \subseteqq \boldsymbol{R}^1$, the representations of surjective linear isometries between $C_0^{(1)}(X)$ and $C_0^{(1)}(Y)$ had been studied by Cambern and Pathak [1] (complex case only), for $m=1$, $n \geqq 1$ and $X=Y=[0, 1]$, by Pathak [2] (complex case only), and for $m=1$, $n \geqq 1$ and X, $Y \subseteqq \boldsymbol{R}^1$ with some conditions by

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the author [3] (real case and complex case) In this paper, we try to consider the most general case: m_1, m_2, n_1, $n_2 \geqq 1$ and $X \subseteqq \boldsymbol{R}^{m_1}$, $Y \subseteqq \boldsymbol{R}^{m_2}$ are locally compact and NIP. Particularly, when X and Y are open sets, a complete representation of linear isometries from $C_0^{(n_1)}(X)$ onto $C_0^{(n_2)}(Y)$ is obtained (Theorem 3.5), the results are true in both the real case and the complex case, extending the results of all the papers mentioned above.

We shall begin the discussion in section § 1 with the representation of extreme points of the unit ball of $C_0^{(n)}(X)^*$, which is very important for the construction of the map Φ_T in the next section. By using the basic lemmas established in section § 2, we state and prove the representations of surjective linear isometries between $C_0^{(n)}(X)$ type spaces in section § 3. Finally, as an application, we consider the isometry group of $C_0^{(n)}(X)$ and give some interesting examples in section § 4.

It is easy to check that $fg \in C_0^{(n)}(X)$ and $\|fg\| \leqq \|f\|\|g\|$ for all f, $g \in C_0^{(n)}(X)$, thus $C_0^{(n)}(X)$ is a Banach algebra when it is complete[1].

§ 1. The extreme points of the unit ball of $C_0^{(n)}(X)^*$

PROPOSITION 1.1. *For any* $\boldsymbol{x}_0 \in \boldsymbol{R}^m$ *and any* ε, $\delta > 0$, *there exists an* $f \in S_{n,\boldsymbol{R}^m}$ *such that* $\operatorname{supp}(f) \subseteqq N_\delta(\boldsymbol{x}_0)$ *and* $(1/n!)|\partial^n f/\partial x_1^n(\boldsymbol{x}_0)| > 1-\varepsilon$.

Proof. For any $\delta > 0$, take a $\varphi \in C_0^{(n)}(\boldsymbol{R}^m)$ with $\operatorname{supp}(\varphi) \subseteqq N_{\delta/2}(\boldsymbol{0})$ and $(\partial^n \varphi/\partial x_1^n)(\boldsymbol{0}) \neq 0$ (e.g., we can take $\psi \in C_0^{(n)}(\boldsymbol{R}^m)$ such that $\operatorname{supp}(\psi) \subseteqq N_{\delta/2}(\boldsymbol{0})$ and $\psi(U)=1$ for some open neighbourhood U of $\boldsymbol{0}$, then $\varphi(\boldsymbol{x}) = \psi(\boldsymbol{x}) x_1^n$ ($\forall \boldsymbol{x}=(x_1, \cdots, x_m) \in \boldsymbol{R}^m$) has the desired properties!). For any $k \geqq 1$, define

$$g_k(\boldsymbol{x}) = \varphi(kx_1, x_2, \cdots, x_m), \quad \boldsymbol{x}=(x_1, \cdots, x_m) \in \boldsymbol{R}^m.$$

It is easy to see that $\operatorname{supp}(g_k) \subseteqq \operatorname{supp}(\varphi) \subseteqq N_{\delta/2}(\boldsymbol{0})$ and

$$\frac{\partial^{r_1+\cdots+r_m} g_k(\boldsymbol{x})}{\partial x_1^{r_1} \cdots \partial x_m^{r_m}} = k^{r_1} \frac{\partial^{r_1+\cdots+r_m}\varphi}{\partial x_1^{r_1} \cdots \partial x_m^{r_m}}(kx_1, x_2, \cdots, x_m).$$

From which, we can show

$$\frac{k^n}{n!}\left|\frac{\partial^n \varphi(\boldsymbol{0})}{\partial x_1^n}\right| \leqq \|g_k\| \leqq k^n \|\varphi\|, \quad k \geqq 1.$$

Set $f_k = (g_k/\|g_k\|) \in S_{n,\boldsymbol{R}^m}$ ($k \geqq 1$). Then, $\operatorname{supp}(f_k) = \operatorname{supp}(g_k) \subseteqq N_{\delta/2}(\boldsymbol{0})$ and

$$\begin{aligned}\sum_{\substack{|\boldsymbol{r}| \leq n \\ r_1 \neq n}} \frac{|f_k^{(\boldsymbol{r})}(\boldsymbol{x})|}{\boldsymbol{r}!} &= \frac{1}{\|g_k\|} \sum_{\substack{|\boldsymbol{r}| \leq n \\ r_1 \neq n}} \frac{|g_k^{(\boldsymbol{r})}(\boldsymbol{x})|}{\boldsymbol{r}!} \\ &\leqq \frac{1}{\dfrac{1}{n!}\left|\dfrac{\partial^n \varphi(\boldsymbol{0})}{\partial x_1^n}\right| k^n} \sum_{\substack{|\boldsymbol{r}| \leq n \\ r_1 \neq n}} \frac{k^{r_1}|\varphi^{(\boldsymbol{r})}(kx_1, x_2, \cdots, x_m)|}{\boldsymbol{r}!}\end{aligned}$$

[1] For the completeness of $C_0^{(n)}(X)$ type spaces ($n \geqq 1$), see [4].

$$\leqq \frac{n!}{\left|\frac{\partial^n \varphi(\mathbf{0})}{\partial x_1^n}\right| k} \sum_{\substack{|\boldsymbol{r}|\leqq n \\ r_1\neq n}} \frac{|\varphi^{(\boldsymbol{r})}(kx_1, x_2, \cdots, x_m)|}{\boldsymbol{r}!}$$

$$\leqq \frac{n!}{\left|\frac{\partial^n \varphi(\mathbf{0})}{\partial x_1^n}\right| k} \|\varphi\| \to 0 \quad (\text{as } k\to\infty).$$

For any $\varepsilon>0$, there exists a $k_0\geqq 1$ such that

$$\sum_{\substack{|\boldsymbol{r}|\leqq n \\ r_1\neq n}} \frac{|f_{k_0}^{(\boldsymbol{r})}(\boldsymbol{x})|}{\boldsymbol{r}!} < \varepsilon, \quad \forall \boldsymbol{x}\in \boldsymbol{R}^m. \tag{1.1}$$

Take a $\boldsymbol{y}_0\in \text{supp}(f_{k_0})\subseteqq N_{\delta/2}(\mathbf{0})$ with

$$\sum_{|\boldsymbol{r}|\leqq n} \frac{|f_{k_0}^{(\boldsymbol{r})}(\boldsymbol{y}_0)|}{\boldsymbol{r}!} = \|f_{k_0}\| = 1,$$

then, from (1.1) we have

$$\frac{1}{n!}\left|\frac{\partial^n f_{k_0}(\boldsymbol{y}_0)}{\partial x_1^n}\right| = 1 - \sum_{\substack{|\boldsymbol{r}|\leqq n \\ r_1\neq n}} \frac{|f_{k_0}^{(\boldsymbol{r})}(\boldsymbol{y}_0)|}{\boldsymbol{r}!} > 1-\varepsilon$$

Finally, the function f defined by

$$f(\boldsymbol{x}) = f_{k_0}(\boldsymbol{x}-\boldsymbol{x}_0+\boldsymbol{y}_0), \quad \forall \boldsymbol{x}\in \boldsymbol{R}^m,$$

belonge to $S_{n,\boldsymbol{R}^m}$ with

$$\text{supp}(f)\subseteqq N_{\delta/2}(\boldsymbol{x}_0-\boldsymbol{y}_0)\subseteqq N_\delta(\boldsymbol{x}_0)$$

and

$$\frac{1}{n!}\left|\frac{\partial^n f(\boldsymbol{x}_0)}{\partial x_1^n}\right| = \frac{1}{n!}\left|\frac{\partial^n f_{k_0}(\boldsymbol{y}_0)}{\partial x_1^n}\right| > 1-\varepsilon. \qquad \square$$

Let X be a locally compact and NIP subset of $\boldsymbol{R}^m$ $(m\geqq 1)$. For any $n\geqq 1$, define

$$\mathcal{S}_{m,n} = \{\boldsymbol{\alpha}=(\alpha_r)\in \boldsymbol{K}^{\#\Gamma} : |\alpha_r|=1 \ (\forall \boldsymbol{r}\in\Gamma)\}$$

$$W = X\times \mathcal{S}_{m,n}$$

where Γ is as before. Then, W is a locally compact Hausdorff space with the product topology We use $C_0(W)$ to denote the Banach space of continuous functions on W vanishing at infinity, with the sup norm.

If $f\in C_0^{(n)}(X)$, define $\tilde{f}\in C_0(W)$ by

$$\tilde{f}(\boldsymbol{x}, \boldsymbol{\alpha}) = \sum_{\boldsymbol{r}\in\Gamma} \frac{\alpha_r f^{(\boldsymbol{r})}(\boldsymbol{x})}{\boldsymbol{r}!}, \quad \forall (\boldsymbol{x}, \boldsymbol{\alpha})\in W,$$

then the mapping $f\mapsto\tilde{f}$ is clearly a linear isometry of $C_0^{(n)}(X)$ onto a (linear) subspace A of $C_0(W)$ (We look $C_0^{(n)}(X)$ and A as the same space). We can prove, in a routine way, that any extreme point f^* of the unit ball of $A^*=C_0^{(n)}(X)^*$ is the restriction on A of some $g^*\in\operatorname{ext} B_{C_0(W)^*}$, henceforth, $g^*=\lambda\delta_w$ for some $w=(\boldsymbol{x}, \boldsymbol{\alpha})\in W$ and some complex number $|\lambda|=1$ (See W. Rudin's book [5]). That is

$$f^*(f)=g^*(\tilde{f})=\lambda\delta_w(\tilde{f})=\lambda\sum_{r\in\Gamma}\frac{\alpha_r f^{(r)}(\boldsymbol{x})}{\boldsymbol{r}!}=\sum_{r\in\Gamma}\frac{\beta_r f^{(r)}(\boldsymbol{x})}{\boldsymbol{r}!}, \quad \forall f\in C_0^{(n)}(X)$$

where $\boldsymbol{\beta}=\lambda\boldsymbol{\alpha}$. That means any $f^*\in\operatorname{ext} B_{C_0^{(n)}(X)^*}$ is of the form

$$f^*(f)=\delta_{(\boldsymbol{x},\boldsymbol{\beta})}(f)=\sum_{r\in\Gamma}\frac{\beta_r f^{(r)}(\boldsymbol{x})}{\boldsymbol{r}!}, \quad \forall f\in C_0^{(n)}(X)$$

for some $(\boldsymbol{x}, \boldsymbol{\beta})\in W$.

Now, we go to show the inverse.

LEMMA 1.2. *For any $w_0=(\boldsymbol{x}_0, \boldsymbol{\alpha})\in W$, the linear functional δ_{w_0} on $C_0^{(n)}(X)$ defined by*

$$\delta_{w_0}(f)=\sum_{r\in\Gamma}\frac{\alpha_r}{\boldsymbol{r}!}f^{(r)}(\boldsymbol{x}_0), \quad \forall f\in C_0^{(n)}(X),$$

is an extreme point of the unit ball $B_{C_0^{(n)}(X)^*}$ of $C_0^{(n)}(X)^*$.

