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KODAI MATH. J.
19 (1996), 62–86

PSEUDOHERMITIAN IMMERSIONS, PSEUDO-EINSTEIN STRUCTURES, AND THE LEE CLASS OF A CR MANIFOLD

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Any nondegenerate CR manifold carrying a fixed contact 1-form is known to possess (cf. N. Tanaka [T], S. Webster [W1]) a canonical linear connection (the *Tanaka-Webster connection*) parallelizing the Levi form and the maximal complex structure. This leads to an (already widely exploited, cf. D. Jerison & J.M. Lee [JL1], [JL2], J.M. Lee [L1], [L2], H. Urakawa [U1], [U2], etc.) analogy between CR geometry on one hand, and both Hermitian and conformal geometry on the other.

To describe our point of view, let M and A be two CR manifolds of CR dimensions n and $N=n+k$, $k\geqq 1$, respectively. A *CR immersion* $f: M\rightarrow A$ is an immersion and a CR map. If f is the inclusion then M is a *CR submanifold* of A (a *CR hypersurface* when $k=1$). For instance, let M^{2n+1} be the intersection between the sphere S^{2n+3} and a transverse complex hypersurface in $\boldsymbol{C}^{n+2}$. Then M^{2n+1} is a CR hypersurface of S^{2n+3} (in particular M^{2n+1} is strictly pseudoconvex). Let M be a CR submanifold of A. Then M is *rigid* in A if any CR diffeomorphism F: $M\rightarrow M'$ onto another CR submanifold M' of A (e.g. F may be the restriction of a biholomorphic mapping) extends to a CR automorphism of A (e.g. if $A=S^{2n+3}$ then F should extend to a fractional linear, or projective, transformation preserving S^{2n+3}). A theory of CR immersions has been initiated by S. Webster [W2]. There it is shown that S^{2n+1} is rigid in S^{2n+3} if $n\geqq 2$. Also, if $n\geqq 3$ then any CR hypersurface of S^{2n+3} is rigid. The basic idea in [W2] is to endow the ambient space S^{2n+3} with the Tanaka-Webster connection (rather than the Levi-Civita connection associated with the canonical Riemannian structure) and obtain CR analogues of the Gauss-Weingarten (respectively Gauss-Ricci-Codazzi) equations (from the theory of isometric immersions between Riemannian manifolds). In the end, these could be used to show that the intrinsic geometry determines the (CR analogue of the) second fundamental form of the given CR immersion. The main inconvenience of this approach seems to be the nonuniqueness of choice of a canonical connection on the CR submanifold (i.e. the induced and the 'intrinsic' Tanaka-Webster connections of the submanifold do not coincide, in general). In [D1] we compensate

Received September 28, 1994, revised April 7, 1995.

for this inadequacy by restricting ourselves to a smaller class of CR immersions, as follows. Let $f: M \to A$ be a CR immersion between two strictly pseudoconvex CR manifolds on which contact 1-forms θ and Θ have been fixed. Then $f^*\Theta=\lambda\theta$ for some C^∞ function $\lambda: M \to \boldsymbol{R}$. If $\lambda \equiv 1$ then f is called *isopseudohermitian* (following the terminology in [J]). An isopseudohermitian immersion $f: M \to A$ is a *pseudohermitian immersion* if $f(M)$ is tangent to the characteristic direction of (A, Θ). If this is the case then (by a result in [D1]) f is an isometry (with respect to the Webster metrics of (M, θ) and (A, Θ)). Also one may use the axiomatic description (of the Tanaka-Webster connection) in [T] to show that the induced and intrinsic connections on M coincide. Moreover, by a result of H. Urakawa (any CR map $f: M \to A$ satisfying $f_* T=\lambda T_A$ for some $\lambda \in C^\infty(M)$ with $T(\lambda)=0$ is harmonic with respect to the Webster metrics of M and A, cf. Corollary 3.2 in [U3], p. 236) any pseudohermitian immersion is actually minimal. Cf. also Theorem 7 in [D1]. The present note is an application of this theory in connection with the problem of the existence of *pseudo-Einstein* pseudohermitian structures (i.e. for which the pseudohermitian Ricci tensor of the Tanaka-Webster connection is proportional to the Levi form, cf. J. M. Lee, [L1]) on (locally realizable) CR manifolds. As in [D1], our main tool consists of pseudohermitian analogues of the Gauss and Weingarten equations. In particular, we introduce the concept of *normal Tanaka-Webster connection* $\nabla^\perp$ (of a given pseudohermitian immersion between two strictly pseudoconvex CR manifolds). When $\nabla^\perp$ is flat we use the (pseudohermitian analogues of the) Gauss-Ricci-Codazzi equations to relate the pseudohermitian Ricci tensors of the Tanaka-Webster connections of the submanifold and ambient space (cf. Theorem 2). As a corollary, we may regard the *Lee class* $\gamma(M)$ (a cohomology class in the first cohomology group of the given (locally realizable) CR manifold M with coefficients in the sheaf of CR-pluriharmonic functions [L1]) as an obstruction toward the existence of pseudohermitian immersions $f: M \to S^{2N+1}$ with a flat normal Tanaka-Webster connection of a strictly pseudoconvex CR manifold M in an odd dimensional sphere. Our methods are similar to those in B. Y. Chen & H. S. Lue [CL] (where holomorphic immersions between Kaehler manifolds are dealt with). We exploit the symmetries of the curvature tensor field of the Tanaka-Webster connection (rather than the Riemannian-Christoffel tensor field in [CL]) and deal with the highly complicated character (due to the presence of torsion terms there) of the Bianchi identities (cf. e.g. (40)). The key points (leading from (52) to (26) in Theorem 2) are Lemma 2 (the (0, 2)-tensor field E_a is proportional to the Levi form of the submanifold) and a nontrivial cancellation of torsion terms.

As a byproduct of the considerations in section 6 we show (cf. Theorem 4) the nonexistence of pseudohermitian immersions of $H_n(s)$ (a quotient of the Heisenberg group by a discrete group of dilations, carrying the contact form discovered in [D2] in analogy with the Boothby metric of a complex Hopf manifold, cf. [D3]) into a Tanaka-Webster flat strictly pseudoconvex CR manifold (e.g. H_N or $U_{\alpha,\beta}$). The extension to which one may exploit the analogy with the case of holomorphic immersions in [CL] (cf. also our Theorem 3) is demonstrated at the close of the same section.

Several examples of CR immersions are examinated in section 9. In particular, CR immersions

$$\partial D_{(\alpha_1,\cdots,\alpha_n),\beta}\to\partial D_{(\alpha_1,\cdots,\alpha_N),\beta},$$

(between boundaries of pseudo-Siegel domains in $\boldsymbol{C}^{n+1}$ and $\boldsymbol{C}^{N+1}$ respectively) arise when looking at the weak pseudoconvexity locus of $D_{\alpha,\beta}=\{(z, w)\in\boldsymbol{C}^{N+1}: \sum_{j=1}^{N}|z_j|^{2\alpha_j}+\mathrm{Im}(w^{\beta})-1<0\}$, cf. [BP]. The authors are grateful to the referee for drawing their attention upon the works by H. Urakawa and for suggestions which improved the first version of the present paper.

