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ON THE UNIVERSAL COVERING OF PROJECTIVE MANIFOLDS OF GENERAL TYPE

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1. Introduction

Around 1970, S. Kobayashi proposed the following conjecture ([3]).

CONJECTURE 1. *Let M be a compact Kähler manifold. Suppose that M is measure hyperbolic. Then M is of general type.*

We note that a compact complex manifold M of general type is always measure hyperbolic ([4, p. 9, Lemma 1]).

Recently M. Gromov introduced the notion of Kähler hyperbolicity and proved that Kähler hyperbolic manifolds are projective of general type ([2]). The main tools in the paper are Atiyah's L^2-index theorem and a Lefschetz type theorem. We note that Kähler hyperbolicity is a property of the universal covering manifold.

Although Gromov's theorem is a partial affirmative answer of Kobayashi's conjecture it seems to be hard to check that a given complex manifold is Kähler hyperbolic.

In this paper we shall give a partial affirmative answer for Kobayashi's conjecture for a compact quotient of a universal covering of a projective manifold of general type.

THEOREM 1. *Let X be a projective manifold of general type and let $\pi: D\rightarrow X$ be the universal covering of X. Then any compact unramified quotient of D is of general type.*

Remark 1. If the canonical bundle of X is ample, X carries a Kähler-Einstein metric g_E of negative Ricci curvature by the solution of Calabi's conjecture ([1, 7]). By Yau's Schwarz lemma ([8]), π^*g_E is invariant under the action of Aut(D). Hence every compact unramified quotient of D carries a metric of strictly negative Ricci curvature. This implies that every compact quotient of D has ample canonical bundle. In particular such a manifold is of general type.

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2. Proof of Theorem 1

Let X be a projective manifold of general type and let $\pi: D \to X$ be the universal covering of X.

DEFINITION 1. Let M be a complex manifold of dimension n and let L be a line bundle on M. L is said to be big, if

$$\limsup_{m \to +\infty} m^{-n} \dim H^0(M, \mathcal{O}_M(L^{\otimes m})) > 0$$

holds

LEMMA 1 (Kodaira's lemma) ([5, Appendix]). *Let M be a smooth projective variety and let D be a big divisor on X. Then there exists an effective $\boldsymbol{Q}$-divisor E on X such that $D-E$ is an ample $\boldsymbol{Q}$-divisor.*

Since K_X is big, there exists an effective $\boldsymbol{Q}$-divisor E such that $K_X - E$ is ample. Let $E = \sum_{i=1}^{k} a_i E_i$ be the irreducible decomposition of E. Let $\sigma_i \in H^0(X, \mathcal{O}_X(E_i))$ be a holomorphic section such that $(\sigma_i) = E_i$. Then by Kodaira's lemma there exists C^∞ hermitian metrics h_0 on K_X and h_i on $\mathcal{O}_X(E_i)$ $(1 \leqq i \leqq k)$ respectively such that

$$\omega_X = \mathrm{curv}\, h_0 - \sum_{i=1}^{k} a_i \,\mathrm{curv}\, h_i$$

is a Kähler form on X, where curv denotes $\sqrt{-1}\,\bar{\partial}\partial \log$ (operator which takes the curvature form of a hermitian metric).

DEFINITION 2. Let M be a complex manifold and let L be a holomorphic line bundle on M. h is said to be a singular hermitian metric on L if there exists a C^∞ hermitian metric h_0 on L and a locally L^1 function φ such that

$$h = e^{-\varphi} h_0$$

holds. We define the curvature current curv h by

$$\mathrm{curv}\, h = \mathrm{curv}\, h_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi$$

where $\mathrm{curv}\, h_0 = \sqrt{-1}\,\bar{\partial}\partial \log h_0$ is the usual curvature form and $\partial\bar{\partial}$ of φ is taken in the sense of current.

DEFINITION 3. Let T be a closed positive (1, 1) current on a complex manifold M. T is said to be strictly positive, if for every point $x \in M$, there exists an open neighborhood U_x and a C^∞ Kähler form ω_x such that $T - \omega_x$ is a closed positive current on U_x.

