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A REMARK ON ARTAL'S PAPER

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Introduction

In his paper [A], Artal introduced the notion of a Zariski pairs, whose definition is as follows:

DEFINITION 0.1. A Zariski pair is a pair of two plane curves, C_1 and C_2 enjoying the following three conditions:

(i) $\deg C_1 = \deg C_2$,
(ii) There exists a diffeomorphism $(T(C_1), C_1) \to (T(C_2), C_2)$ where $T(C_i)$ is a smooth neighborhood of C_i for each i, and
(iii) The pairs $(\boldsymbol{P}^2, C_1)$ and $(\boldsymbol{P}^2, C_2)$ are not homeomorphic.

The second condition seems to be difficult to deal with. However, we can replace it by another condition, which seems to be rather tractable. For detail, see [A].

The first example of a Zariski pair is found in [Z1]: sextic curves C_1 and C_2 with six (2, 3) such that

(i) the six cusps are on a conic for C_1, and
(ii) there is no conic which through the six cusps on C_2.

This example is also intensively studied in [O].

In [A], Artal gave some of new examples of Zariski pairs. The purpose of this note is to give a new proof for one of Artal's examples by using the method developed in [T].

We shall first review one of his example, which we shall study in this article. Let C_i $(i=1, 2)$ be sextic curves both of which have four irreducible components, a smooth cubic and three lines. These two curves satisfy the following properties:

If we write C_i $(i=1, 2)$ in a way such that

$$C_i = l_1^{(i)} \cup l_2^{(i)} \cup l_3^{(i)} \cup E_i,$$

where $l_j^{(i)}$ is a line and E_i is a smooth cubic, then

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(i) $l_j^{(i)}$ ($j=1, 2, 3$) is a tangent line at an inflection point, $p_j^{(i)}$, of E_i,

(ii) $l_1^{(i)}$, $l_2^{(i)}$ and $l_3^{(i)}$ do not intersect at one point, and

(iii) For C_1, the inflection points $p_1^{(1)}$, $p_2^{(1)}$ and $p_3^{(1)}$ are collinear, while the inflection points $p_1^{(2)}$, $p_2^{(2)}$ and $p_3^{(2)}$ are not collinear for C_2.

In [A], the idea to prove the pair, C_1 and C_2, to be a Zariski pair is to study the irregularities of cyclic coverings, Z_1 and Z_2, of $\boldsymbol{P}^2$ branched along C_1 and C_2, respectively. In order to calculate the irregularities of such cyclic coverings, Artal generalized the method developed by Zariski [Z2] and Esnault [E]. These ideas and techniques seem to be rather classical.

In this article, our basic tool is an $\mathcal{S}_3$ covering. We shall also make use of some geometry of double coverings and elliptic K3 surfaces (*cf.* [MP], [P], [S2]). These points make our approach more interesting. Let us start with the definition of $\mathcal{S}_3$ coverings:

DEFINITION 0.2. Let Y be a smooth projective variety. A normal variety, X, with a finite morphism $\pi: X \to Y$ is called an $\mathcal{S}_3$ covering of Y if the rational function field, $\boldsymbol{C}(X)$, of X is a Galois extension of that of Y, $\boldsymbol{C}(Y)$, having the third symmetric group, $\mathcal{S}_3=\langle \sigma, \tau \mid \sigma^2=\tau^3=(\sigma\tau)^2=1\rangle$, as its Galois group.

With the notations in Definition 0.2, let $\boldsymbol{C}(X)^\tau$ be the invariant subfield of $\boldsymbol{C}(X)$ by τ. As $\boldsymbol{C}(X)^\tau$ is a quadratic extension of $\boldsymbol{C}(Y)$, the $\boldsymbol{C}(X)^\tau$-normalization of Y is a double covering. We denote it by $D(X/Y)$ and its covering morphism by β_1. Also, X is a cyclic triple covering of $D(X/Y)$, and β_2 denotes the covering morphism from X to $D(X/Y)$. By their definition, $\pi=\beta_1\circ\beta_2$.