1. Definitions and basic formulae

Let M be a real $(2n+1)$-dimensional C^{∞} manifold. A *CR structure* (of CR dimension n) on M is a complex subbundle $T_{1,0}(M)$, of complex rank n, of the complexified tangent bundle $\boldsymbol{C}TM=T(M)\otimes\boldsymbol{C}$ so that

$$(1)\qquad T_{1,0}(M)\cap T_{0,1}(M)=(0),$$

and

$$(2)\qquad [\Gamma^{\infty}(T_{1,0}(M)), \Gamma^{\infty}(T_{1,0}(M))]\subseteq\Gamma^{\infty}(T_{1,0}(M)).$$

Here $T_{0,1}(M)=\overline{T_{1,0}(M)}$ (throughout an overbar denotes complex conjugation). Also, if $E\to M$ is a vector bundle over M then $\Gamma^{\infty}(E)$ denotes the module of C^{∞} cross-section in E (defined on some open set $U\subseteq M$, to be understood from the context) and E_x is the fibre in E over $x\in M$. A pair $(M, T_{1,0}(M))$ is a *CR manifold* (of CR dimension n). Its *Levi distribution*

$$H(M)=\mathrm{Re}\{T_{1,0}(M)\oplus T_{0,1}(M)\}$$

carries the complex structure $J: H(M)\to H(M)$ given by

$$(3)\qquad J(Z+\bar{Z})=i(Z-\bar{Z}),$$

for any $Z\in T_{1,0}(M)$. Here $i=\sqrt{-1}$. Let $K\subset T^*(M)$ be the annihilator of $H(M)$, i.e. $K_x=\{\omega\in T_x^*(M): \mathrm{Ker}(\omega)\supseteq H(M)_x\}$ for any $x\in M$. Then $K\to M$ is a real line subbundle of $T^*(M)$. Assume from now on that M is orientable. Then K admits globally defined nowhere zero sections $\theta\in\Gamma^{\infty}(K)$ each of which is referred to as a *pseudohermitian structure* on M. The *Levi form* G_θ of $(M, T_{1,0}(M), \theta)$ is given by

$$(4)\qquad G_\theta(X, Y)=d\theta(X, JY),$$

for any $X, Y\in\Gamma^{\infty}(H(M))$ and $(M, T_{1,0}(M))$ is *nondegenerate* if G_θ is nondegenerate for some $\theta\in\Gamma^{\infty}(K)$ (and thus for all). If $(M, T_{1,0}(M))$ is nondegenerate and a pseudohermitian structure θ has been fixed, then there is a unique globally defined nowhere zero tangent vector field T (the *characteristic direction* of $d\theta$) on M transverse to $H(M)$ and satisfying

$$T \rfloor \theta = 1, \quad T \rfloor d\theta = 0. \tag{5}$$

Here $X \rfloor$ denotes the interior product with the tangent vector field X. Clearly

$$T(M) = H(M) \oplus \boldsymbol{R}T. \tag{6}$$

Therefore one may extend J to a bundle morphism $J : T(M) \to T(M)$ by requesting that $JT = 0$. Also, let g_θ be the *Webster metric*, i.e. the semi-Riemannian metric given by $g_\theta(X, Y) = G_\theta(X, Y)$, $g_\theta(X, T) = 0$ and $g_\theta(T, T) = 1$ for any $X, Y \in H(M)$. The CR manifold $(M, T_{1,0}(M))$ is *strictly pseudoconvex* if G_θ is positive definite for some $\theta \in \Gamma^\infty(K)$. If this is the case then g_θ is a Riemannian metric and, as it has been pointed out elsewhere (cf. e.g. [D1]) the synthetic object (J, T, θ, g_θ) is a contact metric structure on M (in the sense of D.E. Blair [B], p. 25). In general (J, T, θ, g_θ) is not normal, and the obstruction to normality is the pseudohermitian torsion, a fragment of the torsion field of the *Tanaka-Webster connection* which we now recall. Cf. [T], [W1], any nondegenerate CR manifold M on which a pseudohermitian structure has been specified carries a canonical linear connection ∇ satisfying the following axioms:

i) $H(M)$ is parallel with respect to ∇,
ii) $\nabla J = 0$,
iii) $\nabla g_\theta = 0$,
iv) $\pi_+ \mathrm{Tor}(Z, W) = 0$ for any $Z \in T_{1,0}(M)$, $W \in \boldsymbol{C}TM$,

where $\pi_+ : \boldsymbol{C}TM \to T_{1,0}(M)$ is the natural projection associated with the direct sum decomposition:

$$\boldsymbol{C}TM = T_{1,0}(M) \oplus T_{0,1}(M) \oplus \boldsymbol{C}T, \tag{7}$$

and Tor is the torsion tensor field of ∇. The *pseudohermitian torsion* τ of the Tanaka-Webster connection is the vector bundle valued 1-form on M given by

$$\tau X = \mathrm{Tor}(T, X), \tag{8}$$

for any $X \in H(M)$. Cf. [D1], trace$(\tau) = 0$ and τ is self-adjoint with respect to the Webster metric g_θ. Also (J, T, θ, g_θ) in normal (in the sense of [B], p. 48) iff $\tau = 0$.

Let $(M, T_{1,0}(M), \theta)$ be nondegenerate and let ∇^θ be the Levi-Civita connection of (M, g_θ). Then

$$\nabla^\theta = \nabla + \left(\frac{1}{2}\Omega_\theta - A\right) \otimes T + \tau \otimes \theta + \theta \odot J. \tag{9}$$

Here $\Omega_\theta(X, Y) = g_\theta(X, JY)$ and $A(X, Y) = g_\theta(\tau X, Y)$ for any $X, Y \in H(M)$. Also $\odot$ denotes the symmetric product (e.g. $(\theta \odot J)(X, Y) = 1/2\{\theta(X)JY + \theta(Y)JX\}$).

Furthermore, we shall need the identities

$$\mathrm{Tor}=2\theta\wedge\tau-\Omega_{\theta}\otimes T, \tag{10}$$

$$\nabla T=0, \tag{11}$$

$$\tau J+J\tau=0. \tag{12}$$

Cf. [T]. Let R be the curvature tensor field of the Tanaka-Webster connection ∇ of (M, θ). Let $\mathrm{Ric}(X, Y)=\mathrm{trace}\{Z\mapsto R(Z, X)Y\}$ for any tangent vector fields X, Y on M. If $\{T_1, \cdots, T_n\}$ is a (local) frame of $T_{1,0}(M)$, the *pseudohermitian Ricci tensor* $R_{\alpha,\bar{\beta}}$ of (M, θ) is given by