Let σ_i be a holomorphic section of $\mathcal{O}_X(E_i)$ with divisor E_i respectively. We set

$$h=\frac{h_0}{\prod_{i=1}^{k} h_i(\sigma_i, \sigma_i)^{a_i}}.$$

h is a singular hermitian metric on X and satisfies

$$\operatorname{curv} h=\omega+\sum_{i=1}^{k} a_i E_i.$$

In particular h has strictly positive curvature current. Then $\pi_X^* h$ is a singular hermitian metric of K_D with strictly positive curvature current. We denote $\pi^*\omega_X$ by ω and $\pi_X^* h$ again by h for simplicity. The following theorem follows from the standard L^2-estimate for $\bar\partial$-operator due to Hörmander.

THEOREM 2 ([6, p. 561]). *Let (M, ω_M) be a complete Kähler manifold and let (L, h_L) be a singular hermitian line bundle on M such that*

$$\operatorname{curv} h_L+\operatorname{Ric}_M \geqq c\omega_M$$

holds for some positive constant c. Let $\mathcal{L}^2(L, h_L)$ denote the sheaf of germs of local L^2 holomorphic sections of (L, h_L). Then we have

$$H^q_{(2)}(M, \mathcal{L}^2(L, h_L))=0$$

holds for every $q\geqq 1$ and $\mathcal{L}^2(L, h_L)$ is a coherent sheaf of $\mathcal{O}_X$-module.

CORORALLY 1.

$$H^0_{(2)}(D, \mathcal{L}^2(K_D^{\otimes m}, h^{\otimes m})) \longrightarrow \mathcal{L}^2(K_D^{\otimes m}, h^{\otimes m})/\mathcal{M}_x\cdot\mathcal{M}_y$$

is surjective for every $x, y\in D$, where $\mathcal{M}_x$ (resp. $\mathcal{M}_y$) denotes the maximal ideal sheaf at x (resp. y).

Proof. The following proof is routine. First we shall consider the case $x\neq y$. Let r_x (resp. r_y) denote the distance function from x (resp. y) with respect to the Kähler form ω. And let U_x (resp. U_y) be a small open neighbourhoods of x (resp. y) let W_x (resp. W_y) be an open neighbourhood of x (resp. y) such that $W_x \Subset U_x$ (resp. $W_y \Subset U_y$). Let ρ be a nonnegative C^∞ function such that $\rho\equiv 1$ on $W_x \cup W_y$ and $\operatorname{Supp}\rho\subset U_x\cup U_y$. We set $\phi=(2n+2)\rho(\log r_x+\log r_y)$. Noting that ϕ is plurisubharmonic on a neighbourhood of x and y, by direct calculation we see that there exists a positive constant c such that

$$\sqrt{-1}\,\partial\bar\partial\phi>-c\omega$$

holds on D, there exists a positive integer m such that

$$m\operatorname{curv} h+\sqrt{-1}\,\partial\bar\partial\log\phi+\operatorname{Ric}_\omega\geqq\omega$$

holds on D. Then by the above vanishing theorem

$$H^q_{(2)}(D, \mathcal{L}^2(L^{\otimes m}, e^{-\phi}h^{\otimes m}))=0$$

holds for every $q \geqq 1$. Hence by the definition of ϕ,

$$H^0_2(D, \mathcal{L}^2(L^{\otimes m}, h^{\otimes m})) \longrightarrow \mathcal{L}^2(L^{\otimes m}, h^{\otimes m})/\mathcal{M}_x \cdot \mathcal{M}_y$$

is surjective. In the case of $x=y$, the proof is similar. Q.E.D.

By Corollary 1, $H^0_{(2)}(D, \mathcal{L}^2(K_D^{\otimes m}, h^{\otimes m}))$ separates generates general points of D.

Let Γ be a discrete cocompact subgroup of $\mathrm{Aut}(D)$ acting D without fixed point. Let σ be a nontrivial L^2 holomorphic section of $K_D^{\otimes m}$. For $k \geqq 2$, we set

$$av(\sigma^{\otimes k}) = \sum_{\gamma \in \Gamma} \gamma^* \sigma^{\otimes k}.$$

At this moment, $av(\sigma^{\otimes k})$ is not well defined because Γ may not be an isometry.