Proof. It is clear that $\|\delta_{w_0}\|\leqq 1$. Suppose that $\delta_{w_0}=(f_1^*+f_2^*)/2$ for some $f_1^*, f_2^*\in B_{C_0^{(n)}(X)^*}$. By the Hahn-Banach Theorem, the functionals f_1^*, f_2^* can be extended to be functionals $g_1^*, g_2^*\in B_{C_0(W)^*}$. Applying the Riesz Representation Theorem (cf. [5]), there are regular Borel measures μ_1, μ_2 on $(W, \mathscr{B}_W)$ such that $|\mu_i|(W)=\|g_i^*\|\leqq 1$ $(i=1, 2)$ and

$$g_i^*(f)=\int_W\tilde{f}(w)d\mu_i(w), \quad \forall f\in C_0^{(n)}(X), \quad i=1, 2. \tag{1.2}$$

From Proposition 1.1, for any $\varepsilon, \delta>0$ there exists an $f\in S_{n,\mathbf{R}^m}$ such that $\operatorname{supp}(f)\subseteqq N_\delta(\boldsymbol{x}_0)$ and

$$\frac{1}{n!}\left|\frac{\partial^n f(\boldsymbol{x}_0)}{\partial x_1^n}\right|>1-\varepsilon.$$

Especially, when $\delta>0$ is small enough $f\in C_0^{(n)}(X)$ and $\|f\|\leqq 1$, therefore,

$$\begin{aligned}|\delta_{w_0}(f)|=\left|\sum_{r\in\Gamma}\frac{\alpha_r f^{(r)}(\boldsymbol{x}_0)}{\boldsymbol{r}!}\right|&\geqq\frac{1}{n!}\left|\frac{\partial^n f(\boldsymbol{x}_0)}{\partial x_1^n}\right|-\sum_{\substack{|r|\leq n\\ r_1\neq n}}\frac{|\alpha_r f^{(r)}(\boldsymbol{x}_0)|}{\boldsymbol{r}!}\\ &>1-\varepsilon-\varepsilon=1-2\varepsilon,\end{aligned}$$

and

$$\left|\frac{g_1^*(f)+g_2^*(f)}{2}\right|=|\delta_{w_0}(f)|>1-2\varepsilon. \tag{1.3}$$

Since $\|g_i^*\|\leqq 1$ $(i=1, 2)$ and $\|f\|\leqq 1$, from (1.3) we have $|g_i^*(f)|\geqq 1-4\varepsilon$ $(i=1, 2)$, thus

(1.4) $$1-4\varepsilon \leqq |g_i^*(f)| = \left|\int_W \tilde{f}(w)d\mu_i(w)\right| = \left|\int_{N_\delta(\boldsymbol{x}_0)\times\mathcal{S}_{m,n}} \tilde{f}(w)d\mu_i(w)\right| \leqq |\mu_i|(N_\delta(\boldsymbol{x}_0)\times\mathcal{S}_{m,n}), \quad i=1, 2.$$

Letting $\varepsilon\to 0$ in (1.4), we get

$$1\leqq |\mu_i|(N_\delta(\boldsymbol{x}_0)\times\mathcal{S}_{m,n}), \quad \forall\delta>0, \quad i=1, 2.$$

Since the Borel measure $|\mu_i|$ is regular and $K=\{\boldsymbol{x}_0\}\times\mathcal{S}_{m,n}$ is compact, setting $\delta\to 0$ we get

$$1\leqq |\mu_i|(K)\leqq |\mu_i|(W)=\|g_i^*\|\leqq 1, \quad i=1, 2$$

which implies

(1.5) $$|\mu_i|(W)=|\mu_i|(K)=1, \quad |\mu_i|(K^c)=0, \quad i=1, 2.$$

By (1.2) and (1.5), we obtain

(1.6) $$g_i^*(g)=\int_W \tilde{g}(w)d\mu_i(w)=\int_K \tilde{g}(w)d\mu_i(w), \quad \forall g\in C_0^{(n)}(X), \quad i=1, 2.$$

Take a $\varphi\in C_0^{(n)}(\boldsymbol{R}^m)$ such that

$$\text{supp}(\varphi)\subseteqq N_{\delta_0}(\boldsymbol{x}_0) \quad \text{and} \quad \varphi(U)=1,$$

where $\delta_0>0$ is small enough so that $\overline{N_{\delta_0}(\boldsymbol{x}_0)}\cap X$ is compact and U is an open neighbourhood of $\boldsymbol{x}_0$. Write

$$\psi(\boldsymbol{x})=\sum_{\boldsymbol{r}\in\Gamma}\bar{\alpha}_{\boldsymbol{r}}(x_1-x_{01})^{r_1}\cdots(x_m-x_{0m})^{r_m}, \quad \boldsymbol{x}=(x_1, \cdots, x_m)\in\boldsymbol{R}^m$$

$$\boldsymbol{x}_0=(x_{01}, \cdots, x_{0m})\in\boldsymbol{R}^m$$

and define

$$g(\boldsymbol{x})=\varphi(\boldsymbol{x})\psi(\boldsymbol{x}), \quad \forall\boldsymbol{x}\in\boldsymbol{R}^m.$$

Then $g\in C_0^{(n)}(X)$ and

(1.7) $$\alpha_{\boldsymbol{r}}g^{(\boldsymbol{r})}(\boldsymbol{x}_0)>0, \quad \forall\boldsymbol{r}\in\Gamma.$$

Therefore, from (1.2), (1.6) and (1.7) we have

$$\sum_{\boldsymbol{r}\in\Gamma}\frac{|g^{(\boldsymbol{r})}(\boldsymbol{x}_0)|}{\boldsymbol{r}!}=\sum_{\boldsymbol{r}\in\Gamma}\frac{\alpha_{\boldsymbol{r}}g^{(\boldsymbol{r})}(\boldsymbol{x}_0)}{\boldsymbol{r}!}=\delta_{w_0}(g)=\frac{f_1^*(g)+f_2^*(g)}{2}=\frac{g_1^*(g)+g_2^*(g)}{2}$$
$$=\frac{1}{2}\left[\int_K \tilde{g}(w)d\mu_1(w)+\int_K \tilde{g}(w)d\mu_2(w)\right].$$

Since $|\mu_1|(K)=|\mu_2|(K)=1$ and

$$|\tilde{g}(w)|\leqq\sum_{\boldsymbol{r}\in\Gamma}\frac{|g^{(\boldsymbol{r})}(\boldsymbol{x}_0)|}{\boldsymbol{r}!}, \quad \forall w\in K,$$

we must have

$$(1.8)\qquad \int_K \tilde{g}(w)d\mu_i(w)=\sum_{r\in\Gamma}\frac{|g^{(r)}(x_0)|}{r!}=\delta_{w_0}(g),\quad i=1,\ 2.$$

For any $w\in K_0=K\setminus\{(x_0,\ \lambda\alpha):|\lambda|=1\}$, from (1.7) we can show that

$$|\tilde{g}(w)|<\sum_{r\in\Gamma}\frac{|g^{(r)}(x_0)|}{r!}.$$

Therefore, from $\|\mu_i\|=1$ and (1.8) we must have $|\mu_i|(K_0)=0$ ($i=1,\ 2$). Thus, from (1.8),

$$(1.9)\qquad \int_{K_1}\tilde{g}(w)d\mu_i(w)=\int_K \tilde{g}(w)d\mu_i(w)=\sum_{r\in\Gamma}\frac{|g^{(r)}(x_0)|}{r!}=\delta_{w_0}(g)$$

where $K_1=K\setminus K_0=\{(x_0,\ \lambda\alpha):|\lambda|=1\}$.

Finally, for any $h\in C_0^{(n)}(X)$ and $w=(x_0,\ \lambda\alpha)\in K_1$, an easy calculation shows that

$$\tilde{h}(w)=\sum_{r\in\Gamma}\frac{\lambda\alpha_r h^{(r)}(x_0)}{r!}=\lambda\delta_{w_0}(h)=\delta_{w_0}(h)\frac{\tilde{g}(w)}{\delta_{w_0}(g)},$$

thus, by (1.2), (1.5) and (1.9),

$$f_i^*(h)=g_i^*(h)=\int_W \tilde{h}(w)d\mu_i(w)=\int_K \tilde{h}(w)d\mu_i(w)=\int_{K_1}\tilde{h}(w)d\mu_i(w)$$
$$=\frac{\delta_{w_0}(h)}{\delta_{w_0}(g)}\int_{K_1}\tilde{g}(w)d\mu_i(w)=\delta_{w_0}(h),\quad \forall h\in C_0^{(n)}(X),\ i=1,\ 2,$$

that is, $f_1^*=f_2^*=\delta_{w_0}$ and $\delta_{w_0}\in\operatorname{ext} B_{C_0^{(n)}(X)^*}$. □

Remark 1. We do not know whether for any $w=(x,\ \alpha)\in W$ there exists a "*peak function*" $f\in C_0^{(n)}(X)$ such that

$$\sum_{r\in\Gamma}\frac{|f^{(r)}(y)|}{r!}<\sum_{r\in\Gamma}\frac{|f^{(r)}(x)|}{r!},\quad \forall y\neq x,\ y\in X$$

and $\alpha_r f^{(r)}(x)>0$ ($r\in\Gamma$), henceforth, we can not directly use the Lemma 3.2 of [6] as being used by many other authors, but the results concerning the representations of the extreme points of $C_0^{(n)}(X)^*$ are the same.

THEOREM 1.3. *Let $m,\ n\geqq 1$ and $X,\ W$ be the same as before. Then,*

$$f^*\in\operatorname{ext} B_{C_0^{(n)}(X)^*}\Longleftrightarrow f^*=\delta_w \text{ for some } w\in W.$$

Moreover, the map $\xi: w\mapsto\delta_w$ is a homeomorphism of W onto ($\operatorname{ext} B_{C_0^{(n)}(X)^*}$, *weak**), *where weak* means the weak-star topology.*

Proof. The first part has been proved. We only prove the latter part. Let us note that the map ξ is one-to-one and onto, we only need to show the continuities of ξ and ξ^{-1}. From the definition, the continuity of ξ is trivial.

Now, if $\delta_{w_d} \xrightarrow{\text{weak*}} \delta_w$ ($w_d=(\boldsymbol{x}_d, \boldsymbol{\alpha}_d)$, $w=(\boldsymbol{x}, \boldsymbol{\alpha})\in W$), by taking $g\in C_0^{(n)}(X)$ so that $\mathrm{supp}(g)\subseteq\overline{N_\delta(\boldsymbol{x})}\cap X$ is compact and $\delta_w(g)\neq 0$, then $\delta_{w_d}(g)\neq 0$ ($\forall d\geqq d_0$) for some d_0. Thus, $\boldsymbol{x}_d\in\mathrm{supp}(g)$ ($d\geqq d_0$), without loss of generality we assume that $\lim_d \boldsymbol{x}_d=\boldsymbol{y}\in\mathrm{supp}(g)$ and $\lim_d \boldsymbol{\alpha}_d=\boldsymbol{\beta}\in\mathcal{S}_{m,n}$. For any $h\in C_0^{(n)}(X)$ we have

$$\delta_{(\boldsymbol{y},\boldsymbol{\beta})}(h)=\lim_d \delta_{w_d}(h)=\delta_w(h),$$

which implies that $(\boldsymbol{y}, \boldsymbol{\beta})=w$ and $\lim_d \xi^{-1}(\delta_{w_d})=\lim_d w_d=(\boldsymbol{y}, \boldsymbol{\beta})=w=\xi^{-1}(\delta_w)$. The continuity of ξ^{-1} is proved. □

§ 2. Some basic lemmas

In this section, we always assume that $n_1, n_2, m_1, m_2\geqq 1$ are integers, $X\subseteq\boldsymbol{R}^{m_1}$ and $Y\subseteq\boldsymbol{R}^{m_2}$ are locally compact and NIP, and $T: C_0^{(n_1)}(X)\to C_0^{(n_2)}(Y)$ is a surjective linear isometry.