$$R_{\alpha\bar{\beta}}=\mathrm{Ric}(T_{\alpha}, T_{\bar{\beta}}),$$

where $T_{\bar{\alpha}}=\bar{T}_{\alpha}$. Set also

$$h_{\alpha\bar{\beta}}=G_{\theta}(T_{\alpha}, T_{\bar{\beta}}).$$

Then θ is *(globally) pseudo-Einstein* if

$$R_{\alpha\bar{\beta}}=\lambda h_{\alpha\bar{\beta}}, \tag{13}$$

for some C^{∞} function λ, i.e. the pseudohermitian Ricci tensor of (M, θ) is proportional to the Levi form (cf. [L1]). If this is the case then $\lambda=(1/n)R$ where $R=h^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}}$ is the *pseudohermitian scalar curvature* of (M, θ). The pseudo-Einstein condition (13) is not so rigid as its Riemannian counterpart. Indeed, the IInd Bianchi identity (associated with the Tanaka-Webster connection) no longer implies $R=\mathrm{const.}$ (due to the presence of torsion terms). It should also be pointed out that (unlike the case of Kaehler geometry) $R_{\alpha\bar{\beta}}$ is only a fragment of Ric (Ric is determined by $R_{\alpha\bar{\beta}}$ and certain covariant derivatives of τ, cf. [D1]). Any odd dimensional sphere $S^{2n+1}\subset \boldsymbol{C}^{n+1}$ endowed with the standard CR structure $T_{1,0}(S^{2n+1})=T^{1,0}(\boldsymbol{C}^{n+1})\cap \boldsymbol{C}TS^{2n+1}$ admits the pseudo-Einstein pseudohermitian structure $\iota^*\theta$ where $\theta=i/2(\bar{\partial}-\partial)|z|^2$ and $\iota: S^{2n+1}\subset \boldsymbol{C}^{n+1}$. Throughout, if X is a complex manifold, then $T^{1,0}(X)$ denotes its holomorphic tangent bundle. Also, if $H_n=\boldsymbol{C}^n\times\boldsymbol{R}$ is the Heisenberg group (cf. e.g. [FS], p. 434-435) and $\delta_s: H_n-\{0\}\rightarrow H_n-\{0\}$ the dilation by $0<s<1$ then $G_s=\{\delta_s^m: m\in\boldsymbol{Z}\}$ acts freely on $H_n-\{0\}$ as a properly discontinuous group of CR automorphisms of $H_n-\{0\}$ so that (cf. [D2], p. 36) the quotient space $H_n(s)=(H_n-\{0\})/G_s$ is a compact CR manifold (of CR dimension n) diffeomorphic to $\Sigma^{2n}\times S^1$, where $\Sigma^{2n}=\{x\in H_n: |x|=1\}$ and $|x|=(|z|^4+t^2)^{1/4}$ is the *Heisenberg norm* of $x=(z, t)$. By a result in [D2] the pseudohermitian structure $\theta=|x|^{-2}\{dt+2\sum_{\alpha=1}^{n}(x^{\alpha}dy^{\alpha}-y^{\alpha}dx^{\alpha})\}$ on $H_n(s)$ is pseudo-Einstein (here $x=(z, t)$, $z=(z^1, \cdots, z^n)$, $z^{\alpha}=x^{\alpha}+iy^{\alpha}$).

2. Pseudohermitian immersions

Let $(M, T_{1,0}(M))$ and $(A, T_{1,0}(A))$ be two CR manifolds of CR dimensions n and $N=n+k$, respectively. A C^{∞} map $f: M\rightarrow A$ is a *CR map* if

$$f_*T_{1,0}(M)\subset T_{1,0}(A).$$

Let us assume from now on that $(M, T_{1,0}(M))$ and $(A, T_{1,0}(A))$ are strictly pseudoconvex and specify pseudohermitian structures θ and Θ, on M and A respectively, so that G_θ and G_Θ are positive definite. Let $f: M\to A$ be a CR map. Then

$$f^*\Theta=\mu\theta,$$

for some C^∞ function $\mu>0$. Assume from now on that f is a CR immersion (i.e. an immersion and a CR map). A theory of CR immersions (between strictly pseudoconvex CR manifolds) has been built in [D1]. We recall that in general f is not an isometry with respect to the Webster metrics g_θ and g_Θ. Yet, if f is *isopseudohermitian* (i.e. $\mu=1$), then $f^*g_\Theta=g_\theta$ iff $\mathrm{nor}(T_A)=0$. Here T_A is the characteristic direction of $d\Theta$ and $\mathrm{nor}_x: T_{f(x)}(A)\to\nu^{2k}(f)_x$ is the natural projection with respect to the direct sum decomposition

$$T_{f(x)}(A)=[(d_xf)T_x(M)]\oplus\nu^{2k}(f)_x,$$

for any $x\in M$. Here $\nu^{2k}(f)\to M$ denotes the normal bundle of the given immersion. Such $f: M\to A$ (i.e. a CR immersion with $f^*\Theta=\theta$ and $\mathrm{nor}(T_A)=0$) is termed *pseudohermitian immersion*. If this is the case then $f_*T=T_A$. Also (cf. [D1]) there are natural CR analogues of the Gauss and Weingarten formulae

$$\nabla^A_{f_*X}f_*Y=f_*\nabla_XY+\alpha(f)(X, Y), \tag{14}$$

$$\nabla^A_{f_*X}\xi=-f_*a_\xi X+\nabla^\perp_X\xi, \tag{15}$$

for any $X, Y\in\Gamma^\infty(TM)$, $\xi\in\Gamma^\infty(\nu^{2k}(f))$. Here ∇, ∇^A are the Tanaka-Webster connections of (M, θ), (A, Θ), respectively. Also $\alpha(f)$ and a are bilinear and $\nabla^\perp$ is a connection in $\nu^{2k}(f)$, referred to as the *normal Tanaka-Webster connection* of f. Unlike the second fundamental form of f, its CR analogue $\alpha(f)$ is not symmetric, i.e.

$$\alpha(f)(X, Y)-\alpha(f)(Y, X)=\mathrm{nor}\{\mathrm{Tor}_A(f_*X, f_*Y)\}, \tag{16}$$

for any $X, Y\in T(M)$. Here Tor_A is the torsion tensor field of ∇^A. Since $\nabla^AT_A=0$ it follows that

$$\alpha(f)(X, T)=0,$$

for any $X\in T(M)$. We consider the normal bundle valued 1-form $Q(f)$ on M given by

$$Q(f)X=\alpha(f)(T, X),$$

for any $X\in T(M)$. If τ_A is the pseudohermitian torsion of the Tanaka-Webster connection ∇^A then

$$\tau_Af_*X=f_*\tau X+Q(f)X. \tag{17}$$

Taking into account (10) and (17), the identity (16) may be also written

$$\alpha(f)(X, Y)-\alpha(f)(Y, X)=2(\theta\wedge Qf)(X, Y),$$

for any $X, Y\in T(M)$. The equations (14)-(15) lead to CR analogues of the Gauss-Codazzi-Ricci equations

$$\tan\{R^A(f_*X, f_*Y)f_*Z\}=R(X, Y)Z+a_{\alpha(f)(X,Z)}Y-a_{\alpha(f)(Y,Z)}X, \tag{18}$$

$$\begin{aligned}&\operatorname{nor}\{R^A(f_*X, f_*Y)f_*Z\}\\ &=(\nabla_X\alpha(f))(Y, Z)-(\nabla_Y\alpha(f))(X, Z)+\alpha(f)(\mathrm{Tor}(X, Y), Z),\end{aligned} \tag{19}$$

$$\begin{aligned}&g_\Theta(R^A(f_*X, f_*Y)\xi, \eta)\\ &=g_\theta(R^\perp(X, Y)\xi, \eta)+g_\theta(a_\eta Y, a_\xi X)-g_\theta(a_\eta X, a_\xi Y),\end{aligned} \tag{20}$$

for any $X, Y, Z\in T(M)$ and any $\xi, \eta\in\nu^{2k}(f)$. Here $\tan_x: T_{f(x)}(A)\to T_xM$ is the natural projection, $x\in M$, and R^A, $R^\perp$ are the curvature tensor fields of ∇^A, $\nabla^\perp$, respectively. Note that

$$g_\theta(a_\xi X, Y)=g_\Theta(\alpha(f)(X, Y), \xi). \tag{21}$$

Therefore (on account of (16)), unlike the Weingarten operator of f, its CR analogue a_ξ is not self-adjoint (unless $Q(f)=0$). Also (by (21)) a_ξ is $H(M)$-valued. As f is a CR map

$$f_*H(M)\subset H(A),$$

$$f_*\circ J=J_A\circ f_*,$$

where $J_A: H(A)\to H(A)$ denotes the complex structure of $H(A)$. Next $\nabla^A J_A=0$ and (14)-(15) yield

$$\alpha(f)(X, JY)=J_A\alpha(f)(X, Y), \tag{22}$$

$$a_{J_A\xi}X=Ja_\xi X, \tag{23}$$

$$\nabla^\perp J_A=0, \tag{24}$$

for any $X, Y\in T(M)$, $\xi\in\nu^{2k}(f)$. Cf. [D1], $f^*g_\Theta=g_\theta$ yields $\nu^{2k}(f)\subset H(A)$ so that $J_A\xi$ makes sense *a priori* (i.e. before the extension of J_A to a (1, 1)-tensor field on A by requesting that $J_AT_A=0$). Conversely, if $\nu_H^{2k}(f)_x$ is the orthogonal complement (with respect to the inner product $g_{\Theta,f(x)}$) of $(d_xf)H(M)_x$ in $H(A)_{f(x)}$ and $\nu_H^{2k}(f)_x=\nu^{2k}(f)_x$ for any $x\in M$, then f is an isometry with respect to the Webster metrics of (M, θ) and (A, Θ).