Now we are in position to state our main result:

THEOREM 0.3. *Let C_1 and C_2 be two sextic curves described as above. Then there exists an $\mathcal{S}_3$ covering of $\boldsymbol{P}^2$ branched along C_1, while there is no $\mathcal{S}_3$ covering of $\boldsymbol{P}^2$ branched along C_2.*

From Theorem 0.3, we can easily see that the pair (C_1, C_2) is a Zariski pair. As we shall show in § 5, Theorem 0.3 easily follows from next two propositions as below:

PROPOSITION 0.4. *Let C_1 be as above, and let $f_1: W_1 \to \boldsymbol{P}^2$ be a double covering branched along C_1. Then there exists an $\mathcal{S}_3$ covering, S_1, of $\boldsymbol{P}^2$ branched along C_1 with $D(S_1/\boldsymbol{P}^2)=W_1$.*

PROPOSITION 0.5. *Let C_2 be as above, and let $f_2: W_2 \to \boldsymbol{P}^2$ be a double covering branched along C_2. Then there is no $\mathcal{S}_3$ covering, S_2, of $\boldsymbol{P}^2$ branched along C_2 with $D(S_2/\boldsymbol{P}^2)=W_2$.*

Now we shall explain the contents of this article. The first section starts with a brief summary on $\mathcal{S}_3$ coverings. In § 2, we shall look into the canonical resolutions of the double coverings, W_1 and W_2, introduced in Propositions 0.4 and 0.5. § 3 will devote to prove Proposition 0.4, and § 4 will devote to prove Proposition 0.5. In § 5, we shall prove Theorem 0.3.

Notations and Conventions. Throughout this article, the ground field will always be the complex number field $\boldsymbol{C}$.

$\boldsymbol{C}(X)$:=the rational function field of X.

Let X be a normal variety, and let Y be a smooth variety. Let $\pi : X \to Y$ be a finite morphism from X to Y. We define the branch locus of f, which we denote by $\Delta(X/Y)$, as follows:

$$\Delta(X/Y) = \{y \in Y \mid \#(\pi^{-1}(y)) < \deg \pi\}.$$

For a divisor D on Y, $\pi^{-1}(D)$ denotes the set-theoretic inverse image of D, while $\pi^*(D)$ denotes the ordinary pullback. Also Supp D means the supporting set of D.

Let $\pi : X \to Y$ be an $\mathcal{S}_3$ covering of Y. Morphisms, β_1 and β_2, and the variety $D(X/Y)$ always mean those defined in Introduction.

Let S be a finite double covering of a smooth projective surface Σ. The "*canonical resolution*" of S always means the resolution given by Horikawa in [H].

Let S be an elliptic surface over C. We call S *minimal* if the fibration is relatively minimal. In this paper, we always assume that an elliptic surface is minimal. For singular fibers of an elliptic surface, we use the notation of Kodaira [K].

Let D_1, D_2 be divisors.

$D_1 \sim D_2$: linear equivalence of divisors.

$D_1 \approx D_2$: algebraic equivalence of divisors.

$D_1 \approx_{\boldsymbol{Q}} D_2$: $\boldsymbol{Q}$-algebraic equivalence of divisors.

For singularities of a plane curve, we shall use the same notations as those in [P].

§ 1. A summary on $\mathcal{S}_3$ coverings

We shall start with the following proposition.

PROPOSITION 1.1. *Let $f : Z \to Y$ be a smooth finite double covering of a smooth projective variety Y. Let σ be the involution determined by the covering transformation of f. Let D_1, D_2, and D_3 be effective divisors on Z. Suppose that*

(a) *D_1 is reduced, and D_1 and $\sigma^* D_1$ have no common component,*

(b) *$D_1 + 3D_2 \sim \sigma^* D_1 + 3D_3$.*

Then there exists an $\mathcal{S}_3$ covering, X, of Y such that (i) *$D(X/Y) = Z$, and* (ii) Supp$(D_1 + \sigma^* D_1)$ *is the branch locus of β_2.*

For a proof, see [T].

The conditions in Proposition 1.1 seem very complicate as well as intractable. These are, however, essential to consider $\mathcal{S}_3$ coverings, because we have the following proposition saying that the "inverse" of Proposition 1.2 holds.