Denote

$$W_1=X\times\mathcal{S}_{m_1,n_1} \quad\text{and}\quad W_2=Y\times\mathcal{S}_{m_2,n_2}.$$

For any $(\boldsymbol{x}, \boldsymbol{\alpha})=w_1\in W_1$, since $(T^{-1})^*: C_0^{(n_1)}(X)^*\to C_0^{(n_2)}(Y)^*$ is a surjective linear isometry and $\delta_{w_1}\in\mathrm{ext}\, B_{C_0^{(n_1)}(X)^*}$, we have $(T^{-1})^*(\delta_{w_1})=\delta_{w_2}\in\mathrm{ext}\, B_{C_0^{(n_2)}(Y)^*}$ for some unique $w_2\in W_2$. Define

$$\Phi_T(w_1)=w_2.$$

Remark 2. It is evident that

$$\delta_{\Phi_T(w)}(Tf)=\delta_w(f),\quad \forall w\in W_1,\ f\in C_0^{(n_1)}(X).$$

LEMMA 2.1. *$\Phi_T: W_1\to W_2$ is a homeomorphism.*

Proof. Let

$$\xi_1: W_1\to(\mathrm{ext}\, B_{C_0^{(n_1)}(X)^*},\ \text{weak*})$$

$$\xi_2: W_2\to(\mathrm{ext}\, B_{C_0^{(n_2)}(Y)^*},\ \text{weak*})$$

be as in Theorem 1.3. Since ξ_1, ξ_2 are homeomorphic and $(T^{-1})^*$ is a surjective linear isometry and a weak-star isomorphism, from the following commutative diagram

$$\begin{array}{ccc} W_1 & \xrightarrow{\ \xi_1\ } & (\mathrm{ext}\, B_{C_0^{(n_1)}(X)^*},\ \text{weak*}) \\ \downarrow \Phi_T & & \downarrow (T^{-1})^* \\ W_2 & \xleftarrow{\ \xi_2^{-1}\ } & (\mathrm{ext}\, B_{C_0^{(n_2)}(Y)^*},\ \text{weak*}) \end{array}$$

we can easily see that Φ_T is a homeomorphism. □

LEMMA 2.2. *For any* $\boldsymbol{x}_0 \in X$, *define*

$$A(\boldsymbol{x}_0)=\{\boldsymbol{y}\in Y : \Phi_T(\boldsymbol{x}_0, \boldsymbol{\alpha})=(\boldsymbol{y}, \boldsymbol{\beta}) \text{ for some } \boldsymbol{\alpha}\in\mathcal{S}_{m_1, n_1} \text{ and } \boldsymbol{\beta}\in\mathcal{S}_{m_2, n_2}\}.$$

Then ${}^{\#}A(\boldsymbol{x}_0)=1$.

Proof. Take $\boldsymbol{\alpha}_1=(\alpha_{1r})$, $\boldsymbol{\alpha}_2=(\alpha_{2r})\in\mathcal{S}_{m_1, n_1}$ such that

$$\alpha_{10}=\alpha_{20}=1 \quad \text{and} \quad \alpha_{1r}=-\alpha_{2r} \ (1\leqq |\boldsymbol{r}|\leqq n_1).$$

Then,

$$\delta_x=\frac{1}{2}[\delta_{(x, \boldsymbol{\alpha}_1)}+\delta_{(x, \boldsymbol{\alpha}_2)}], \quad \forall \boldsymbol{x}\in X.$$

Set $\Phi_T(\boldsymbol{x}_0, \boldsymbol{\alpha}_1)=(\boldsymbol{y}_1, \boldsymbol{\beta}_1)$ and $\Phi_T(\boldsymbol{x}_0, \boldsymbol{\alpha}_2)=(\boldsymbol{y}_2, \boldsymbol{\beta}_2)$.

Suppose that $\boldsymbol{y}_3\in A(\boldsymbol{x}_0)\setminus\{\boldsymbol{y}_1, \boldsymbol{y}_2\}$. Let U and V be two open subsets of Y satisfying

$$U\cap V=\emptyset, \quad \boldsymbol{y}_1, \boldsymbol{y}_2\in U, \quad \boldsymbol{y}_3\in V. \tag{2.1}$$

Since the mappings

$$\boldsymbol{x}\mapsto Q_Y\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha}_1)$$

$$\text{and} \quad \boldsymbol{x}\mapsto Q_Y\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha}_2)$$

are continuous (where $Q_Y : W_2\to Y$ is the natural projection), there exists an open neighbourhood O of $\boldsymbol{x}_0$ so that

$$Q_Y\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha}_1), \quad Q_Y\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha}_2)\in U, \quad \forall \boldsymbol{x}\in O. \tag{2.2}$$

Let $\Phi_T(\boldsymbol{x}_0, \boldsymbol{\alpha}_3)=(\boldsymbol{y}_3, \boldsymbol{\beta}_3)$ and take a $g\in C_0^{(n_2)}(Y)$ such that

$$\delta_{(\boldsymbol{y}_3, \boldsymbol{\beta}_3)}(g)\neq 0, \quad \operatorname{supp}(g)\subseteqq V. \tag{2.3}$$

Then, from (2.1), (2.2) and (2.3) we have

$$\begin{aligned} T^{-1}(g)(\boldsymbol{x})=\delta_x(T^{-1}(g))&=\frac{1}{2}[\delta_{(x, \boldsymbol{\alpha}_1)}+\delta_{(x, \boldsymbol{\alpha}_2)}](T^{-1}(g)) \\ &=\frac{1}{2}[\delta_{\Phi_T(x, \boldsymbol{\alpha}_1)}(g)+\delta_{\Phi_T(x, \boldsymbol{\alpha}_2)}(g)]=0, \quad \forall \boldsymbol{x}\in O. \end{aligned}$$

Therefore,

$$T^{-1}(g)^{(\boldsymbol{r})}(\boldsymbol{x}_0)=0, \quad |\boldsymbol{r}|\leqq n_1$$

which implies that

$$0\neq\delta_{(\boldsymbol{y}_3, \boldsymbol{\beta}_3)}(g)=\delta_{(x_0, \boldsymbol{\alpha}_3)}(T^{-1}(g))=0,$$

a contradiction. Thus, $A(\boldsymbol{x}_0)\subseteqq\{\boldsymbol{y}_1, \boldsymbol{y}_2\}$.

When the scalar field is $\boldsymbol{C}^1$, $A(\boldsymbol{x}_0)$ is the range of the continuous map $Q_Y\Phi_T(\boldsymbol{x}_0, \cdot)$ on the connected domain $\mathcal{S}_{m_1, n_1}$, henceforth, connected, thus ${}^{\#}A(\boldsymbol{x}_0)=1$. When the scalar field is $\boldsymbol{R}^1$, let $P: W_2\to\mathcal{S}_{m_2, n_2}$ be the natural projection, then, from the continuity of $P\Phi_T$ and the discreteness of $\mathcal{S}_{m_2, n_2}$ there exists an open neighbourhood O_1 of $\boldsymbol{x}_0$ such that

$$P\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha}_1)=\boldsymbol{\beta}_1, \quad \forall \boldsymbol{x}\in O_1 \tag{2.4}$$
$$P\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha}_2)=\boldsymbol{\beta}_2, \quad \forall \boldsymbol{x}\in O_1 .$$

If $\boldsymbol{y}_1 \neq \boldsymbol{y}_2$, we can take two disjoint open subsets U_1 and V_1 of Y such that $\boldsymbol{y}_1\in U_1$ and $\boldsymbol{y}_2\in V_1$. There exists an open neighbourhood O_2 of $\boldsymbol{x}_0$ so that

$$Q_Y\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha}_1)\in U_1, \quad Q_Y\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha}_2)\in V_1, \quad \forall \boldsymbol{x}\in O_2 . \tag{2.5}$$

Let $g\in C_0^{(n_2)}(Y)$ satisfy $\mathrm{supp}(g)\subseteqq U_1$ and $g(U_2)=\beta_{1(0,\dots,0)}^{-1}$ for some open neighbourhood U_2 of $\boldsymbol{y}_1$ (where $\boldsymbol{\beta}_1=(\beta_{1r})$). Take an open neighbourhood O_3 of $\boldsymbol{x}_0$ so that

$$Q_Y\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha}_1)\in U_2, \quad \forall \boldsymbol{x}\in O_3 . \tag{2.6}$$

Then, from (2.4), (2.5) and (2.6) we get

$$\begin{aligned} T^{-1}(g)(\boldsymbol{x}) &= \boldsymbol{\delta}_{\boldsymbol{x}}(T^{-1}(g))=\frac{1}{2}[\boldsymbol{\delta}_{(\boldsymbol{x}, \boldsymbol{\alpha}_1)}+\boldsymbol{\delta}_{(\boldsymbol{x}, \boldsymbol{\alpha}_2)}](T^{-1}(g)) \\ &= \frac{1}{2}[\boldsymbol{\delta}_{(Q_Y\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha}_1), \beta_1)}(g)+\boldsymbol{\delta}_{(Q_Y\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha}_2), \beta_2)}(g)] \\ &= \frac{1}{2}[1+0]=\frac{1}{2}, \quad \forall \boldsymbol{x}\in O_1\cap O_2\cap O_3 , \end{aligned} \tag{2.7}$$

in particular,

$$T^{-1}(g)^{(\boldsymbol{r})}(\boldsymbol{x}_0)=0, \quad 1\leqq |\boldsymbol{r}|\leqq n_1 . \tag{2.8}$$

Thus, from (2.7), (2.8) and noting that $\boldsymbol{y}_2\notin\mathrm{supp}(g)$, we obtain that

$$0=|\boldsymbol{\delta}_{(\boldsymbol{y}_2, \boldsymbol{\beta}_2)}(g)|=|\boldsymbol{\delta}_{(\boldsymbol{x}_0, \boldsymbol{\alpha}_2)}(T^{-1}(g))|=|T^{-1}(g)(\boldsymbol{x}_0)|=\frac{1}{2},$$

which is a contradiction.

We have shown that $^{\#}A(\boldsymbol{x}_0)=1$ for all $\boldsymbol{x}_0\in X$. □

LEMMA 2.3. *There exists a homeomorphism* $\tau: X\to Y$ *such that*

$$\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha})=(\tau(\boldsymbol{x}), *), \quad \boldsymbol{x}\in X, \quad \boldsymbol{\alpha}\in\mathcal{S}_{m_1, n_1}$$

where $*$ *is an element in* $\mathcal{S}_{m_2, n_2}$ *depending on* $(\boldsymbol{x}, \boldsymbol{\alpha})$.