3. CR-pluriharmonic functions and the Lee class

Let $M\subset \boldsymbol{C}^{n+1}$ be a real hypersurface. Then $T_{1,0}(M)=T^{1,0}(\boldsymbol{C}^{n+1})\cap \boldsymbol{C}TM$ is a CR structure of CR dimension n on M. Such $(M, T_{1,0}(M))$ is referred to as an *embedded CR manifold*. A CR manifold $(M, T_{1,0}(M))$ is *locally realizable* if each point of M admits a neighborhood which is CR isomorphic to some embedded CR manifold. If $(M, T_{1,0}(M))$ is a locally realizable CR manifold then pseudo-Einstein pseudohermitian structures exist (locally) in some neighborhood of every point of M (cf. Corollary B of [L1]) but there may be obstructions to the existence of global pseudo-Einstein structures. Let $\mathcal{P}$ be the sheaf of *CR-pluriharmonic* functions on M, i.e. if $U \subseteq M$ is open then $u \in \mathcal{P}(U)$ iff $u = \mathrm{Re}(F)$ for some CR-holomorphic function $F: U \to \boldsymbol{C}$. Then there is a CR-invariant cohomology class $\gamma(M) \in H^1(M, \mathcal{P})$ (referred hereafter as the *Lee class* of M) which vanishes iff M admits a global pseudo-Einstein structure (cf. [L1], p. 172). A complex valued q-form η on M is a $(0, q)$-form if $T \rfloor \eta = 0$ and $T_{1,0}(M) \rfloor \eta = 0$. For instance, if $\{\theta^1, \cdots, \theta^n\}$ is an admissible coframe, i.e. θ^α are the (local) 1-forms determined by $T_\beta \rfloor \theta^\alpha = \delta^\alpha_\beta$, $T \rfloor \theta^\alpha = 0$ and $T_{\bar\beta} \rfloor \theta^\alpha = 0$, then any (0, 1)-form η may be written locally as $\eta = \eta_{\bar\alpha} \theta^{\bar\alpha}$, where $\theta^{\bar\alpha} = \overline{\theta^\alpha}$. Let $\Lambda^{0,q}(M)$ be the bundle of $(0, q)$-forms on M. The *tangential Cauchy-Riemann operator* is the differential operator $\bar\partial_M : \Gamma^\infty(\Lambda^{0,q}(M)) \to \Gamma^\infty(\Lambda^{0,q+1}(M))$ defined as follows. Let η be a $(0, q)$-form on M. Then $\bar\partial_M \eta$ is the unique $(0, q+1)$-form which coincides with $d\eta$ when restricted to $T_{0,1}(M) \otimes \cdots \otimes T_{0,1}(M)$ ($q+1$ factors). A $(0, q)$-form η is *CR-holomorphic* if it satisfies the *tangential Cauchy-Riemann equations*

$$\bar\partial_M \eta = 0 .$$

Let $f: M \to A$ be a pseudohermitian immersion. Then

$$\bar\partial_M f^* \eta = f^* \bar\partial_A \eta , \tag{25}$$

for any $(0, q)$-form η on A. Let $\mathcal{P}_A$ be the sheaf of CR-pluriharmonic functions on A. Assume for the rest of this section that f is a homeomorphism on its image. As a consequence of (25), if $D \subseteq A$ is open and $v \in \mathcal{P}_A(D)$ then $v \circ f \in \mathcal{P}(V)$, where $V = f^{-1}(D \cap f(M))$. We need to recall the construction of the CR-invariant cohomology class $\gamma(A) \in H^1(A, \mathcal{P}_A)$ built in [L1], p. 172. Assume from now on that A is locally realizable (e.g. if either A is compact or $N > 2$, then by results in L. Boutet De Monvel [BM] (for the compact case) and M. Kuranishi [K], T. Akahori [A] (for the noncompact case) it follows that $(A, T_{1,0}(A))$ is locally realizable). Then, by a result in [L1], p. 158, there is an open covering $\mathcal{D} = \{D_j\}_{j \in \Sigma}$ of A and a pseudo-Einstein pseudohermitian structure Θ_j on each D_j, $j \in \Sigma$. If $I_{ij} : D_i \cap D_j \to D_j$ are inclusions, then $I^*_{ij} \Theta_j = \exp(2U_{ji}) I^*_{ji} \Theta_i$ for some C^∞ functions $U_{ji} : D_i \cap D_j \to \boldsymbol{R}$. By Proposition 5.1 of [L1], p. 172, $U_{ji} \in \mathcal{P}_A(D_i \cap D_j)$. Let $N(\mathcal{D})$ be the nerve of $\mathcal{D}$ (we use the notations and conventions in S. Goldberg [G], p. 272-275). Let $C \in C^1(N(\mathcal{D}), \mathcal{P}_A)$ be the 1-cochain mapping each 1-simplex $\sigma = (D_i D_j)$ of $N(\mathcal{D})$ in $U_{ji} \in \mathcal{P}_A(\cap \sigma)$. Then $C \in Z^1(N(\mathcal{D}), \mathcal{P}_A)$, i.e. C so built is a 1-cocycle with coefficients in $\mathcal{P}_A$. Finally $\gamma(A) \in H^1(A, \mathcal{P}_A)$ is the equivalence class of $[C] \in H^1(N(\mathcal{D}), \mathcal{P}_A)$. Note that each pseudohermitian immersion $f: M \to A$ (so that $f: M \to f(M)$ is a homeomorphism) induces a map on cohomology $f^*: H^p(A, \mathcal{P}_A) \to H^p(M, \mathcal{P})$. Let $\mathrm{Cov}(A)$ be the set of all open coverings of A. Let $\Gamma \in H^p(A, \mathcal{P}_A)$. Since

$$H^p(A, \mathcal{P}_A)=\varinjlim H^p(N(\mathcal{D}), \mathcal{P}_A),$$

there is $\mathcal{D}\in\mathrm{Cov}(A)$ and $h\in H^p(N(\mathcal{D}), \mathcal{P}_A)$ so that $\Gamma=[h]$. Let $V_j=f^{-1}(D_j\cap f(M))$ and set $\mathcal{V}=\{V_j\}_{j\in\Sigma}$. Then $\mathcal{V}\in\mathrm{Cov}(M)$. Set $f^*\Gamma=[f^*h]$ where $f^*: H^p(N(\mathcal{D}), \mathcal{P}_A)\to H^p(N(\mathcal{V}), \mathcal{P})$ is described as follows. Let $c\in Z^p(N(\mathcal{D}), \mathcal{P}_A)$ so that $h=[c]$ and set $f^*h=[f^*c]$ where $f^*: C^p(N(\mathcal{D}), \mathcal{P}_A)\to C^p(N(\mathcal{V}), \mathcal{P})$ is described as follows. Let $\sigma=(V_{j_0}\cdots V_{j_p})$ be a p-simplex of $N(\mathcal{V})$ and set

$$(f^*c)\sigma=\rho_{f_*\sigma, \sigma}c(f_*\sigma),$$

where $f^*\sigma=(D_{j_0}\cdots D_{j_p})$ while $\rho_{f_*\sigma, \sigma}: \mathcal{P}_A(\cap f_*\sigma)\to\mathcal{P}(\cap\sigma)$ is given by

$$\rho_{f_*\sigma, \sigma}(v)=v\circ f,$$

for any CR-pluriharmonic function $v: D_{j_0}\cap\cdots\cap D_{j_p}\to\boldsymbol{R}$. It is an elementary matter to check that the definition of f^* doesn't depend (at the various stages) on the choice of representatives. We may state the following