PROPOSITION 1.2. *Let $\pi: X \to Y$ be an $\mathcal{S}_3$ covering and let σ denote the involution on $D(X/Y)$ coming from the covering transformation of β_1. Suppose that $D(X/Y)$ is smooth. Then there exist three effective divisors D_1, D_2 and D_3 on $D(X/Y)$ such that*

(i) *D_1 is reduced, and D_1 and $\sigma^* D_1$ have no common component*
(ii) *$D_1 + 3D_2 \sim \sigma^* D_1 + 3D_3$, and*
(iii) $\mathrm{Supp}(D_1 + \sigma^* D_1)$ *is the branch locus of β_2.*

For a proof, see [T].

COROLLARY 1.3. *Let $\pi: S \to \Sigma$ be an $\mathcal{S}_3$ covering of a smooth projective surface Σ, and let D be an irreducible component of $\beta_1(\Delta(S/D(S/\Sigma)))$. If we denote x by any intersection point of D and $\Delta(D(S/\Sigma)/\Sigma)$. Then the intersection multiplicity at x is $\geqq 2$.*

Proof. This is immediate from Proposition 1.2.

§ 2. An elliptic K3 surface arising from W_i

We shall use the same notations as those in Introduction. Let $\mathcal{E}_i \to W_i$ be the canonical resolution of W_i ($i=1, 2$). By its construction, $\mathcal{E}_i$ satisfies the following diagram:

$$\begin{array}{ccc} W_i & \xleftarrow{\mu_i} & \mathcal{E}_i \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle \tilde{f}_i} \\ \boldsymbol{P}^2 & \xleftarrow[q_i]{} & \Sigma_i \end{array}$$

where Σ is obtained by a succession of blowing-ups.

As the singular points of C_i are of types a_1 and a_5, singularities of W_i are all rational double point. Hence, $\mathcal{E}_i$ is a K3 surface. In order to apply the results in the preceding section to $\tilde{f}_i$, we shall look into $\mathcal{E}_i$ ($i=1, 2$) in detail.

Let x_i be the intersection point of $l_1^{(i)}$ and $l_2^{(i)}$. Then, lines passing through x_i define an elliptic fibration on $\mathcal{E}_i$. This fibration is called the standard fibration centered at x_i (see [P], p. 282), which we denote by $\varphi_i: \mathcal{E}_i \to \boldsymbol{P}^1$. By its construction, the line $l_3^{(i)}$ determines the section of φ_i, which we denote by $s_0^{(i)}$. For the configuration of singular fibers of φ_i, we have

LEMMA 2.1. *The configuration of the singular fibers of $\varphi_1: \mathcal{E} \to \boldsymbol{P}^1$ is I_6, IV^*, IV^*, I_1, I_1.*

Proof. Let l be a line through x_1. Suppose that $l \neq l_1^{(1)}, l_2^{(1)}$, the line connecting x_1 and $p_3^{(1)}$, which we denote by l_{x_1}. Then, l determines a smooth fiber of φ_1 unless l is tangent to E_1. From [MP], Table 6.2, if l is tangent to E_1 at an inflection point, then the corresponding singular fiber is of type II, and if l is tangent to E_1 at a non-inflection point, the corresponding singular fiber is of type I_1.

CLAIM 2.2. *Let l be as above. Then l is not tangent to E_1 at an inflection point.*

Proof of Claim 2.2. Without loss of generality, we may assume that $x_1 = (0:1:0)$ and $l_3^{(1)}$ is the line at infinity. From the assumption that three a_5 singularities are collinear, we may also assume that $l_1^{(1)}$, $l_2^{(1)}$ and E_1 are defined by the affine equations:

$$l_1^{(1)}: y = ax+b$$

$$l_2^{(1)}: y = -(ax+b)$$

$$E_1: y^2 = x^3 + (ax+b)^2$$

$$a, b \in \boldsymbol{C}, \quad a \neq 0.$$

With this coordinate (x, y), x_1 has a coordinate $(-b/a, 0)$. Suppose that l is tangent to E_1 at an inflection point and let (x_0, y_0) denote its coordinate. Then, since E_1 is symmetric with respect to the x axis, $(x_0, -y_0)$ is also an inflection point and the tangent line at $(x_0, -y_0)$ also passes through $(-a/b, 0)$. Thus, four tangent lines at four distinct inflection points meet at $(-a/b, 0)$. On the other hand, the degree of the dual curve of E_1 is 6 and each inflection point corresponds to a cusp on it. Our situation that the four tangent lines at four distinct inflection points intersect at $(-a/b, 0)$ means that four different cusps on the dual curve of E_1 are collinear, but this is impossible as the degree of the dual curve is 6.