Proof. For any $\boldsymbol{x}\in X$, from Lemma 2.2, we can define

$$\tau(\boldsymbol{x})=Q_Y\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha})$$

which does not depend on the choice of $\boldsymbol{\alpha}\in\mathcal{S}_{m_1, n_1}$. Then,

$$\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha})=(\tau(\boldsymbol{x}), *), \quad \forall \boldsymbol{\alpha}\in\mathcal{S}_{m_1, n_1}$$

where $*$ depends on $(\boldsymbol{x}, \boldsymbol{\alpha})\in W_1$. Because that Φ_T is homeomorphic, we can verify that $\tau: X\to Y$ is homeomorphic. □

Remark 3. When $X\subseteqq \boldsymbol{R}^{m_1}$ and $Y\subseteqq \boldsymbol{R}^{m_2}$ are open subsets, from Lemma 2.3, we must have $m_1=m_2$.

COROLLARY 2.4. *For any $f\in C_0^{(n_1)}(X)$ and $\boldsymbol{x}\in X$,*

$$\sum_{|\boldsymbol{r}|\leq n_1}\frac{|f^{(\boldsymbol{r})}(\boldsymbol{x})|}{\boldsymbol{r}!}=\sum_{|\boldsymbol{r}|\leq n_2}\frac{|Tf^{(\boldsymbol{r})}(\tau(\boldsymbol{x}))|}{\boldsymbol{r}!}.$$

Proof. Let $\boldsymbol{\alpha}\in\mathcal{S}_{m_1, n_1}$ satisfy $\delta_{(\boldsymbol{x}, \boldsymbol{\alpha})}(f)=\sum_{|\boldsymbol{r}|\leq n_1}|f^{(\boldsymbol{r})}(\boldsymbol{x})|/\boldsymbol{r}!$ and set $\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha})=(\tau(\boldsymbol{x}), \boldsymbol{\beta})$, then

$$\sum_{|\boldsymbol{r}|\leq n_1}\frac{|f^{(\boldsymbol{r})}(\boldsymbol{x})|}{\boldsymbol{r}!}=\delta_{(\boldsymbol{x}, \boldsymbol{\alpha})}(f)=\delta_{(\tau(\boldsymbol{x}), \boldsymbol{\beta})}(Tf)\leqq\sum_{|\boldsymbol{r}|\leq n_2}\frac{|Tf^{(\boldsymbol{r})}(\tau(\boldsymbol{x}))|}{\boldsymbol{r}!}. \tag{2.9}$$

On the other hand, if $\boldsymbol{\beta}^*\in\mathcal{S}_{m_2, n_2}$ such that $\delta_{(\tau(\boldsymbol{x}), \boldsymbol{\beta}^*)}(Tf)=\sum_{|\boldsymbol{r}|\leq n_2}|Tf^{(\boldsymbol{r})}(\tau(\boldsymbol{x}))|/\boldsymbol{r}!$, let $\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha}^*)=(\tau(\boldsymbol{x}), \boldsymbol{\beta}^*)$, then

$$\sum_{|\boldsymbol{r}|\leq n_2}\frac{|Tf^{(\boldsymbol{r})}(\tau(\boldsymbol{x}))|}{\boldsymbol{r}!}=\delta_{(\tau(\boldsymbol{x}), \boldsymbol{\beta}^*)}(Tf)=\delta_{(\boldsymbol{x}, \boldsymbol{\alpha}^*)}(f)\leqq\sum_{|\boldsymbol{r}|\leq n_1}\frac{|f^{(\boldsymbol{r})}(\boldsymbol{x})|}{\boldsymbol{r}!}. \tag{2.10}$$

From (2.9) and (2.10) we can get the desired equality. □

LEMMA 2.5. *Let U be an open subset of X. If $f\in C_0^{(n_1)}(X)$ satisfies $f|_U=1$, then $|Tf(\boldsymbol{y})|=1$ $(\forall \boldsymbol{y}\in\tau(U))$ and $Tf^{(\boldsymbol{r})}(\boldsymbol{y})=0$ $(\forall \boldsymbol{y}\in\tau(U),\ 1\leqq|\boldsymbol{r}|\leqq n_2)$. If $f, g\in C_0^{(n_1)}(X)$ satisfy $f|_U=g|_U=1$, then $Tf|_{\tau(U)}=Tg|_{\tau(U)}$. Furthermore, for any $h\in C_0^{(n_1)}(X)$ and $\boldsymbol{x}\in X$, we have*

$$h(\boldsymbol{x})=0 \Longleftrightarrow Th(\tau(\boldsymbol{x}))=0.$$

Proof. Suppose that $f|_U=1$ for some $f\in C_0^{(n_1)}(X)$ and some open subset U of X. For any $\boldsymbol{x}\in U$ and $\boldsymbol{\beta}\in\mathcal{S}_{m_2, n_2}$, letting $\Phi_T(\boldsymbol{x}, \boldsymbol{\alpha})=(\tau(\boldsymbol{x}), \boldsymbol{\beta})$, from

$$|\delta_{(\tau(\boldsymbol{x}), \boldsymbol{\beta})}(Tf)|=|\delta_{(\boldsymbol{x}, \boldsymbol{\alpha})}(f)|=1=\sum_{|\boldsymbol{r}|\leq n_1}\frac{|f^{(\boldsymbol{r})}(\boldsymbol{x})|}{\boldsymbol{r}!}=\sum_{|\boldsymbol{r}|\leq n_2}\frac{|Tf^{(\boldsymbol{r})}(\tau(\boldsymbol{x}))|}{\boldsymbol{r}!}. \tag{2.11}$$

We can see that $\{Tf^{(\boldsymbol{r})}(\tau(\boldsymbol{x})): |\boldsymbol{r}|\leqq n_2\}$ has at most one non-zero term and

$$|Tf(\tau(\boldsymbol{x}))|=0 \text{ or } 1, \quad \boldsymbol{x}\in U.$$

For any $\boldsymbol{y}_0\in\tau(U)$, by the continuity of Tf, there exists an open neighbourhood V of $\boldsymbol{y}_0$ such that $V\subseteqq\tau(U)$ and

$$|Tf(\boldsymbol{y})|=0, \quad \boldsymbol{y}\in V \tag{I}$$

$$\text{or} \quad |Tf(\boldsymbol{y})|=1, \quad \boldsymbol{y}\in V. \tag{II}$$

The case (I) does not exist. Otherwise,

$$Tf^{(r)}(y)=0, \quad y\in V, \quad |r|\leqq n_2,$$

a contradiction to (2.11).

For the case (II), from (2.11) we have

$$Tf^{(r)}(y_0)=0, \quad y_0\in\tau(U), \quad 1\leqq|r|\leqq n_2.$$

For any $h\in C_0^{(n_1)}(X)$ and $x\in X$ that satisfy $h(x)=0$, take $\alpha_1=(\alpha_{1r})$, $\alpha_2=(\alpha_{2r})\in\mathcal{S}_{m_1,n_1}$ such that $\delta_{(x,\alpha_1)}(h)=\sum_{|r|\leqq n_1}|h^{(r)}(x)|/r!$ and

$$\alpha_{10}=-\alpha_{20}=1, \quad \alpha_{1r}=\alpha_{2r} \ (1\leqq|r|\leqq n_1).$$

Set $\Phi_T(x, \alpha_i)=(\tau(x), \beta_i)$ $(i=1, 2)$. Take an $f\in C_0^{(n_1)}(X)$ so that $f|_U=1$ for some open neighbourhood U of x, then $|Tf(\tau(x))|=1$, $Tf^{(r)}(\tau(x))=0$ $(1\leqq|r|\leqq n_2)$ and

$$0=[\delta_{(x,\alpha_1)}+\delta_{(x,\alpha_2)}](f)=[\delta_{(\tau(x),\beta_1)}+\delta_{(\tau(x),\beta_2)}](Tf)=(\beta_{10}+\beta_{20})Tf(\tau(x)),$$

thus, $\beta_{10}+\beta_{20}=0$. Now, from

$$\delta_{(\tau(x),\beta_1)}(Th)=\delta_{(x,\alpha_1)}(h)=\sum_{|r|\leqq n_1}\frac{|h^{(r)}(x)|}{r!}=\sum_{|r|\leqq n_2}\frac{|Th^{(r)}(\tau(x))|}{r!}$$

$$\delta_{(\tau(x),\beta_2)}(Th)=\delta_{(x,\alpha_2)}(h)=\sum_{|r|\leqq n_1}\frac{|h^{(r)}(x)|}{r!}=\sum_{|r|\leqq n_2}\frac{|Th^{(r)}(\tau(x))|}{r!},$$

we have

$$\begin{aligned}\sum_{|r|\leqq n_2}\frac{|Th^{(r)}(\tau(x))|}{r!}&=\frac{1}{2}[\delta_{(\tau(x),\beta_1)}+\delta_{(\tau(x),\beta_2)}](Th)\\&=\sum_{|r|\leqq n_2}\frac{\beta_{1r}+\beta_{2r}}{2}\frac{Th^{(r)}(\tau(x))}{r!}\\&=\sum_{1\leqq|r|\leqq n_2}\frac{\beta_{1r}+\beta_{2r}}{2}\frac{Th^{(r)}(\tau(x))}{r!}\\&\leqq\sum_{1\leqq|r|\leqq n_2}\left|\frac{\beta_{1r}+\beta_{2r}}{2}\right|\frac{|Th^{(r)}(\tau(x))|}{r!}\\&\leqq\sum_{1\leqq|r|\leqq n_2}\frac{|Th^{(r)}(\tau(x))|}{r!}.\end{aligned}$$

It follows that $|Th(\tau(x))|=0$. By a symmetric consideration with respect to T^{-1}, we can also show that

$$Th(\tau(x))=0 \Longrightarrow h(x)=0.$$

Hence, for any $x\in X$ and $h\in C_0^{(n_1)}(X)$,

$$h(x)=0 \Longleftrightarrow Th(\tau(x))=0.$$

If $f|_U=g|_U=1$ for some f, $g\in C_0^{(n_1)}(X)$ and some open subset $U\subseteq X$, then $(f-g)(\boldsymbol{x})=0$ $(\boldsymbol{x}\in U)$, from the above we have $(Tf-Tg)(\tau(\boldsymbol{x}))=0$ $(\boldsymbol{x}\in U)$, i.e., $Tf|_{\tau(U)}=Tg|_{\tau(U)}$. □

§ 3. Representations of isometries

THEOREM 3.1. *Let n_1, n_2, m_1, $m_2\geqq 1$ be integers and $X\subseteq\boldsymbol{R}^{m_1}$, $Y\subseteq\boldsymbol{R}^{m_2}$ be locally compact and NIP. Suppose that $T: C_0^{(n_1)}(X)\to C_0^{(n_2)}(Y)$ is a surjective linear isometry. Then there exists a homeomorphism $\sigma: Y\to X$ and a continuous modular function $\theta(\boldsymbol{y})$ on Y such that*

(1) *$\theta^{(r)}=0$ for all $|\boldsymbol{r}|\geqq 1$, and*

(2) *for any $f\in C_0^{(n_1)}(X)$,*

$$Tf(\boldsymbol{y})=\theta(\boldsymbol{y})f(\sigma(\boldsymbol{y})),\quad \forall\boldsymbol{y}\in Y. \tag{3.1}$$

Proof. Let $\tau: X\to Y$ be the same as in section § 2 and set $\sigma=\tau^{-1}$. For any $\boldsymbol{y}\in Y$, by Lemma 2.5, there exists a $\theta(\boldsymbol{y})$ in S^1, the set of numbers of absolute value 1 in $\boldsymbol{K}^1$, so that

$$\theta(\boldsymbol{y})=Tf(\boldsymbol{y}),$$

for all $f\in C_0^{(n_1)}(X)$ such that $f|_U=1$ for some open neighbourhood U of $\tau^{-1}(\boldsymbol{y})$ $=\sigma(\boldsymbol{y})$. From Lemma 2.5, $\theta(\boldsymbol{y})$ is well defined, continuous and

$$\theta^{(r)}(\boldsymbol{y})=0,\quad 1\leqq|\boldsymbol{r}|\leqq n_2.$$

Thus, $\theta: Y\to S^1$ is a continuous modular function satisfying (1).