THEOREM 1. *Let $f: M\to A$ be a pseudohermitian immersion (so that $f: M\to f(M)$ is a homeomorphism) between two strictly pseudoconvex CR manifolds M and A of CR dimensions n and $N=n+k$. Assume that both M, A are locally realizable (e.g. either M, A are compact on $n>2$). Then*

$$f^*\gamma(A)-\gamma(M)\in\mathrm{Ker}(j),$$

where $j: H^1(M, \mathcal{P})\to H^1(M, \mathcal{E})$ is the map induced on cohomology by the natural sheaf morphism $\mathcal{P}\to\mathcal{E}$ (and $\mathcal{E}$ is the sheaf of C^∞ functions on M). Set $\varphi_j=f^*\Theta_j$, $V_j=f^{-1}(D_j\cap f(M))$, $j\in\Sigma$. If each (V_j, φ_j) is pseudo-Einstein then $f^*\gamma(A)=\gamma(M)$; in particular, if A admits a global pseudo-Einstein structure, then so does M.

Given a pseudohermitian immersion, between two strictly pseudoconvex CR manifolds (M, θ) and (A, Θ) so that Θ is pseudo-Einstein, it is natural to ask (on account of Theorem 1) whether θ is pseudo-Einstein, as well. We obtain the following

THEOREM 2. *Let $f: M\to A$ be a pseudohermitian immersion between two strictly pseudoconvex CR manifolds (M, θ) and (A, Θ). If the normal Tanaka-Webster connection is flat (i.e. $R^\perp=0$) then*

$$R_{\alpha\bar\beta}=\mathrm{trace}\{Z\mapsto R^A(Z, f_*T_\alpha)f_*T_{\bar\beta}\}. \tag{26}$$

In particular, if Θ is pseudo-Einstein then θ is pseudo-Einstein, too.

COROLLARY 1. *Let $M\to S^{2N+1}$ be a pseudohermitian immersion with a flat normal Tanaka-Webster connection, of a strictly pseudoconvex CR manifold M in the standard sphere. Then $\gamma(M)=0$.*

4. Consequences of the embedding equations

We shall need the following

LEMMA 1. *For any $X, Y \in T(M)$ and any $\xi \in \nu^{2k}(f)$ the following identity holds*

$$g_\theta(a_\xi JX + Ja_\xi X, Y) = g_\Theta(\mathrm{Tor}_A(f_*X, f_*Y), J_A\xi) + g_\Theta(\mathrm{Tor}_A(f_*JX, f_*Y), \xi). \tag{27}$$

Proof. Using (21), (16), (22) and again (16) we may conduct the following calculation

$$\begin{aligned} g_\theta(a_\xi JX, Y) &= g_\Theta(\alpha(f)(JX, Y), \xi) \\ &= g_\Theta(\alpha(f)(Y, JX) + \mathrm{Tor}_A(f_*JX, f_*Y), \xi) \\ &= g_\Theta(J_A\alpha(f)(Y, X) + \mathrm{Tor}_A(f_*JX, f_*Y), \xi) \\ &= g_\Theta(J_A\alpha(f)(X, Y), \xi) + g_\Theta(\mathrm{Tor}_A(f_*JX, f_*Y), \xi) \\ &\quad - g_\Theta(J_A \,\mathrm{nor}\{\mathrm{Tor}_A(f_*X, f_*Y)\}, \xi). \end{aligned}$$

Finally

$$J_A^2 = -I + \Theta \otimes T_A, \tag{28}$$

$$g_\Theta(J_AX, J_AY) = g_\Theta(X, Y) - \Theta(X)\Theta(Y), \tag{29}$$

lead to (27). Q.E.D.

Let $\xi \in \nu^{2k}(f)$ so that $R^\perp(X, Y)\xi = 0$ for any $X, Y \in T(M)$. Then (20) and (23) furnish

$$R^A(f_*X, f_*Y\,;\,\xi, J_A\xi) = g_\theta(Ja_\xi Y, a_\xi X) - g_\theta(Ja_\xi X, a_\xi Y). \tag{30}$$

Throughout $R(X, Y\,;\,Z, W) = g_\theta(R(X, Y)Z, W)$, etc.. Note that (16) may be restated as

$$g_\theta(a_\xi X, Y) = g_\theta(X, a_\xi Y) + g_\Theta(\mathrm{Tor}_A(f_*X, f_*Y), \xi). \tag{31}$$

By (31) and $J^2 = -I + \theta \otimes T$ we obtain

$$g_\theta(Ja_\xi Y, a_\xi X) = -g_\theta(a_\xi Ja_\xi X, Y) - g_\Theta(\mathrm{Tor}_A(f_*Y, f_*Ja_\xi X), \xi). \tag{32}$$

Let us replace X by $a_\xi X$ in (27) of Lemma 1 so that to yield

$$g_\theta(a_\xi Ja_\xi X, Y) = -g_\theta(Ja_\xi^2 X, Y) + g_\Theta(\mathrm{Tor}_A(f_*a_\xi X, f_*Y), J_A\xi) + g_\Theta(\mathrm{Tor}_A(f_*Ja_\xi X, f_*Y), \xi). \tag{33}$$

Substitution from (33) into (32) now leads to

$$g_\theta(Ja_\xi Y, a_\xi X)=g_\theta(Ja_\xi^2 X, Y)-g_\theta(\mathrm{Tor}_A(f_*a_\xi X, f_*Y), J_A\xi). \tag{34}$$

On the other hand we may replace X by Y and Y by $Ja_\xi X$ in (31). The resulting identity and (33) furnish

$$g_\theta(Ja_\xi X, a_\xi Y)=-g_\theta(Ja_\xi^2 X, Y)+g_\theta(Tor_A(f_*a_\xi X, f_*Y), J_A\xi). \tag{35}$$

Finally, by (34)-(35) the (CR analogue of) Ricci's equation (30) becomes

$$R^A(f_*X, f_*Y; \xi, J_A\xi)=2g_\theta(Ja_\xi^2 X, Y)-2g_\theta(Tor_A(f_*a_\xi X, f_*Y), J_A\xi), \tag{36}$$

for any $X, Y\in T(M)$ and $\xi\in\nu^{2k}(f)$ with the property $R^\perp(X, Y)\xi=0$. Let $\{\xi_1, \cdots, \xi_k, J_A\xi_1, \cdots, J_A\xi_k\}$ be a local orthonormal frame of $\nu^{2k}(f)$ and $\{E_1, \cdots, E_{2n+1}\}$ a local orthonormal frame of $T(M)$, with $E_{2n+1}=T$ and $E_j\in H(M)$, $1\leqq j\leqq 2n$. Let $K(Z, W)=\mathrm{trace}\{V\mapsto R^A(V, Z)W\}$. It is our purpose of compute $K(f_*X, f_*Y)$ for any $X, Y\in T(M)$. To this end, note that (18) may be restated as follows

$$\begin{aligned} R^A(f_*X, f_*Y; f_*Z, f_*W) \\ =R(X, Y; Z, W)+g_\theta(\alpha(f)(Y, W), \alpha(f)(X, Z)) \\ -g_\theta(\alpha(f)(X, W), \alpha(f)(Y, Z)), \end{aligned} \tag{37}$$

for any $X, Y, Z, W\in T(M)$. To compute traces we use

$$K(f_*X, f_*Y)=\sum_{i=1}^{2n+1} R^A(f_*E_i, f_*X; f_*Y, f_*E_i)$$
$$+\sum_{a=1}^{k} R^A(\xi_a, f_*X; f_*Y, \xi_a)+R^A(J_A\xi_a, f_*X; f_*Y, J_A\xi_a)\}.$$