By Claim 2.2, if $l \neq l_1^{(1)}, l_2^{(1)}, l_{x_1}$, l determines either a smooth fiber or a singular fiber of type I_1. Now we shall go on to the remaining three cases. In the case of $l = l_1^{(1)}$ or $l_2^{(2)}$, we can easily check that l determines an irreducible component of a singular fiber of type IV^* by looking into the process of the canonical resolution $\mu_1 : \mathcal{E}_1 \to W_1$. We next consider the case of $l = l_{x_1}$. By looking into the process of the canonical resolution $\mu_1 : \mathcal{E}_1 \to W_1$, we see l_{x_1} determines an irreducible component of a singular fiber of type I_6 if l_{x_1} meets E_1 at two distinct points other than $p_3^{(1)}$. Hence, it is enough to show that l_{x_1} is not tangent to E_1. Suppose that l_{x_1} is tangent to E_1 at a point P. If we take the same affine coordinate as those in the proof of Claim 2.2, we can easily see that P is on the x-axis, and l_{x_1} is a line parallel to the y-axis. But this is impossible as $(-b/a, 0)$ which is different from P, is also on l_{x_1}. Thus, the configuration of singular fibers of $\mathcal{E}_1$ is I_6, IV^*, IV^*, and some I_1's. By

[K], Theorem 12.2, the number of I_1 fibers is 2, and we have our lemma. ■

For the configuration of singular fibers of $\mathcal{E}_2$, we have the following lemma.

LEMMA 2.4. *Let $\varphi_2 : \mathcal{E}_2 \to \boldsymbol{P}^1$ be the elliptic fibration centered at $x_2 = l_1^{(2)} \cap l_2^{(2)}$. Then its configuration of singular fibers is either* (i) I_6, IV^*, IV^*, I_1, I_1, (ii) I_6, IV^*. IV^*, II, *or* (iii) I_7, IV^*, IV^*, I_1.

Proof. Our statement easily follows from the proof of Lemma 2.2. ■

Remark 2.5. Note that every exceptional curve except one arising from the singularity at x_1 (resp. x_2) is an irreducible component of a singular fiber of $\mathcal{E}_1$ (resp. $\mathcal{E}_2$).

We shall end this section with the following theorem by Shioda [S2], which will play a key role in proving Propositions 0.4 and 0.5. We here need some notations to refer his theorem. Let $\varphi : \mathcal{E} \to C$ be a minimal elliptic surface over a curve C. Assume that

(i) φ has a section s_0, and
(ii) there is at least one singular fiber.

Let T denote a subgroup of the Néron-Severi group, $NS(\mathcal{E})$, of $\mathcal{E}$ generated by the section s_0 and all irreducible components of fibers, and let $MW(\mathcal{E})$ be the Mordell-Weil group (the group of sections) of $\varphi : \mathcal{E} \to C$, where s_0 corresponds to the zero element. With these notations, we have

THEOREM 2.6. *Let $\varphi : S \to C$ be as above. Then there is an isomorphism*

$$\psi : NS(\mathcal{E})/T \cong MW(\mathcal{E}).$$

For details, see [S2].

It is clear that both elliptic surfaces $\varphi_i : \mathcal{E}_i \to \boldsymbol{P}^1$ ($i=1, 2$) satisfy the conditions for Theorem 2.6. Hence, the assertion in Theorem 2.6 holds for $\varphi_i : \mathcal{E}_i \to \boldsymbol{P}^1$.

§ 3. Proof of Proposition 0.4

We shall use the same notations as those in § 2. The goal of this section is that there exist an $\mathcal{S}_3$ covering $\pi_1 : S_1 \to \boldsymbol{P}^2$ branched along C_1 with $D(S_1/\boldsymbol{P}^2) = W_1$.