For any $g\in C_0^{(n_1)}(X)$ and $\boldsymbol{y}\in Y$, let f be as above, then the function $h=g-g(\sigma(\boldsymbol{y}))f\in C_0^{(n_1)}(X)$ satisfies $h(\sigma(\boldsymbol{y}))=0$, applying Lemma 2.5 we get

$$Th(\boldsymbol{y})=Tg(\boldsymbol{y})-g(\sigma(\boldsymbol{y}))Tf(\boldsymbol{y})=Tg(\boldsymbol{y})-g(\sigma(\boldsymbol{y}))\theta(\boldsymbol{y})=0,$$

that is,

$$Tg(\boldsymbol{y})=\theta(\boldsymbol{y})g(\sigma(\boldsymbol{y})),\quad \forall\boldsymbol{y}\in Y.$$ □

In order to find out the relations of m_1, m_2, n_1 and n_2, we need the following lemma.

LEMMA 3.2. *Under the same conditions as in Theorem 3.1, set*

$$\Gamma_1=\{\boldsymbol{r}=(r_1,\cdots,r_{m_1})\in\boldsymbol{Z}_+^{m_1}: r_1+\cdots+r_{m_1}\leqq n_1\}$$

$$\Gamma_2=\{\boldsymbol{r}=(r_1,\cdots,r_{m_2})\in\boldsymbol{Z}_+^{m_2}: r_1+\cdots+r_{m_2}\leqq n_2\}.$$

Then, (1) $^{\#}\Gamma_1={}^{\#}\Gamma_2$, (2) *for any $f\in C_0^{(n_1)}(X)$ and $\boldsymbol{x}\in X$,*

$$^{\#}\{\boldsymbol{r}\in\Gamma_1: f^{(r)}(\boldsymbol{x})=0\}={}^{\#}\{\boldsymbol{r}\in\Gamma_2: Tf^{(r)}(\tau(\boldsymbol{x}))=0\}$$

$$^{\#}\{\boldsymbol{r}\in\Gamma_1: f^{(r)}(\boldsymbol{x})\neq 0\}={}^{\#}\{\boldsymbol{r}\in\Gamma_2: Tf^{(r)}(\tau(\boldsymbol{x}))\neq 0\}$$

where τ *is as in Lemma* 2.3.

Proof. (1) Let Φ_T be as in section § 2. By Lemma 2.1 and 2.3, $P\Phi_T(x, \cdot)$ is a homeomorphism from $\mathcal{S}_{m_1,n_1}=S^{\#\Gamma_1}$ onto $\mathcal{S}_{m_2,n_2}=S^{\#\Gamma_2}$ (where $P: W_2\to\mathcal{S}_{m_2,n_2}$ is the natural projection), thus, $^{\#}\Gamma_1={}^{\#}\Gamma_2$.

(2) For any $f\in C_0^{(n_1)}(X)$ and $x\in X$, let

$$k_1={}^{\#}\{r\in\Gamma_1: f^{(r)}(x)=0\}, \quad k_2={}^{\#}\{r\in\Gamma_2: Tf^{(r)}(\tau(x))=0\},$$

then S^{k_1} is homeomorphic to $C=\{\alpha\in\mathcal{S}_{m_1,n_1}: \delta_{(x,\alpha)}(f)=\sum_{|r|\leq n_1}|f^{(r)}(x)|/r!\}$ and S^{k_2} is homeomorphic to $D=\{\beta\in\mathcal{S}_{m_2,n_2}: \delta_{(\tau(x),\beta)}(Tf)=\sum_{|r|\leq n_2}|Tf^{(r)}(\tau(x))|/r!\}$. But, from $\sum_{|r|\leq n_1}|f^{(r)}(x)|/r!=\sum_{|r|\leq n_2}|Tf^{(r)}(\tau(x))|/r!$ (Corollary 2.4) and $\delta_{(x,\alpha)}(f)=\delta_{(\tau(x),P\Phi_T(x,\alpha))}(Tf)$, we can see that C and D are homeomorphic under the map $P\Phi_T(x, \cdot)$. Therefore, S^{k_1} and S^{k_2} are homeomorphic, hence $k_1=k_2$, i.e.,

$${}^{\#}\{r\in\Gamma_1: f^{(r)}(x)=0\}={}^{\#}\{r\in\Gamma_2: Tf^{(r)}(\tau(x))=0\}.$$

It follows that

$$\begin{aligned}{}^{\#}\{r\in\Gamma_1: f^{(r)}(x)\neq0\}&={}^{\#}\Gamma_1-{}^{\#}\{r\in\Gamma_1: f^{(r)}(x)=0\}\\&={}^{\#}\Gamma_2-{}^{\#}\{r\in\Gamma_2: Tf^{(r)}(\tau(x))=0\}\\&={}^{\#}\{r\in\Gamma_2: Tf^{(r)}(\tau(x))\neq0\}.\end{aligned}$$ □

THEOREM 3.3. *Under the same conditions as in Theorem* 3.1, *we have* $m_1=m_2$ *and* $n_1=n_2$.

Proof. Let θ and $\sigma=\tau^{-1}$ be the same as in Theorem 3.1 and $\sigma(y_0)=x_0$ be fixed. There exists an $a\in R^{m_1}$ and an open neighbourhood U of x_0 such that

$$x_j-a_j>0, \quad \forall x=(x_1, \cdots, x_{m_1})\in U, \ j=1, \cdots, m_1.$$

Take $\{f_j\}_{j=1}^{m_1}\subseteq C_0^{(n_1)}(X)$ so that

$$f_j(x)=x_j-a_j, \quad \forall x=(x_1, \cdots, x_{m_1})\in V, \ j=1, \cdots, m_1$$

for some open neighbourhood $V\subseteq U$ of x_0. From Theorem 3.1, we have

$$Tf_j(\tau(x))=\theta(\tau(x))f_j(x)\neq0, \quad \forall x\in V, \ j=1, \cdots, m_1. \tag{3.2}$$

By Lemma 3.2 and the choice of f_j, we know that

$$\begin{aligned}&{}^{\#}\{r\in\Gamma_2: Tf_j^{(r)}(\tau(x))\neq0\}\\&={}^{\#}\{r\in\Gamma_1: f_j^{(r)}(x)\neq0\}=2, \quad \forall x\in V, \ 1\leq j\leq m_1.\end{aligned} \tag{3.3}$$

CLAIM. *If* $x\in V$, $r\in\Gamma_2$ *and* $Tf_j^{(r)}(\tau(x))\neq0$, *then* $|r|\leq1$.

Otherwise, there is a $r^*\in\Gamma_2$ with $1\leq|r^*|=|r|-1$ and

$$Tf_j^{(r)}(\boldsymbol{y})=\frac{\partial Tf_j^{(r^*)}(\boldsymbol{y})}{\partial y_i}, \quad \forall \boldsymbol{y}\in Y$$

for some $1\leqq i\leqq m_2$. By the continuity of $Tf_j^{(r)}$,

$$Tf_j^{(r)}(\boldsymbol{y})\neq 0, \quad \forall \boldsymbol{y}\in O \tag{3.4}$$

for some open neighbourhood $O\subseteqq\tau(V)$ of $\tau(\boldsymbol{x})$. Clearly, $Tf_j^{(r^*)}(\boldsymbol{y})\not\equiv 0$ on O. We can take a $\boldsymbol{y}^*=\tau(\boldsymbol{x}^*)$ $(\boldsymbol{x}^*\in V)$ such that $Tf_j^{(r^*)}(\boldsymbol{y}^*)\neq 0$, then together with (3.2) and (3.4) we have

$$Tf_j(\boldsymbol{y}^*)\neq 0, \quad Tf_j^{(r^*)}(\boldsymbol{y}^*)\neq 0, \; Tf_j^{(r)}(\boldsymbol{y}^*)\neq 0$$

which contradicts with (3.3). The claim is true.

Now, define

$$f=f_1+\cdots+f_{m_1}\in C_0^{(n_1)}(X),$$

then for any $\boldsymbol{r}\in\Gamma_2$ with $|\boldsymbol{r}|>1$, from the above claim,

$$Tf^{(r)}(\boldsymbol{y}_0)=Tf_1^{(r)}(\boldsymbol{y}_0)+\cdots+Tf_{m_1}^{(r)}(\boldsymbol{y}_0)=0.$$

Besides, from the properties of $\{f_j: 1\leqq j\leqq m_1\}$ we can calculate

$$\#\{\boldsymbol{r}\in\Gamma_1: f^{(r)}(\boldsymbol{x}_0)\neq 0\}=1+m_1.$$

Therefore, by Lemma 3.2 and the claim,

$$\begin{aligned}1+m_1&=\#\{\boldsymbol{r}\in\Gamma_1: f^{(r)}(\boldsymbol{x}_0)\neq 0\}\\&=\#\{\boldsymbol{r}\in\Gamma_2: Tf^{(r)}(\boldsymbol{y}_0)\neq 0\}\\&\leqq\#\{\boldsymbol{r}\in\Gamma_2: |\boldsymbol{r}|\leqq 1\}=1+m_2,\end{aligned}$$

that is, $m_1\leqq m_2$. Similarly, by considering T^{-1}, we can also get $m_2\leqq m_1$, thus $m_1=m_2$.

Finally, if we set $m_1=m_2=m$, then from

$$\begin{aligned}&\#\{\boldsymbol{r}=(r_1,\cdots,r_m)\in\boldsymbol{Z}_+^m: r_1+\cdots+r_m\leqq n_1\}=\#\Gamma_1=\#\Gamma_2\\&=\#\{\boldsymbol{r}=(r_1,\cdots,r_m)\in\boldsymbol{Z}_+^m: r_1+\cdots+r_m\leqq n_2\}\end{aligned}$$

we can easily see that $n_1=n_2$. □

From now on, we only consider the cases where $m_1=m_2$ and $n_1=n_2$. Suppose that $Y\subseteqq\boldsymbol{R}^m$ is NIP and F is a map from Y into $\boldsymbol{R}^m$. Recall that the Jacobian matrix $J(F)$ of F at the point $\boldsymbol{y}\in Y$ is defined by

$$J(F)(\boldsymbol{y})=\begin{pmatrix} \frac{\partial F_1(\boldsymbol{y})}{\partial y_1} & \frac{\partial F_2(\boldsymbol{y})}{\partial y_1} & \cdots & \frac{\partial F_m(\boldsymbol{y})}{\partial y_1} \\ \frac{\partial F_1(\boldsymbol{y})}{\partial y_2} & \frac{\partial F_2(\boldsymbol{y})}{\partial y_2} & \cdots & \frac{\partial F_m(\boldsymbol{y})}{\partial y_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_1(\boldsymbol{y})}{\partial y_m} & \frac{\partial F_2(\boldsymbol{y})}{\partial y_m} & \cdots & \frac{\partial F_m(\boldsymbol{y})}{\partial y_m} \end{pmatrix}$$

where $F(\boldsymbol{y})=(F_1(\boldsymbol{y}), F_2(\boldsymbol{y}), \cdots, F_m(\boldsymbol{y}))$.