We may assume that $E_{\alpha+n}=JE_\alpha$, $1\leqq\alpha\leqq n$. Consequently

$$\sum_{i=1}^{2n+1}\alpha(f)(E_i, E_i)=0.$$

Here $\alpha(f)$ is not the second fundamental form of f (with respect to the Webster metrics of M and A) but rather its pseudohermitian analogue. Nevertheless (as observed in the introduction) the 'true' second fundamental form of f is traceless as well (and f is a minimal isometric immersion). This is natural since pseudohermitian immersions appear to behave very much like holomorphic isometric immersions between Kaehlerian manifolds. The implications of minimality have been discussed in [D1] (cf. Theorems 7, 8 and 12 there). Next (37) leads to

$$\begin{aligned} \mathrm{Ric}(X, Y)=K(f_*X, f_*Y) \\ -\sum_{a=1}^{k}\{R^A(\xi_a, f_*X; f_*Y, \xi_a)+R^A(J_A\xi_a, f_*X; f_*Y, J_A\xi_a)\} \\ -\sum_{i=1}^{2n+1} g_\theta(\alpha(f)(X, E_i), \alpha(f)(E_i, Y)), \end{aligned} \tag{38}$$

for any $X, Y \in T(M)$.

5. Proof of Theorem 2

We shall need the 1st Bianchi identity for ∇^A (cf. e.g. S. Kobayashi & K. Nomizu [KN], vol. I, p. 135)

$$\sum_{VZW} R^A(V, Z)W = \sum_{VZW} \{(\nabla^A_V \mathrm{Tor}_A)(Z, W) + \mathrm{Tor}_A(\mathrm{Tor}_A(V, Z), W)\}, \tag{39}$$

for any $V, Z, W \in T(A)$. Here $\sum_{VZW}$ denotes the cyclic sum over V, Z, W. Set $V = f_*X$, $Z = J_A f_* Y$ and $W = \xi_a$ in (39) and take the inner product of the resulting identity with $J_A \xi_a$. This procedure leads to

$$\begin{aligned} &R^A(f_*X, J_A f_* Y ; \xi a, J_A \xi a) \\ &= R^A(\xi_a, J_A f_* Y ; f_* X, J_A \xi_a) - R^A(\xi_a, f_* X ; J_A f_* Y, J_A \xi_a) + E_a(X, Y), \end{aligned} \tag{40}$$

where

$$\begin{aligned} E_a(X, Y) = &g_\theta((\nabla^A_{f_*X} \mathrm{Tor}_A)(J_A f_* Y, \xi_a), J_A \xi_a) \\ &+ g_\theta((\nabla^A_{J_A f_* Y} \mathrm{Tor}_A)(\xi_a, f_* X), J_A \xi_a) \\ &+ g_\theta((\nabla^A_{\xi_a} \mathrm{Tor}_A)(f_* X, J_A f_* Y), J_A \xi_a) \\ &+ g_\theta(\mathrm{Tor}_A(\mathrm{Tor}_A(f_* X, J_A f_* Y), \xi_a), J_A \xi_a) \\ &+ g_\theta(\mathrm{Tor}_A(\mathrm{Tor}_A(J_A f_* Y, \xi_a), f_* X), J_A \xi_a) \\ &+ g_\theta(\mathrm{Tor}_A(\mathrm{Tor}_A(\xi_a, f_* X), J_A f_* Y), J_A \xi_a). \end{aligned}$$

Note that

$$R^A(V, Z) J_A W = J_A R^A(V, Z) W, \tag{41}$$

(as a consequence of $\nabla^A J_A = 0$) for any $V, Z, W \in T(A)$. By (41) and $\Theta(\xi_a) = 0$ we obtain

$$R^A(\xi_a, f_* X ; J_A f_* Y, J_A \xi_a) = R^A(\xi_a, f_* X ; f_* Y, \xi_a). \tag{42}$$

Next, replace ξ by ξ_a and Y by JY in (36) so that to obtain (provided $R^\perp = 0$)

$$\begin{aligned} &R^A(f_* X, f_* JY ; \xi_a, J_A \xi_a) \\ &= 2g_\theta(a^2_{\xi_a} X, Y) - 2g_\theta(\mathrm{Tor}_A(f_* a_{\xi_a} X, f_* JY), J_A \xi_a), \end{aligned} \tag{43}$$

for any $X, Y \in T(M)$. At this point we may use (42)-(43) such that to write (40) as follows

$$\begin{aligned} &2g_\theta(a^2_{\xi_a} X, Y) - 2g_\theta(\mathrm{Tor}(f_* a_{\xi_a} X, f_* JY), J_A \xi_a) \\ &= R^A(\xi_a, J_A f_* Y ; f_* X, J_A \xi_a) - R^A(\xi_a, f_* X ; f_* Y, \xi_a) + E_a(X, Y), \end{aligned} \tag{44}$$

for any $X, Y \in T(M)$. To deal with the torsion terms in (44) we need the following

LEMMA 2. *Let* $T_\alpha = 1/2(E_\alpha - iJE_\alpha)$, $1 \leqq \alpha \leqq n$. *Then*

$$E_a(T_\alpha, T_{\bar\beta}) = ig_\theta(\tau_A \xi_a, J_A \xi_a) h_{\alpha\bar\beta}. \tag{45}$$

The proof of Lemma 2 is a straightforward consequence of

$$\mathrm{Tor}_A(Z, W) = \mathrm{Tor}_A(\bar Z, \bar W) = 0,$$
$$\mathrm{Tor}_A(Z, \bar W) = iG_\theta(Z, \bar W) T_A,$$
$$\tau_A Z \in T_{0,1}(A),$$

for any $Z, W \in T_{1,0}(A)$.

LEMMA 3. *For any* $X, YZ, W \in H(A)$ *the following identity holds*

$$\begin{aligned} &R^A(X, Y; Z, W) \\ &= R^A(Z, W; X, Y) + A_\theta(Y, Z)\Omega_\theta(W, X) + A_\theta(X, W)\Omega_\theta(Z, Y) \\ &\quad + A_\theta(W, Y)\Omega_\theta(X, Z) + A_\theta(Z, X)\Omega_\theta(Y, W), \end{aligned} \tag{46}$$

where $A_\theta(X, Y) = g_\theta(\tau_A X, Y)$.