Suppose that there exists an $\mathcal{S}_3$ covering $\pi_1 : S_1 \to \boldsymbol{P}^2$ with $\Delta(S_1/\boldsymbol{P}^2) = C_1$ and $D(S_1/\boldsymbol{P}^2) = W_1$. Then the $\boldsymbol{C}(S_1)$-normalization, $\tilde{S}_1$, of Σ_1 is an $\mathcal{S}_3$ covering of Σ_1 such that

(i) $D(\tilde{S}_1/\Sigma_1) = \mathcal{E}_1$,
(ii) $q_1(\Delta(\tilde{S}_1/\Sigma_1)) = C_1$.

Conversely, if there exists an $\mathcal{S}_3$ covering, $\tilde{S}_1$, enjoying the above two conditions, then the $\boldsymbol{C}(\tilde{S}_1)$-normalization, S_1, of $\boldsymbol{P}^2$ is an $\mathcal{S}_3$ covering branched along C_1 with $D(S/\boldsymbol{P}^2)=W_1$.

Thus, it is enough to show that there exists an $\mathcal{S}_3$ covering of Σ satisfying the above two conditions. As $\tilde{f}_1:\mathcal{E}_1\to\Sigma_1$ is a smooth double covering, we can apply Proposition 1.1 to this case. Now we reduce our problem to find three effective divisors D_1, D_2 and D_3, on $\mathcal{E}_1$ such that

(i) these three divisors satisfy three conditions in Proposition 1.1, and

(ii) every irreducible component of D_1 is that of the exceptional divisor of the resolution $\mu_1:\mathcal{E}_1\to W_1$.

To find these three divisors, we shall make use of the elliptic fibration $\varphi_1:\mathcal{E}_1\to\boldsymbol{P}^1$, and label irreducible components of singular fibers as below:

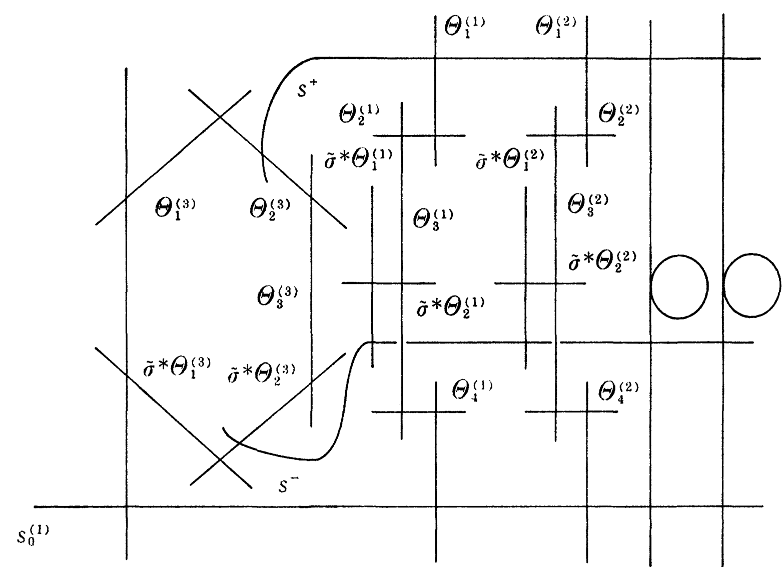

Figure 1

where s^+, s^- denote sections arising from the line through $p_1^{(1)}$, $p_2^{(1)}$ and $p_3^{(1)}$. Note that $\Theta_4^{(i)}$ $(i=1, 2)$ and $\Theta_3^{(3)}$ are irreducible components arising from $l_i^{(1)}$ $(i=1, 2)$ and l_{x_1}, respectively, and all other labeled irreducible components of singular fibers are those of the exceptional divisor of the resolution $\mu_1\ \mathcal{E}_1\to W_1$.

Under these notations, we have

LEMMA 3.1. *If we put*

$$D_1=\sum_{i=1,2}(\Theta_1^{(i)}+\tilde{\sigma}^*\Theta_2^{(i)})+\Theta_2^{(3)}+\tilde{\sigma}^*\Theta_1^{(3)},$$

$$D_2 = s^+ + \sum_{i=1,2} \{\Theta_1^{(i)} + \Theta_4^{(i)} + \tilde{\sigma}^*\Theta_1^{(i)} + \tilde{\sigma}^*\Theta_2^{(i)}$$

$$+ 2(\Theta_2^{(i)} + \Theta_3^{(i)})\} + \Theta_1^{(3)} + \Theta_2^{(3)} + \Theta_3^{(3)} + \tilde{\sigma}^*\Theta_2^{(3)}, \textit{ and}$$

$$D_3 = s_0^{(1)} + 2F,$$

then (i) *these three divisors satisfy the conditions in Proposition* 1.1 *and* (ii) *every component of* D_1 *is that of the exceptional divisors of* μ_1.