THEOREM 3.4. *Let $m, n\geqq 1$ be integers and $X, Y\subseteqq \boldsymbol{R}^m$ be locally compact and NIP. Suppose that $T: C_0^{(n)}(X)\to C_0^{(n)}(Y)$ is a surjective linear isometry. Let $\sigma: Y\to X$ be the homeomorphism as in Theorem 3.1. Then for any $\boldsymbol{y}_0\in Y$ there exists an open neighbourhood O of $\boldsymbol{y}_0$ and a permutation π on $\{1, 2, \cdots, m\}$ such that the Jacobian matrix $J(\sigma)$ is a constant matrix on O with the property*

$$\frac{\partial\sigma_j(\boldsymbol{y}_0)}{\partial y_{\pi(j)}}\in\{-1, 1\}\,(1\leqq j\leqq m) \quad \text{and} \quad \frac{\partial\sigma_j(\boldsymbol{y}_0)}{\partial y_i}=0 \quad (i\neq\pi(j)). \tag{3.5}$$

Proof. For any $\boldsymbol{y}_0\in Y$, set $\boldsymbol{x}_0=\sigma(\boldsymbol{y}_0)\in X$. As in the proof of Theorem 3.3, we can take an $\boldsymbol{a}\in\boldsymbol{R}^m$ and $\{f_j\}_{j=1}^m\subseteqq C_0^{(n)}(X)$ such that

$$f_j(\boldsymbol{x})=x_j-a_j>0, \quad \forall\boldsymbol{x}\in U, \ 1\leqq j\leqq m$$

for some open neighbourhood U of $\boldsymbol{x}_0$. We have known that

$$Tf_j(\boldsymbol{y})=\theta(\boldsymbol{y})f_j(\sigma(\boldsymbol{y}))=\theta(\boldsymbol{y})(\sigma_j(\boldsymbol{y})-a_j)\neq 0, \quad \forall\boldsymbol{y}\in\tau(U) \tag{3.6}$$

where θ is as in Theorem 3.1, $\tau=\sigma^{-1}$ is as in section §2 and $\sigma=(\sigma_1, \cdots, \sigma_m)$. From (3.6), we know that

$$\sigma_j(\boldsymbol{y})=a_j+\frac{1}{\theta(\boldsymbol{y})}Tf_j(\boldsymbol{y}), \quad \forall\boldsymbol{y}\in\tau(U)$$

which implies $\sigma_j^{(r)}$ is continuous at $\boldsymbol{y}_0$, henceforth continuous on Y ($\boldsymbol{r}\in\Gamma$). As shown in the proof of Theorem 3.3, for any $1\leqq j\leqq m$ and $\boldsymbol{y}\in\tau(U)$ we can show that there is a unique $\boldsymbol{r}(j, \boldsymbol{y})\in\Gamma$ with $|\boldsymbol{r}|=1$ and

$$Tf_j^{(\boldsymbol{r}(j,\boldsymbol{y}))}(\boldsymbol{y})\neq 0, \quad Tf_j^{(\boldsymbol{r})}(\boldsymbol{y})=0 \ (|\boldsymbol{r}|\geqq 1, \ \boldsymbol{r}\neq\boldsymbol{r}(j, \boldsymbol{y})).$$

By Corollary 2.4 and (3.6), letting $\boldsymbol{y}=\tau(\boldsymbol{x})$ ($\boldsymbol{x}\in U$), we can calculate

$$\begin{aligned} 1&=\sum_{1\leq|\boldsymbol{r}|\leq n}\frac{|f_j^{(\boldsymbol{r})}(\boldsymbol{x})|}{\boldsymbol{r}!}=\sum_{|\boldsymbol{r}|\leq n}\frac{|f_j^{(\boldsymbol{r})}(\boldsymbol{x})|}{\boldsymbol{r}!}-|f_j(\boldsymbol{x})| \\ &=\sum_{|\boldsymbol{r}|\leq n}\frac{|Tf_j^{(\boldsymbol{r})}(\boldsymbol{y})|}{\boldsymbol{r}!}-|Tf_j(\boldsymbol{y})|=\sum_{1\leq|\boldsymbol{r}|\leq n}\frac{|Tf_j^{(\boldsymbol{r})}(\boldsymbol{y})|}{\boldsymbol{r}!} \\ &=|Tf_j^{(\boldsymbol{r}(j,\boldsymbol{y}))}(\boldsymbol{y})|=|\sigma_j^{(\boldsymbol{r}(j,\boldsymbol{y}))}(\boldsymbol{y})|, \quad \forall\boldsymbol{y}\in\tau(U). \end{aligned} \tag{3.7}$$

By the uniqueness of $r(j, y)$, we can also show that the map $r(j, \cdot): \tau(U) \to \Gamma$ is continuous (where Γ equips with the discrete topology). That is, there exists an open neighbourhood $O \subseteqq \tau(U)$ of y_0 so that

$$r(j, y) = r(j, y_0), \quad \forall y \in O, \ 1 \leqq j \leqq m.$$

Now, let us show that

$$r(j, y_0) \neq r(k, y_0), \quad \text{if } j \neq k. \tag{3.8}$$

In fact, if $r(j, y_0) = r(k, y_0) = r_0$ $(j \neq k)$, since $\sigma_j^{(r_0)}(y_0)$ and $\sigma_k^{(r_0)}(y_0)$ are real numbers and belong to $\{-1, 1\}$ (from (3.7)) we can take a real number $c \in \{-1, 1\}$ with $\sigma_j^{(r_0)}(y_0) + c\sigma_k^{(r_0)}(y_0) = 0$. Considering $f = f_j + cf_k \in C_0^{(n)}(X)$, we can calculate

$$\begin{aligned}
2 &= \sum_{1 \leq |r| \leq n} \frac{|f^{(r)}(x_0)|}{r!} = \sum_{|r| \leq n} \frac{|f^{(r)}(x_0)|}{r!} - |f(x_0)| \\
&= \sum_{|r| \leq n} \frac{|Tf^{(r)}(y_0)|}{r!} - |Tf(y_0)| \\
&= \sum_{1 \leq |r| \leq n} \frac{|Tf^{(r)}(y_0)|}{r!} = |Tf_j^{(r_0)}(y_0) + cTf_k^{(r_0)}(y_0)| \\
&= |\theta(y_0)\sigma_j^{(r_0)}(y_0) + c\theta(y_0)\sigma_k^{(r_0)}(y_0)| = 0,
\end{aligned}$$

a contradiction. Thus, (3.8) is proved. Let

$$r(j, y_0) = (0, \cdots, 0, \underset{\langle \pi(j) \rangle}{1}, 0, \cdots, 0), \quad 1 \leqq j \leqq m,$$

then from (3.8), $\pi(\cdot): \{1, 2, \cdots, m\} \to \{1, 2, \cdots, m\}$ is a permutation and

$$\frac{\partial \sigma_j(y_0)}{\partial y_{\pi(j)}} = \sigma_j^{(r(j, y_0))}(y_0) \in \{-1, 1\} \quad (1 \leqq j \leqq m),$$

$$\frac{\partial \sigma_j(y_0)}{\partial y_i} = \sigma_j^{(r)}(y_0) = 0 \quad (i \neq \pi(j))$$

where $r = (0, \cdots, 0, \underset{\langle i \rangle}{1}, 0, \cdots, 0) \in \Gamma$.

Define

$$O_1 = \bigcap_{j=1}^{m} \left\{ y \in O : \sigma_j^{(r(j, y_0))}(y) = \sigma_j^{(r(j, y_0))}(y_0) \right\}.$$

Since $\sigma_j^{(r(j, y_0))}(y) = \sigma_j^{(r(j, y))}(y) \in \{-1, 1\}$, by the continuities of $\sigma_j^{(r(j, y_0))}$ $(1 \leqq j \leqq m)$, we can see that O_1 is an open neighbourhood of y_0. For any $y \in O_1$,

$$\frac{\partial \sigma_j(y)}{\partial y_{\pi(j)}} = \sigma_j^{(r(j, y_0))}(y) = \sigma_j^{(r(j, y_0))}(y_0) \in \{-1, 1\}, \quad 1 \leqq j \leqq m$$

and

$$\frac{\partial \sigma_j(y)}{\partial y_i} = \sigma_j^{(r)}(y_0) = 0, \quad i \neq \pi(j)$$

where $\boldsymbol{r}=(0, \cdots, 0, \underset{\langle i \rangle}{1}, 0, \cdots, 0)\in\Gamma$ with $\boldsymbol{r}\neq\boldsymbol{r}(j, \boldsymbol{y})=\boldsymbol{r}(j, \boldsymbol{y}_0)$. Therefore, the Jacobian matrix $J(\sigma)$ of σ is a constant matrix on O_1 and satisfies (3.5). □

As we know (for example, from [7]) that every (linear) isometry I on $(\boldsymbol{R}^m, \|\cdot\|_{l^1})$ is of the form

$$(3.9)\qquad I(\boldsymbol{x})=(a_1x_{\pi(1)}, \cdots, a_m x_{\pi(m)}), \quad \forall \boldsymbol{x}=(x_1, \cdots, x_m)\in\boldsymbol{R}^m$$

for some $a_1, \cdots, a_m\in\{-1, 1\}$ and some permutation π on $\{1, \cdots, m\}$. For our convenience, we shall call the map I of the form (3.9) *a permutation* (of axes) on $\boldsymbol{R}^m$. If we use IOR(m) to denote the isometry group of $(\boldsymbol{R}^m, \|\cdot\|_{l^1})$, then for the σ in Theorem 3.4, looking as a linear operator on $\boldsymbol{R}^m$, $J(\sigma)(\boldsymbol{y})$ is a linear isometry on $(\boldsymbol{R}^m, \|\cdot\|_{l^1})$ $(\boldsymbol{y}\in Y)$, therefore $J(\sigma): Y\to \mathrm{IOR}(m)$ is a locally constant map. Now, let us keep the above notation, we can state the conditions for a linear operator T to be an onto isometry between $C_0^{(n)}(X)$ and $C_0^{(n)}(Y)$, especially, the conditions for which $C_0^{(n)}(X)\cong C_0^{(n)}(Y)$.