We shall prove Lemma 3 later on. Using (46) we may compute the first curvature term in (44) as

$$\begin{aligned} &R^A(\xi_a, f_* JY; f_* X, J_A \xi_a) \\ &= R^A(f_* X, J_A \xi_a; \xi_a, f_* JY) + A_\theta(f_* JY, f_* X)\Omega_\theta(J_A \xi_a, \xi_a) \\ &\quad + A_\theta(\xi_a, J_A \xi_a)\Omega_\theta(f_* X, f_* JY), \end{aligned} \tag{47}$$

for any $X, Y \in H(M)$. Also

$$R^A(f_* X, J_A \xi_a; \xi_a, f_* JY) = -R^A(J_A \xi_a, f_* X; f_* Y, J_A \xi_a). \tag{48}$$

Let us substitute from (47)-(48) into (44) and use the identities

$$A_\theta(f_* X, f_* Y) = A(X, Y),$$
$$\Omega_\theta(f_* X, f_* JY) = -g_\theta(X, Y),$$

so that to yield

$$\begin{aligned} &2g_\theta(a_{\xi_a}^2 X, Y) - 2g_\theta(\mathrm{Tor}_A(f_* a_{\xi_a} X, f_* JY), J_A \xi_a) \\ &= -R^A(J_A \xi_a, f_* X; f_* Y, J_A \xi_a) - R^A(\xi_a, f_* X; f_* Y, \xi_a) \\ &\quad + A(X, JY) - g_\theta(X, Y) A_\theta(\xi_a, J_A \xi_a) + E_a(X, Y), \end{aligned} \tag{49}$$

for any $X, Y \in H(M)$. On the other hand (using (27)) one may show that

$$\sum_{i=1}^{2n+1} g_{\Theta}(\alpha(f)(X, E_i), \alpha(f)(E_i, Y)) \tag{50}$$
$$= \sum_{a=1}^{k} \{2g_{\theta}(a^2_{\xi_a}X, Y) + g_{\Theta}(\mathrm{Tor}_A(f_*Y, f_*a_{\xi_a}X), \xi_a)$$
$$- g_{\Theta}(\mathrm{Tor}_A(f_*Y, f_*Ja_{\xi_a}X), J_A\xi_a)\}.$$

Finally, substitution from (49)-(50) into (38) gives

$$\mathrm{Ric}(X, Y) = K(f_*X, f_*Y) - \sum_{a=1}^{k} \{g_{\Theta}(\mathrm{Tor}_A(f_*Ja_{\xi_a}X, f_*Y), J_A\xi_a) \tag{51}$$
$$- g_{\Theta}(\mathrm{Tor}_A(f_*a_{\xi_a}X, f_*Y), \xi_a) - 2g_{\theta}(\mathrm{Tor}_A(f_*a_{\xi_a}, f_*JY), J_A\xi_a)$$
$$- A(X, JY) + g(X, Y)A_{\Theta}(\xi_a, J_A\xi_a) - E_a(X, Y)\},$$

for any $X, Y \in H(M)$. Let us extend both sides of (51) by $\boldsymbol{C}$-linearity to $H(M)\otimes\boldsymbol{C}$. It follows that (51) holds for any $X, Y \in H(M)\otimes\boldsymbol{C}$ (as both sides are $\mathcal{E}\otimes\boldsymbol{C}$-linear and coincide on real vectors). Set $X=Z$, $Y=\overline{W}$, with $Z, W \in T_{1,0}(M)$. We obtain

$$\mathrm{Ric}(Z, \overline{W}) = K(f_*Z, f_*\overline{W}) + \sum_{a=1}^{k} \{A_{\Theta}(\xi_a, J_A\xi_a)g_{\theta}(Z, \overline{W}) - E_a(Z, \overline{W})\}. \tag{52}$$

Finally, we set $Z=T_\alpha$ and $W=T_\beta$ in (52) and use (45) of Lemma 2 so that to yield (26). Q.E.D.

6. Pseudohermitian Ricci curvature and the first Chern class of the normal bundle

Let (M, θ) and (A, Θ) be two strictly pseudoconvex CR manifolds and $f: M \to A$ a pseudohermitian immersion. The purpose of the present section is the converse of Theorem 1, i.e. it may be asked whether (26) yields $R^\perp=0$. We establish the following weaker result. Let $\nu^{2k}(f) \to M$ be the normal bundle of f. By a result in [D1], $\nu^{2k}(f)_x \subset H(A)_{f(x)}$ for any $x \in M$ so that J_A descends to a complex structure $J^\perp$ in $\nu^{2k}(f)$. Extend $J^\perp$ by complex linearity to $\nu^{2k}(f)\otimes\boldsymbol{C}$ and let $\nu^{2k}(f)^{1,0}$ be the eigenbundle corresponding to the eigenvalue i. We may state

THEOREM 3. *Let $f: M \to A$ be a pseudohermitian immersion with the property $R_{\alpha\bar\beta} = K_{\alpha\bar\beta}$, where $K_{\alpha\bar\beta} = K(f_*T_\alpha, f_*T_{\bar\beta})$. If the Tanaka-Webster connection of A has parallel pseudohermitian torsion ($\nabla^A\tau_A=0$) then*

$$c_1(\nu^{2k}(f)^{1,0}) = 0.$$

Throughout, if $E \to M$ is a $\boldsymbol{C}$-vector bundle then $c_1(E) \in H^2(M; \boldsymbol{R})$ denotes its first Chern class. To prove Theorem 3 we need the following

LEMMA 4. *Let $f: M \to A$ be a pseudohermitian immersion. If the ambient space A has parallel pseudohermitian torsion then*

$$(\nabla_X A)(Y, Z) = g_\Theta(\alpha(f)(X, Z), (Qf)Z) + g_\Theta((Qf)Y, \alpha(f)(X, Z)), \tag{53}$$

for any $X, Y, Z \in T(M)$.

The proof of Lemma 4 follows from $\nabla^A \tau_A = 0$ and (14)-(15), (17) in a straightforward manner. Recall that $c_1(T_{1,0}M)$ is represented by $(i/2\pi)d\omega_\alpha^\alpha$ where

$$d\omega_\alpha^\alpha = R_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta} + W_{\alpha\beta}^\alpha \theta^\beta \wedge \theta - W_{\bar\alpha\bar\beta}^{\bar\alpha}\theta^{\bar\beta} \wedge \theta,$$

and

$$W_{\beta\gamma}^\alpha = A_{\beta\gamma,\bar\sigma} h^{\alpha\bar\sigma},$$

where

$$A_{\alpha\beta,\bar\gamma} = (\nabla_{T_{\bar\gamma}} A)(T_\alpha, T_\beta)$$

are the covariant derivatives of the pseudohermitian torsion (with respect to the Tanaka-Webster connection). Also ω_β^α are the connection 1-forms of ∇. Cf. [L1], p. 162. Let $\{\Theta^1, \cdots, \Theta^N\}$ be the admissible coframe dual to $\{T_1, \cdots, T_n, \zeta_1, \cdots, \zeta_k\}$ where $\zeta_a = 1/2(\xi_a - iJ_A\xi_a)$. Then $f^*\Theta^\alpha = \theta^\alpha$ and $f^*\Theta^{a+n} = 0$. Next $c_1(T_{1,0}A)$ is represented by $(i/2\pi)d\Omega_j^j$ where Ω_j^i are the connection 1-forms of ∇^A and $(A_\Theta)_{ij,\bar k} = 0$ yields

$$d\Omega_j^j = K_{j\bar k}\Theta^j \wedge \Theta^{\bar k}.$$

Finally (53) gives $A_{\alpha\beta,\bar\gamma} = 0$ so that $f^*c_1(T_{1,0}A) = c_1(T_{1,0}M)$ and the direct sum decomposition

$$T_{1,0}(A)_{f(x)} = [(d_x f)T_{1,0}(M)_x] \oplus \nu^{2k}(f)_x^{1,0},$$

for each $x \in M$, yields $c_1(\nu^{2k}(f)^{1,0}) = 0$. Q.E.D.