Proof. Regard $s_0^{(1)}$ as the zero element of the Mordell-Weil group, $MW(\mathcal{E}_1)$, of $\mathcal{E}_1$, and let $\langle\, ,\rangle$ denote Shioda's pairing on $MW(\mathcal{E}_1)$ (For detail, see [S2]). Let ψ be the isomorphism in Theorem 2.6. Then, by [S2], Theorem 8.6, we have $\langle\psi(s^+), \psi(s^+)\rangle = 0$. Hence, by [S2], Lemma 8.1, we have

$$s^+ \approx_{\boldsymbol{Q}} s_0^{(1)} + 2F - \sum_{i=1,2}\left(\frac{4}{3}\Theta_1^{(i)} + \frac{5}{3}\Theta_2^{(i)} + 2\Theta_3^{(i)} + \Theta_4^{(i)} + \frac{4}{3}\tilde{\sigma}^*\Theta_2^{(i)} + \frac{2}{3}\tilde{\sigma}^*\Theta_1^{(i)}\right)$$
$$-\left(\frac{2}{3}\Theta_1^{(3)} + \frac{4}{3}\Theta_2^{(3)} + \Theta_3^{(3)} + \frac{2}{3}\tilde{\sigma}^*\Theta_2^{(3)} + \frac{1}{3}\tilde{\sigma}^*\Theta_1^{(3)}\right)$$

Hence, we have

$$3s^+ \approx 3s_0^{(1)} + 6F - \sum_{i=1,2}(4\Theta_1^{(1)} + 5\Theta_2^{(i)} + 6\Theta_3^{(i)} + 3\Theta_4^{(i)} + 4\tilde{\sigma}^*\Theta_2^{(i)} + 2\tilde{\sigma}^*\Theta_1^{(i)})$$
$$-(2\Theta_1^{(3)} + 4\Theta_2^{(3)} + 3\Theta_3^{(3)} + 2\tilde{\sigma}^*\Theta_2^{(3)} + \tilde{\sigma}^*\Theta_1^{(3)}).$$

As $\mathcal{E}_1$ is simply connected, we can replace algebraic equivalence by linear equivalence. By using this linear equivalence of divisors, if we put three effective divisors D_1, D_2 and D_3 as above, it is straightforward to check out that they satisfy the assertions (i) and (ii). ∎

Combining Lemma 3.1 and Proposition 1.1, we have Proposition 0.4.

§4. Proof of Proposition 0.5

We shall also use the same notations as those in §2. The purpose of this section is to show that there exists no $\mathcal{S}_3$ covering, S_2, of $\boldsymbol{P}^2$ branched along C_2 with $D(S_2/\boldsymbol{P}^2) = W_2$.

Suppose that there exists such an $\mathcal{S}_3$ covering of $\boldsymbol{P}^2$. Let $\mathcal{E}_2$ be the canonical resolution of W_2, and let Σ_2 be the surface we have defined in §2. Let $\tilde{S}_2$ be the $\boldsymbol{C}(S_2)$-normalization of Σ_2. Then $\tilde{S}_2$ is an $\mathcal{S}_3$ covering of Σ_2. Moreover, since $\mathcal{E}_2$ is the $\boldsymbol{C}(D(S_2/\boldsymbol{P}^2))$-normalization of Σ_2, we have $\mathcal{E}_2 = D(\tilde{S}_2/\Sigma_2)$. We shall denote the covering morphism from $\tilde{S}_2$ to Σ_2 by $\tilde{\pi}_2$. We shall also denote the canonical morphism from $\mathcal{E}_2$ to Σ_2 and one from $\tilde{S}_2$ to $\mathcal{E}_2$ by $\tilde{\beta}_1$ and $\tilde{\beta}_2$, respectively. $\tilde{\sigma}$ denotes the involution on $\mathcal{E}_2$ determined by $\tilde{\beta}_1$. As $\tilde{\beta}_1 : \mathcal{E}_2 \to \Sigma_2$ is a smooth finite double covering, we can apply Proposition 1.2 to this case.