To show that $av(\sigma^{\otimes k})$ is well defined, we shall use the measure hyperbolicity of D. We shall review the definition of measure hyperbolic manifolds. Let M be an n-dimensional connected complex manifold. Let Δ^n denote the unit open polydisk in $\boldsymbol{C}^n$. Let us take a point $x \in M$. Let $f: \Delta^n \to M$ be a holomorphic mapping such that $f(O)=x$ and f is nondegenerate at 0. Let Ω_0 be the Poincaré volume form on Δ^n defined by

$$\Omega_0 = \prod_{i=1}^{n} \frac{4}{|z_i|^2(\log|z_i|)^2}(\sqrt{-1})^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.$$

By inverse function theorem there exists $0<r<1$ and a neighbourhood U of $x=f(0)$ such that $f|\Delta^n(r): \Delta(r)^n \to U$ is a biholomorphic mapping, where $\Delta^n(r)$ denotes the polydisk of radius r with center O. We set

$$\Psi_{M,f}(x) = (f^{-1}|\Delta(r)^n)^* \Omega_0(O)$$

and

$$\Psi_M(x) = \inf\{\Psi_{M,f}(x)\},$$

where the infimum is taken for all holomorphic mapping $f: \Delta^n \to M$ such that $f(O)=x$ and nondegenerate at O. Then Ψ_M is a pseudo-volume form on M. We call Ψ_M the hyperbolic volume form of M. It is easy to check that Ψ_M is an upper semicontinuous $2n$-form on M. Ψ_M defines a measure μ_M on M. We call the measure μ_M the hyperbolic measure of M. M is said to be measure hyperbolic, if $\mu_M(B)>0$ for any non-empty open subset $B \subset M$. The following propositions are well known (cf. [3]).

PROPOSITION 1. *Let M be a projective manifold of general type. Then M is measure hyperbolic.*

PROPOSITION 2. *Let M be a complex manifold and let $\pi: \tilde{M} \to M$ be an unramified covering. Then $\Psi_{\tilde{M}} = \pi^* \Psi_M$ holds. In particular M is measure hyperbolic if and only if $\tilde{M}$ is measure.*

Using Proposition 1, 2, we see that D is measure hyperbolic and Γ is measure preserving with respect to the hyperbolic measure μ_D.

LEMMA 2. *There exists a constant* $C>1$ *such that*

$$\frac{1}{C}\int_D \|f\|^2 \mu_D \leqq \int_D \|f\|^2 \omega^n \leqq C\int_D \|f\|^2 d\mu_D$$

holds for every $f \in H^0_{(2)}(D, \mathcal{O}_D(K_D^{\otimes m}))$.

Proof. Let F be a fundamental domain of $\pi_1(X)$. Since μ_D and ω are $\pi_1(X)$-invariant, it is sufficient to prove that there exist positive constants C_1, C_2 such that for every $f \in H^0(\bar{F}, \mathcal{O}_D(K_D^{\otimes m}))$ (where $\bar{F}$ denote the closure of F),

$$\int_F \|f\|^2 \mu_D \leqq C_1 \int_F \|f\|^2 \omega^n$$

and

$$\int_F \|f\|^2 \omega^n \leqq C_2 \int_F \|f\|^2 d\mu_D$$

hold. Since Ψ_D is $\pi_1(X)$-invariant, Ψ_D/ω^n is bounded from above by the definition of Ψ_D, this implies the existence of C_1.

Suppose that C_2 does not exist. Then there exists a sequence $\{f_j\}_{j=1}^{\infty}$, $f_j \in H^0(\bar{F}, \mathcal{O}_D(K_D^{\otimes m}))$ such that

$$\int_F \|f_j\|^2 \omega^n = 1$$

and

$$\int_F \|f_j\|^2 d\mu_D \leqq \frac{1}{2^j}.$$

By the plurisubharmonicity of the square of the absolute value of a holomorphic function, we see that for every relatively compact subset W of F, there exists a constant C_W such that

$$\|f\|^2(x) \leqq C_W \int_F \|f\|^2 \omega^n$$

holds for every $x \in W$ and $f \in H^0(F, \mathcal{O}_D(K_D^{\otimes m}))$. Since Ψ_D vanishes only on some measure 0 subset of D, this is a contradiction. Q.E.D.

LEMMA 3. *There exists a positive constant c such that*

$$h \cdot \Psi_D > c$$

holds.