THEOREM 3.5. *Let $m, n\geqq 1$ be integers and $X, Y\subseteq\boldsymbol{R}^m$ be open sets. Then T is a linear isometry of $C_0^{(n)}(X)$ onto $C_0^{(n)}(Y)$ if and only if the followings hold:*

(1) *there exists a continuous modular function $\theta: Y\to S^1$ such that $\theta^{(\boldsymbol{r})}=0$ for all $|\boldsymbol{r}|\geqq 1$;*

(2) *there exists a homeomorphism $\sigma: Y\to X$ such that $J(\sigma): Y\to\mathrm{IOR}(m)$ is locally constant;*

(3) *for any $f\in C_0^{(n)}(X)$,*

$$Tf(\boldsymbol{y})=\theta(\boldsymbol{y})f(\sigma(\boldsymbol{y})), \quad \boldsymbol{y}\in Y.$$

Proof. The "only if" part is the direct consequence of Theorem 3.1 and 3.4. For the "if" part, let θ, σ satisfy (1), (2) and T be defined by (3). First of all, let us show that $Tf\in C_0^{(n)}(Y)$ for all $f\in C_0^{(n)}(X)$. Let $f\in C_0^{(n)}(X)$ be fixed. For any $\boldsymbol{y}_0\in Y$, let $J(\sigma)=(a_{ij}): Y\to\mathrm{IOR}(m)$ be constant on some open neighbourhood O of $\boldsymbol{y}_0$ and π be a permutation on $\{1, \cdots, m\}$ so that

$$(3.10)\qquad a_{\pi(j)j}\in\{-1, 1\}\ (1\leqq j\leqq m), \quad a_{ij}=0\ (i\neq\pi(j)).$$

We can calculate (Noting that $\theta^{(\boldsymbol{r})}=0$ $(|\boldsymbol{r}|\geqq 1)$),

$$\frac{\partial^{r_j}Tf(\boldsymbol{y})}{\partial y_j^{r_j}}=\theta(\boldsymbol{y})\frac{\partial^{r_j}f(\sigma(\boldsymbol{y}))}{\partial x_{\pi^{-1}(j)}^{r_j}}a_{j\pi^{-1}(j)}^{r_j}, \quad \forall\boldsymbol{y}\in O.$$

Thus, for any $\boldsymbol{r}=(r_1, \cdots, r_m)\in\Gamma$ and $\boldsymbol{y}\in O$,

$$(3.11)\qquad \begin{aligned} Tf^{(\boldsymbol{r})}(\boldsymbol{y})&=\theta(\boldsymbol{y})\frac{\partial^{r_1+\cdots+r_m}f(\sigma(\boldsymbol{y}))}{\partial x_{\pi^{-1}(1)}^{r_1}\cdots\partial x_{\pi^{-1}(m)}^{r_m}}a_{1\pi^{-1}(1)}^{r_1}\cdots a_{m\pi^{-1}(m)}^{r_m}\\ &=\theta(\boldsymbol{y})f^{(r_{\pi(1)},\cdots,r_{\pi(m)})}(\sigma(\boldsymbol{y}))a_{1\pi^{-1}(1)}^{r_1}\cdots a_{m\pi^{-1}(m)}^{r_m}. \end{aligned}$$

It follows that $Tf^{(r)}(\boldsymbol{y})$ is continuous on O, especially, $Tf^{(r)}(\boldsymbol{y})$ is continuous at $\boldsymbol{y}_0$. So, $Tf^{(r)}$ is continuous on Y for all $\boldsymbol{r}\in\Gamma$. Since $|\theta(\boldsymbol{y})|=1$ and $a_{j\pi^{-1}(j)}\in\{-1, 1\}$, from (3.11) we can also show

$$\sum_{\boldsymbol{r}\in\Gamma}\frac{|Tf^{(\boldsymbol{r})}(\boldsymbol{y}_0)|}{\boldsymbol{r}!}=\sum_{\boldsymbol{r}\in\Gamma}\frac{|f^{(\boldsymbol{r})}(\sigma(\boldsymbol{y}_0))|}{\boldsymbol{r}!},\quad \boldsymbol{y}_0\in Y. \tag{3.12}$$

For any $\varepsilon>0$, the set

$$\left\{\boldsymbol{y}\in Y:\sum_{\boldsymbol{r}\in\Gamma}\frac{|Tf^{(\boldsymbol{r})}(\boldsymbol{y})|}{\boldsymbol{r}!}\geqq\varepsilon\right\}=\left\{\boldsymbol{y}\in Y:\sum_{\boldsymbol{r}\in\Gamma}\frac{|Tf^{(\boldsymbol{r})}(\sigma(\boldsymbol{y}))|}{\boldsymbol{r}!}\geqq\varepsilon\right\}$$
$$=\sigma^{-1}\left(\left\{\boldsymbol{x}\in X:\sum_{\boldsymbol{r}\in\Gamma}\frac{|f^{(\boldsymbol{r})}(\boldsymbol{x})|}{\boldsymbol{r}!}\geqq\varepsilon\right\}\right)$$

is compact in Y. Therefore, $Tf\in C_0^{\{n\}}(Y)$ and T is well-defined.

From (3), T is linear, and from (3.12), isometric. It remains to prove that T is surjective. Suppose $g\in C_0^{\{n\}}(Y)$. Suffice to show that the function f defined by

$$f(\boldsymbol{x})=\theta(\sigma^{-1}(\boldsymbol{x}))^{-1}g(\sigma^{-1}(\boldsymbol{x})),\quad \forall\boldsymbol{x}\in X$$

belongs to $C_0^{\{n\}}(X)$.

For any $\boldsymbol{x}_0\in X$, let $J(\sigma)=(a_{ij})$ be constant on some open neighbourhood O of $\boldsymbol{y}_0=\sigma^{-1}(\boldsymbol{x}_0)\in Y$ with (3.9). Then the Jacobian matrix $J(\sigma^{-1})$ of σ^{-1} is constant on $\sigma(O)$ with

$$J(\sigma^{-1})=J(\sigma)^{-1}=(b_{ij})$$

and

$$b_{\pi^{-1}(j)j}=a_{j\pi^{-1}(j)}^{-1}\in\{-1, 1\}\ \ (1\leqq j\leqq m),\quad b_{ij}=0\ \ (i\neq\pi^{-1}(j)).$$

Similar calculations as in (3.11) and (3.12), for any $\boldsymbol{x}\in\sigma(O)$ we have

$$f^{(\boldsymbol{r})}(\boldsymbol{x})=\theta(\sigma^{-1}(\boldsymbol{x}))^{-1}g^{(r_{\pi^{-1}(1)},\cdots,r_{\pi^{-1}(m)})}(\sigma^{-1}(\boldsymbol{x}))b_{1\pi(1)}^{r_1}\cdots b_{m\pi(m)}^{r_m}$$

for all $\boldsymbol{r}\in\Gamma$ and

$$\sum_{\boldsymbol{r}\in\Gamma}\frac{|f^{(\boldsymbol{r})}(\boldsymbol{x})|}{\boldsymbol{r}!}=\sum_{\boldsymbol{r}\in\Gamma}\frac{|g^{(\boldsymbol{r})}(\sigma^{-1}(\boldsymbol{x}))|}{\boldsymbol{r}!}.$$

From which we can prove that $f\in C_0^{\{n\}}(X)$. This finishes the proof. □

Remark 4. The results of Theorem 3.5 remain true if $m=1$ and $X, Y\subseteq\boldsymbol{R}^1$ are locally compact subsets without isolated points, under almost the same proof. But, we do not know whether it is also true for $m>1$ and general locally compact and NIP subsets of $\boldsymbol{R}^m$.

Remark 5. As a direct consequence of Theorem 3.5, $C_0^{\{n\}}(X)\cong C_0^{\{n\}}(Y)$ if and only if the condition (2) in the theorem holds.

§ 4. Applications

As an application of the representations of surjective linear isometries (Theorem 3.5), let us consider the isometry group of $C_0^{(n)}(X)$.

THEOREM 4.1. *Let $m, n \geqq 1$ be integers and X is an open subset of $\boldsymbol{R}^m$. Define*

$$\Theta = \{\theta \mid \theta : X \to S^1 \text{ is continuous and } \theta^{(r)} = 0\ (\forall |\boldsymbol{r}| \geqq 1)\}$$

$$\text{and}\quad \Sigma = \left\{\sigma \,\middle|\, \begin{array}{l} \sigma \text{ is a homeomorphism on } X \text{ such that} \\ J(\sigma): X \to \mathrm{IOR}(m) \text{ is locally constant} \end{array}\right\}.$$

Then the isometry group $I_{n,X}$ of $C_0^{(n)}(X)$ $(n \geqq 1)$, with the operator topology, is homeomorphic to $\Theta \times \Sigma$ with the group operation $(\theta_1, \sigma_1) \circ (\theta_2, \sigma_2) = (\theta_1 \cdot (\theta_2 \circ \sigma_1), \sigma_2 \circ \sigma_1)$ and the product topology of $\Theta \times \Sigma$, where Θ equips with the uniform topology (i.e., $d(\theta_1, \theta_2) = \sup\{|\theta_1(\boldsymbol{x}) - \theta_2(\boldsymbol{x})| : \boldsymbol{x} \in X\}$) and Σ equips with the discrete topology.

Proof. From Theorem 3.5, for any $T \in I_{n,X}$, there is a $\theta \in \Theta$ and a $\sigma \in \Sigma$ such that

$$Tf(\boldsymbol{x}) = \theta(\boldsymbol{x}) f(\sigma(\boldsymbol{x})), \quad \forall \boldsymbol{x} \in X,\ f \in C_0^{(n)}(X). \tag{4.1}$$

Clearly, the correspondence $T \leftrightarrow (\theta, \sigma)$ is a bijection between $I_{n,X}$ and $\Theta \times \Sigma$. If $T_1 \leftrightarrow (\theta_1, \sigma_1)$ and $T_2 \leftrightarrow (\theta_2, \sigma_2)$, then

$$\|T_1 - T_2\| = \begin{cases} \geqq 1, & \sigma_1 \neq \sigma_2 \\ d(\theta_1, \theta_2), & \sigma_1 = \sigma_2, \end{cases}$$

from which we can easily show that $T \mapsto (\theta, \sigma)$ is a homeomorphism of $I_{n,X}$ onto $\Theta \times \Sigma$. The group operation is evident from (4.1). □

THEOREM 4.2. *Let $m, n \geqq 1$ be integers and $X, Y \subseteqq \boldsymbol{R}^m$ be connected open subsets. Then, $T : C_0^{(n)}(X) \to C_0^{(n)}(Y)$ is a surjective linear isometry $\Leftrightarrow$ there exists a number $\lambda \in \boldsymbol{K}$ with $|\lambda| = 1$ and a homeomorphism σ of Y onto X of the form: σ = permutation + translation, so that*

$$Tf(\boldsymbol{y}) = \lambda f(\sigma(\boldsymbol{y})), \quad \forall f \in C_0^{(n)}(X), \quad \boldsymbol{y} \in Y.$$

Proof. By Theorem 3.5, the "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part, let θ, σ be the same as in Theorem 3.5, then $\theta(\boldsymbol{y})$ and $J(\sigma)$ are locally constant on Y. Since Y is connected, $\theta(\boldsymbol{y})$ is a constant on Y and $J(\sigma) = (a_{ij})$ is a constant matrix on Y with

$$a_{\pi(j)j} \in \{-1, 1\} \ (1 \leqq j \leqq m), \quad a_{ij} = 0, \ (i \neq \pi(j))$$

for some permutation π on $\{1, \cdots, m\}$. Using the connectedness of Y again, for each $1 \leqq j \leqq m$ we can show that

$$\sigma_j(\boldsymbol{y}) = a_{\pi(j)j} y_{\pi(j)} + c_j, \quad \forall \boldsymbol{y} \in Y$$

for some constant $c_j \in \boldsymbol{R}^1$. Thus,

$$\sigma(\boldsymbol{y}) = (a_1 y_{\pi(1)}, \cdots, a_m y_{\pi(m)}) + \boldsymbol{c}, \quad \forall \boldsymbol{y} \in Y$$

where $\boldsymbol{c} = (c_1, \cdots, c_m) \in \boldsymbol{R}^m$. Set $\lambda = \theta(\boldsymbol{y})$, then λ and σ satisfy what we need. □

COROLLARY 4.3. *Let m, n, X and Y be the same as in Theorem* 4.2. *Then $C_0^{(n)}(X) \cong C_0^{(n)}(Y) \Leftrightarrow X$ and Y are isometric under the l^1-norm*[2] $\Leftrightarrow$ *there exists a map σ on $\boldsymbol{R}^m$ of the form: σ=permutation+translation, so that $\sigma(Y) = X$.*