Hence there exist three effective divisors D_1, D_2 and D_3 on $\mathcal{E}_2$ which satisfy that

(i) D_1 is reduced and D_1 and $\tilde{\sigma}^*D_1$ have no common component,
(ii) $D_1+3D_2\sim\tilde{\sigma}^*D_1+3D_3$, and
(iii) $\mathrm{Supp}(D_1+\tilde{\sigma}^*D_1)$ is the branch locus of $\tilde{\beta}_2$.

Since the stein factorization of $q_2\circ\tilde{\pi}_2:\tilde{S}_2\to\boldsymbol{P}^2$ is S_2, and the support of the branch locus of $\tilde{\beta}_1$ contains the proper transform of C_2, the third condition means that every irreducible component of D_1 is that of the exceptional divisor of μ_2. Also, the condition (i) means that the exceptional curves arising from a_1 singularities of C_2 can not be irreducible components of D_1. Moreover, by Corollary 1.3, any exceptional curve which meets $s_0^{(2)}$ transversely can not be an irreducible component of D_1. Thus, by Remark 2.5, every irreducible component of D_1 is that of singular fibers of $\mathcal{E}_2$ not intersecting the section $s_0^{(2)}$.

Consider $s_0^{(2)}$ as the zero element of the Mordell-Weil group, $MW(\mathcal{E}_2)$, and let $NS(\mathcal{E}_2)$ and T be the Néron-Severi group of $\mathcal{E}_2$ and a subgroup of $NS(\mathcal{E}_2)$ defined by the same way as that in Theorem 2.6, respectively.

If we see the second condition on D_1, D_2 and D_3 in a way that $3(D_3-D_2)\sim D_1-\tilde{\sigma}^*D_1$, by the same argument as that in [T], §4 Claim, the divisor D_2-D_3 gives rise to a torsion element of order three in $NS(\mathcal{E}_2)/T$. Then, by Theorem 2.6, there exists a torsion element of order three in $MW(\mathcal{E}_2)$. Thus, we get a section s of φ_2 which corresponds to the torsion element of order three.

Let $\langle\, ,\rangle$ denote Shioda's pairing on $MW(\mathcal{E}_2)$ (For definition of the pairing, see [S2]), and let ψ denote the isomorphism in Theorem 2.6. In order to calculate the value of $\langle\psi(s),\psi(s)\rangle$ by using of the explicit formula by Shioda (*cf.* [S2]), we label irreducible components of singular fibers of $\mathcal{E}_2$ in such a way as in [S2] pp. 228-229.

As $\mathcal{E}_2$ is a K3 surface, by [S2], Theorem 8.6, we have

$$\langle\psi(s),\psi(s)\rangle=4+2ss_0-\textit{the contribution terms from the singular fibers.}$$

The contribution term arising from a singular fiber of type IV^* is 0 or 4/3. Also, if the section s intersects the i-th component of a singular fiber of type I_b, the contribution term arising from this singular fiber is $i(b-i)/b$.

By Lemma 2.4, there are three cases for the configuration of singular fibers of $\mathcal{E}_2$. As $MW(\mathcal{E}_2)$ has a torsion of order three, the second case in Lemma 2.4 does not occur (see [S1], Remark 1.10).

Now we shall consider the first case in Lemma 2.4. Assume that s intersects the i-th component of the singular fiber of type I_6. Then we have

$$\langle\psi(s),\psi(s)\rangle=4+2ss_0^{(2)}-\frac{i(6-i)}{6}$$
$$-\textit{the contribution terms from the singular fibers of type } IV^*.$$

On the other hand, as s is a torsion, $\langle\phi(s), \phi(s)\rangle=0$ by [S2], Theorem 8.4. Hence, $ss_0^{(2)}=0$ and s intersects either the second or the fourth component of the singular fiber of type I_6, and the both contribution terms from the singular fibers of type IV^* is 4/3. Thus we may assume that s intersects at each singular fiber in the same way as s^+ in Figure 1. Under these circumstances, we have