Proof. We note that h and $d\mu_D$ are both $\pi_1(X)$ invariant. Hence we can identify h^{-1} and $d\mu_D$ volume forms on X. Let $f: \Delta^n \to X$ be a holomorphic mapping. Then since curv h a Kähler form on X, by the maximal principle (Schwarz lemma) there exists a positive constant c independent of f such that

$$f^*h^{-1}\leqq c\Omega_0$$

holds. Then by the definition of $d\mu_D$, we completes the proof of Lemma 2.
Q.E.D.

We note that Ψ_D^{-1} is a Γ-invariant singular hermitian metric of the canonical bundle K_D. By Lemma 2 and Lemma 3, we see that $av(\sigma^{\otimes k})$ is well defined for every $k\geqq 2$. We set

$$\varphi=\frac{av(\sigma^{\otimes 2k})}{av(\sigma^{\otimes 2})^{\otimes k}}\,.$$

φ is well defined by the following lemma.

LEMMA 4. *We may assume that $av(\sigma^{\otimes 2})$ is not identically zero.*

Proof. Let x be a point on D. Let η be a generator of $K_D^{\otimes m}$ around x. We set

$$a_\gamma=\frac{\gamma^*\sigma}{\eta}\,.$$

Suppose that $av(\sigma^{\otimes 2k})\equiv 0$ for every $k\geqq 1$. Then

$$\sum_{\gamma\in\Gamma}a_\gamma^{2k}\equiv 0$$

holds around x for *every* k. But this implies that $a_\gamma\equiv 0$ around x for every γ. This is the contradiction. Hence replacing σ by $\sigma^{\otimes l}$ for some l, if necessary, we may assume that $av(\sigma^{\otimes 2})$ is not identically 0. Q.E.D.

LEMMA 5. *φ is a nonconstant Γ-invariant meromorphic function for some k.*

Proof. Suppose that φ is constant for every k. Let x be a point on D such that $av(\sigma^{\otimes 2})(x)\neq 0$. Let y be a point on D. We set

$$f_\gamma=\frac{\gamma^*\sigma^{\otimes 2}}{av(\sigma^{\otimes 2})}\,.$$

Since φ is constant for every k,

$$\sum_{\gamma\in\Gamma}f_\gamma(x)^k=\sum_{\gamma\in\Gamma}f_\gamma(y)^k$$

holds for every k. This implies that

$$\{f_\gamma(x)\}=\{f_\gamma(y)\}$$

holds. Hence by moving y on a neighbourhood of x, we see that f_γ is constant. Since $av(\sigma^{\otimes 2})$ is Γ invariant and D is noncompact, this contradicts the fact that σ is a L^2-holomorphic section. Q.E.D.

Let K be the function field generated by such φ. Then there exists a projective variety B whose function field corresponds K. Let $r: \Gamma\backslash D- \cdots \to B$ be a rational map induced by the inclusion $K(B) \hookrightarrow K(\Gamma\backslash D)$. Let $\mu: (\Gamma\backslash\tilde{\ } D) \to \Gamma\backslash D$ be a resolution of the base locus of $r: \Gamma\backslash D- \cdots \to B$. Let $\tilde{r}: \Gamma\backslash\tilde{\ } D \to B$ be the natural morphism. Let $\tilde{Y}$ be a general fibre of $\tilde{r}$ and let Y denote $\mu(Y)$. Suppose that $\dim Y \geqq 1$. Let $\pi_\Gamma: D \to \Gamma\backslash D$ be the natural projection. Then $\pi_\Gamma^{-1}(Y)$ is a Γ invariant subvariety of D and every element of K is constant on $\pi_\Gamma^{-1}(Y)$. We note that $H^0_{(2)}(D, \mathcal{L}^2(K_D^{\otimes m}, h^{\otimes m}))$ separates general point of $\pi_\Gamma^{-1}(Y)$, if we take Y sufficiently general. Using this fact, repeating the same argument as above, we can construct an element of K which is nonconstant on $\pi_\Gamma^{-1}(Y)$. This is the constradiction.

In conclusion, K separates the general points of $\Gamma\backslash D$. Hence $\Gamma\backslash D$ is of general type. This completes the proof of Theorem 1.

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