Proof. It is trivial that X and Y are isometric under the l^1-norm if there exists a map σ=permutation+translation, such that $\sigma(Y) = X$. By Theorem 4.2, it is enough to show that any isometry $\sigma: Y \xrightarrow{\text{onto}} X$ under the l^1-norm is of the form: σ=permutation+translation. Now, suppose that $\sigma: Y \to X$ is such a surjective isometry, from the generalized Mazur-Ulam's Theorem (See [8]) σ can be extended to be an affine isometry σ_* on $(\boldsymbol{R}^m, \|\cdot\|_{l^1})$, i.e.,

$$\sigma_*(\boldsymbol{y}) = I(\boldsymbol{y}) + \boldsymbol{c}, \quad \forall \boldsymbol{y} \in \boldsymbol{R}^m$$

for some point $\boldsymbol{c} \in \boldsymbol{R}^m$ and some (surjective) linear isometry I on $(\boldsymbol{R}^m, \|\cdot\|_{l^1})$. Therefore, σ has the desired form. □

COROLLARY 4.4. *Let $m, n \geqq 1$ be integers and $X, Y \subseteqq \boldsymbol{R}^m$ be open subsets. Then $C_0^{(n)}(X) \cong C_0^{(n)}(Y) \Leftrightarrow$ there is a homeomorphism $\sigma: Y \to X$ such that σ is isometric on each connected part of Y under the l^1-norm of $\boldsymbol{R}^m$.*

COROLLARY 4.5. *Let $m, n \geqq 1$ be integers and X be a connected open subset of $\boldsymbol{R}^m$. Then the isometry group of $C_0^{(n)}(X)$ is $S^1 \times \Sigma$ with the group operation $(\lambda_1, \sigma_1) \circ (\lambda_2, \sigma_2) = (\lambda_1 \lambda_2, \sigma_2 \circ \sigma_1)$ and*

$$(\lambda, \sigma)(f)(\boldsymbol{x}) = \lambda f(\sigma(\boldsymbol{x})), \quad \forall \boldsymbol{x} \in X, \ f \in C_0^{(n)}(X),$$

where $\Sigma = \{\sigma \mid \sigma: X \xrightarrow{\text{onto}} X$ is isometric in the l^1-norm of $\boldsymbol{R}^m\}$.

Remark 6. Although we assume that the X and Y are open subsets of $\mathbf{R}^m$ in this section, it is worth to mention that all the results in Theorem 4.1 & 4.2 and Corollary 4.3~4.5 remain true if we replace "open subset(s)" by "locally compact and NIP subset(s) which is(are) contained in the closure(s) of its(their) interior(s)".

Now, let us see some examples.

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[2] That is, there exists a bijection $\varphi: X \to Y$ such that $\|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})\|_{l^1} = \|\boldsymbol{x} - \boldsymbol{y}\|_{l^1}$, $\forall \boldsymbol{x}, \boldsymbol{y} \in X$.

Example 1. Let $n\geqq 1$ and

$$X=\{(x_1, x_2)\in \boldsymbol{R}^2 : |x_1|+|x_2|<\sqrt{2}\}$$

$$Y=\{(x_1, x_2)\in \boldsymbol{R}^2 : |x_1|<1, |x_2|<1\}.$$

Then X and Y are isometric under the l^2-norm of $\boldsymbol{R}^2$, but $C_0^{(n)}(X)\not\cong C_0^{(n)}(Y)$.

Check. Since X is a rotation of Y, X and Y are isometric in the l^2-norm of $\boldsymbol{R}^2$. If $C_0^{(n)}(X)\cong C_0^{(n)}(Y)$, we must have a homeomorphism $\sigma: Y\to X$ of the form: σ=permutation+translation. It is obvious that Y is invariant under permutations. Therefore, Y must be transferred to X by some translation which is obviously impossible. □

When $X\subseteqq \boldsymbol{R}^m$ is connected and open, the isometry group of $C_0^{(n)}(X)$, due to the symmetry of X, may be as large as $S^1\times \mathrm{IOR}(m)$ and also may be as small as S^1. See the next two examples.

Example 2. Let $X_p=\{(x_1, \cdots, x_m)\in \boldsymbol{R}^m : (|x_1|^p+\cdots+|x_m|^p)^{1/p}<1\}$ $(m\geqq 1, 0<p\leqq\infty)$. Then the isometry group I_{n,X_p} of $C_0^{(n)}(X_p)$ $(n\geqq 1)$ is equal to $S^1\times \mathrm{IOR}(m)$ with $(\lambda_1, \sigma_1)\circ(\lambda_2, \sigma_2)=(\lambda_1\lambda_2, \sigma_2\circ\sigma_1)$ and

$$(\lambda, \sigma)(f)(\boldsymbol{x})=\lambda f(\sigma(\boldsymbol{x})), \quad \forall \boldsymbol{x}\in X_p, \ f\in C_0^{(n)}(X_p).$$

Moreover, when $m>1$, $n\geqq 1$ and $p\neq q$ we have $C_0^{(n)}(X_p)\not\cong C_0^{(n)}(X_q)$.

Check. Let Σ be the same as in Corollary 4.5. By the Mazur-Ulam's Theorem [8], each $\sigma\in\Sigma$ can be extended to be a surjective linear isometry on $(\boldsymbol{R}^m, \|\cdot\|_{l^1})$, thus, $\sigma\in\mathrm{IOR}(m)$. On the other hand, $\sigma\in\Sigma$ when $\sigma\in\mathrm{IOR}(m)$. Therefore, $\Sigma=\mathrm{IOR}(m)$ and $I_{n,X_p}=S^1\times\mathrm{IOR}(m)$. The group operation is trivial.

Finally, if $m>1$, $n\geqq 1$ and $p\neq q\in(0, \infty]$, noting that X_q is invariant under permutations and $X_q\neq X_p$, we can see that there is no map σ of the form: σ=permutation+translation, such that $\sigma(X_q)=X_p$, by Corollary 4.3, $C_0^{(n)}(X_p)\not\cong C_0^{(n)}(X_q)$. □

Example 3.

(1) Let $m>1$, $a_i>0$ $(1\leqq i\leqq m)$, $a_i\neq a_j$ $(i\neq j)$ and

$$X=\prod_{i=1}^{m}(-a_i, a_i) \quad \text{or} \quad X=\left\{\boldsymbol{x}=(x_1, \cdots, x_m)\in\boldsymbol{R}^m : \sum_{i=1}^{m}\frac{|x_i|}{a_i}<1\right\}.$$

Then, the isometry group $I_{n,X}$ of $C_0^{(n)}(X)$ $(n\geqq 1)$ is homeomorphic to $S^1\times\{-1, 1\}^m$ and $\mathrm{IOR}(m)\not\subseteq I_{n,X}$.

(2) Let $m>1$, $0<a_1<\cdots<a_{2m}<\infty$ and

$$X=\left\{\begin{matrix}\boldsymbol{x}=(x_1, \cdots, x_m)\in\boldsymbol{R}^m : x_j=-a_{2j-1}\lambda_{2j-1}+a_{2j}\lambda_{2j}, \ \lambda_j>0 \\ (1\leqq j\leqq m), \ \lambda_1+\cdots+\lambda_{2m}=1\end{matrix}\right\}.$$

Then, the isometry group of $C_0^{(n)}(X)$ $(n\geqq 1)$ is S^1 and each surjective linear isometry T on $C_0^{(n)}(X)$ is of the form $Tf=\lambda f$ $(f\in C_0^{(n)}(X))$ for some scalar $|\lambda|=1$.

Check. In both the cases (1) and (2), the lengths of the projections of X to the axes are different and invariant under translations. Thus, any surjective isometry σ (=permutation+translation) on X must keep the axes unchanged and

$$\sigma(\boldsymbol{x})=(b_1x_1, \cdots, b_mx_m)+(c_1, \cdots, c_m), \quad \forall \boldsymbol{x}=(x_1, \cdots, x_m)\in X$$

for some $\boldsymbol{b}=(b_1, \cdots, b_m)\in\{-1, 1\}^m$ and $\boldsymbol{c}=(c_1, \cdots, c_m)\in\boldsymbol{R}^m$.

For the case (1), since $\sigma(\boldsymbol{x})\in X$ for $\boldsymbol{x}=(0, \cdots, 0, x_j, 0, \cdots, 0)\in X$, letting $b_jx_j\to\pm a_j$, we get

$$-a_j\leqq\pm a_j+c_j\leqq a_j, \quad 1\leqq j\leqq m.$$

Thus, $c_j=0$ $(1\leqq j\leqq m)$. On the other hand, for any $\boldsymbol{b}=(b_1, \cdots, b_m)\in\{-1, 1\}^m$, it is obvious that $\sigma_{\boldsymbol{b}}(\boldsymbol{x})=(b_1x_1, \cdots, b_mx_m)$ $(\forall\boldsymbol{x}=(x_1, \cdots, x_m)\in X)$ determines an isometry $\sigma_{\boldsymbol{b}}$ on X in the l^1-norm. By Corollary 4.5, the isometry group $I_{n,X}$ of $C_0^{(n)}(X)$ $(n\geqq 1)$ is homeomorphic to $S^1\times\Sigma=S^1\times\{-1, 1\}^m$. Since IOR$(m)$ is homeomorphic to $\{-1, 1\}^m\times\Pi$ (from (3.9)), where $\Pi=\{$permutation on $\{1, \cdots, m\}\}\neq\{Id\}$, IOR$(m)$ is not contained in $I_{n,X}$.

For the case (2), it is trivial that σ is also an isometry on the closure $\bar{X}$ of X, which can be represented by

$$\bar{X}=\left\{\begin{array}{r}\boldsymbol{x}=(x_1, \cdots, x_m)\in\boldsymbol{R}^m : x_j=-a_{2j-1}\lambda_{2j-1}+a_{2j}\lambda_{2j},\ \lambda_j\geqq 0 \\ (1\leqq j\leqq m),\ \lambda_1+\cdots+\lambda_{2m}=1\end{array}\right\},$$

and the map $t\mapsto b_jt+c_j$ is an isometry from $[-a_{2j-1}, a_{2j}]$ onto $[-a_{2j-1}, a_{2j}]$. For any $1\leqq j\leqq m$, let $t\in[-a_{2j-1}, a_{2j}]$ be such that $b_jt+c_j=a_{2j}$ and $\boldsymbol{x}=(0, \cdots, 0, \underset{\langle j\rangle}{t}, 0, \cdots, 0)\in\bar{X}$. Since $\sigma(\boldsymbol{x})=(c_1, \cdots, c_{j-1}, a_{2j}, c_{j+1}, \cdots, c_m)\in\bar{X}$ with the corresponding $\lambda_{2j}=1$, we must have $\lambda_i=0$ $(\forall i\neq j)$. Therefore, $c_j=0$ $(1\leqq j\leqq m)$.

Because that $\sigma(\boldsymbol{y})=(0, \cdots, 0, \underset{\langle j\rangle}{b_ja_{2j}}, 0, \cdots, 0)\in\bar{X}$, where $\boldsymbol{y}=(0, \cdots, 0, \underset{\langle j\rangle}{a_{2j}}, 0, \cdots, 0)\in\bar{X}$, we have $-a_{2j-1}\leqq b_ja_{2j}$. Together with the condition $0<a_1<\cdots<a_{2m}<\infty$ and $b_j=\pm 1$, we can get that $b_j=1$ $(1\leqq j\leqq m)$. Thus, $\sigma=Id_X$ and $\Sigma=\{Id\}$. By Theorem 4.2, each isometry T on $C_0^{(n)}(X)$ $(n\geqq 1)$ is of the form $Tf=\lambda f$ $(f\in C_0^{(n)}(X))$ for some scalar $|\lambda|=1$. Therefore, the isometry group of $C_0^{(n)}(X)$ is S^1. □

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