CLAIM 4.1. *$q_2\circ\tilde{\beta}_1(s)$ is a line passing through $p_1^{(2)}$, $p_2^{(2)}$ and $p_3^{(2)}$.*

Proof of Claim 4.1. Put $C=q_2\circ\tilde{\beta}_1(s)$. Since $ss_0^{(2)}=\tilde{\sigma}^*ss_0^{(2)}=0$, we have

$$C\cap l_3^{(2)}\subset \mathrm{Sing}(C_2).$$

Let Q_i denote the intersection point of $l_i^{(2)}$ and $l_3^{(2)}$ $(i=1, 2)$. Suppose $Q_i\in C$, then $s\cap(q_2\circ\tilde{\beta}_1)^{-1}(Q_i)\neq\emptyset$. As $(q_2\circ\tilde{\beta}_1)^{-1}(Q_i)=(\beta_1\circ\mu_2)^{-1}(Q_i)$ is the exceptional curve for the A_1 singularity of $D(S_2/\boldsymbol{P}^2)$ lying over Q_i, it is the irreducible component of the corresponding singular fiber of type IV^* which meets $s_0^{(2)}$. But s does not meet such an irreducible component. Hence, $Q_i\notin C$, and this implies $C\cap l_3^{(2)}=\{p_3^{(2)}\}$. Now by looking into the inverse process of the canonical resolution, we can easily see that C intersects $l_3^{(2)}$ transversely and passes through $p_1^{(2)}$ and $p_2^{(2)}$. Hence, $Cl_3^{(2)}=1$, i.e., C is a line, and it passes through $p_1^{(2)}$, $p_2^{(2)}$ and $p_3^{(2)}$.

The statement of Claim contradicts to our assumption. Therefore, the first case in Lemma 2.4 does not occur.

Now we go on to the remaining case. Namely, the configuration of the singular fibers is the case (iii) in Lemma 2.4. We also assume that s intersects the i-th component of the singular fiber of type I_7. Then we have

$$\langle\phi(s), \phi(s)\rangle=4+2ss_0^{(2)}-\frac{i(7-i)}{7}$$

—the contribution terms from the singular fibers of type IV^*.

As $\phi(s)$ is a torsion element of $MW(\mathcal{E}_2)$, we have $\langle\phi(s), \phi(s)\rangle=0$. But, by the above formula, it easy to show that the value $\langle\phi(s), \phi(s)\rangle$ can never be zero for any i $(1\leqq i\leqq 6)$ and any value of the contribution terms from the singular fibers of type IV^*.

Combining both cases, we disprove the existence of the section s. This implies that there do not exist such three divisors as D_1, D_2 and D_3. Therefore, by Proposition 1.2, we obtain Proposition 0.5.

§5. Proof of Theorem 0.3

It is now enough to show the following:

CLAIM 5.1. *There is no $\mathcal{S}_3$ covering of $\boldsymbol{P}^2$ branched along C_2.*

Proof. Suppose that there exists an $\mathcal{S}_3$ covering, S, of $\boldsymbol{P}^2$ branched along C_2. As the branch locus, $\Delta(D(S/\boldsymbol{P}^2)/\boldsymbol{P}^2)$, of $\beta_1: D(S/\boldsymbol{P}^2)\to\boldsymbol{P}^2$ is a curve of even degree, its degree is either 2, 4, or 6. By Proposition 0.5, we have $\Delta(D(S/\boldsymbol{P}^2)/\boldsymbol{P}^2)\neq C_2$. Hence we have $\deg\Delta(D(S/\boldsymbol{P}^2)/\boldsymbol{P}^2)=2$ or 4. But, for each case, there is a line component, l, of C_2 such that

(i) l intersects a line component of $\Delta(D(S/\boldsymbol{P}^2)/\boldsymbol{P}^2)$ transversely, and
(ii) $\beta_1^*(l)$ is contained in the branch locus of β_2.

But this contradicts to Corollary 1.3.

From Proposition 0.4 and Claim 5.1, we have $\pi_1(\boldsymbol{P}^2\setminus C_1)\not\cong\pi_1(\boldsymbol{P}^2\setminus C_2)$. Therefore, the pair (C_1, C_2) is a Zariski pair.

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