Let $f: M \to A$ be a pseudohermitian immersion. Assume that $R^A = 0$ (e.g. $A = H_N$). Then (38) gives

$$\mathrm{Ric}(X, Y) = -\sum_{i=1}^{2n+1} g_\Theta(\alpha(f)(X, E_i), \alpha(f)(E_i, Y)),$$

or (by computing traces)

$$2R = -\|\alpha(f)\|^2 \leqq 0. \tag{54}$$

THEOREM 4. *There is no pseudohermitian immersion of*

$$\Big(H_n(s),\ |x|^{-2}\Big\{dt + 2\sum_{\alpha=1}^{n}(x^\alpha dy^\alpha - y^\alpha dx^\alpha)\Big\}\Big)$$

into a Tanaka-Webster flat strictly pseudoconvex CR manifold.

Proof. By a result of [D2], p. 42, we have

$$R_{\alpha\bar{\beta}}=(n+1)|x|^{-2}|z|^2 h_{\alpha\bar{\beta}}, \tag{55}$$

or (by computing traces)

$$R=n(n+1)|x|^{-2}|z|^2. \tag{56}$$

Assume there is strictly pseudoconvex CR manifold A with $R^A=0$ and a pseudohermitian immersion $f: H_n(s)\to A$. Then (56) contradicts (54) and Theorem 4 is completely proved.

We end this section with a remark regarding the analogy with Kählerian geometry (cf. [CL], p. 554). Let $f: M\to A$ be a pseudohermitian immersion. Assume that $c_1(T_{1,0}(M))=0$. Then, there is a real 1-form η on M so that

$$\Gamma=d\eta, \tag{57}$$

where $\Gamma=(i/2\pi)d\omega_\alpha^\alpha$. A $\boldsymbol{C}$-valued 2-form η on M is a (1, 1)-*form* if $T\rfloor\eta=0$ and $\eta(Z, W)=\eta(\bar{Z}, \bar{W})=0$ for any $Z, W\in T_{1,0}(M)$. Let $\Lambda^{1,1}(M)$ be the bundle of (1, 1)-forms on M. Define $L_\theta: \mathcal{E}(M)\otimes\boldsymbol{C}\to\Lambda^{1,1}(M)$ by setting $L_\theta f=f\Omega_\theta$ for any C^∞ function $f: M\to\boldsymbol{C}$. Next we need $\Lambda_\theta: \Lambda^{1,1}(M)\to\mathcal{E}(M)\otimes\boldsymbol{C}$ given by $(\Lambda_\theta\Psi, f)_\theta=(\Psi, L_\theta f)_\theta$ for any $\Psi\in\Gamma^\infty(\Lambda^{1,1}(M))$. Here $(\ ,\ )_\theta$ is the usual L^2 inner product on (M, θ), i.e.

$$(\phi, \psi)_\theta=\int_M\langle\phi, \psi\rangle\theta\wedge(d\theta)^n,$$

for any (1, 1)-forms ϕ, ψ on M (at least one of compact support) where $\langle\phi, \psi\rangle=\phi_{\alpha\bar{\beta}}\psi^{\alpha\bar{\beta}}$ and $\phi=\phi_{\alpha\bar{\beta}}\theta^\alpha\wedge\theta^{\bar{\beta}}$, $\psi=\psi_{\alpha\bar{\beta}}\theta^\alpha\wedge\theta^{\bar{\beta}}$, and $\psi^{\alpha\bar{\beta}}=\overline{\psi^{\bar{\alpha}\beta}}$, $\psi^{\bar{\alpha}\beta}=\psi_{\lambda\bar{\mu}}h^{\lambda\bar{\alpha}}h^{\bar{\mu}\beta}$. We may extend Λ_θ to an operator $\Lambda_\theta: \Lambda^2T^*M\otimes\boldsymbol{C}\to\mathcal{E}(M)\otimes\boldsymbol{C}$ by declaring it to be zero on $\Lambda^{0,2}\oplus\Lambda^{2,0}$ (a (2, 0)-form η is a $\boldsymbol{C}$-valued 2-form satisfying $T_{0,1}(M)\rfloor\eta=0$). Then

$$\Lambda_\theta\Gamma=-\frac{1}{\pi}R,$$

and we may apply Λ_θ to (57) so that to yield

$$\frac{1}{2\pi}R=n\ \eta_0+i\ \mathrm{div}(Z), \tag{58}$$

where $\eta=\eta_\alpha\theta^\alpha+\eta_{\bar{\alpha}}\theta^{\bar{\alpha}}+\eta_0\theta$ and $Z=Z^{\bar{\alpha}}T_{\bar{\alpha}}-Z^\alpha T_\alpha$ with $Z^\alpha=h^{\alpha\bar{\beta}}\eta_{\bar{\beta}}$. The divergence in (58) is taken with respect to the volume form $\psi=\theta\wedge(d\theta)^n$ (i.e. $\mathrm{div}(Z)\psi=\mathcal{L}_Z\psi$, where $\mathcal{L}$ denotes the Lie derivative). Therefore, if $\int_M\eta_0\psi\geqq0$ then (54) gives $\alpha(f)=0$ and thus $R^\perp=0$ (as a consequence of (20)) provided that $R^A=0$. Yet, by a result of [L1], p. 169, if (M, θ) is pseudo-Einstein, one representative of Γ is $\eta=(1/2\pi n)R\theta$ so that (in view of (54)) the hypothesis $\int_M\eta_0\psi\geqq0$ is generically not satisfied. Indeed, let η' be any other real 1-form so that $\Gamma=d\eta'$. If for instance $H^1(M;\boldsymbol{R})=0$ then $\eta'=\eta+du$ for some C^∞ function $u: M\to\boldsymbol{R}$ and (58) yields $\int_M T(u)\psi=0$, that is $\int_M\eta_0'\psi=\int_M\eta_0\psi\leqq0$.

7. Curvature properties of the Tanaka-Webster connection

The main purpose of this section is to prove Lemma 3. Let (M, θ) be a nondegenerate CR manifold. Let R, R^θ be the curvature tensor fields of ∇, ∇^θ, respectively. Taking into account (9) we may derive the following identity

$$R^\theta(X, Y)Z = R(X, Y)Z - (LX \wedge LY)Z + \theta(Z)S(X, Y) - g_\theta(S(X, Y), Z)T + 2\theta(Z)(\theta \wedge \mathcal{O})(X, Y) - 2g_\theta((\theta \wedge \mathcal{O})(X, Y), Z)T - \frac{1}{2}\Omega_\theta(X, Y)JZ, \tag{59}$$

for any $X, Y, Z \in T(M)$. We need to explain the notations in (59). Firstly $L = \tau + (1/2)J$. Next $(X \wedge Y)Z = g_\theta(Y, Z)X - g_\theta(X, Z)Y$ for any $X, YZ \in T(M)$. Also we set

$$S(X, Y) = (\nabla_X \tau)Y - (\nabla_Y \tau)X. \tag{60}$$

Finally, the operator $\mathcal{O}$ is given by $\mathcal{O} = \tau^2 + J\tau - (1/4)I$, where I denotes the identical transformation. The proof of (59) is a rather lengthy computation based on the identities

$$\nabla J = 0, \quad \theta \circ J = 0, \quad \theta \circ \tau = 0,$$
$$\nabla_X^\theta T = LX, \quad \nabla \Omega_\theta = 0, \quad \nabla \theta = 0,$$
$$A(X, JY) = A(JX, Y),$$
$$\Omega_\theta(X, \tau Y) + \Omega_\theta(\tau X, Y) = 0,$$
$$L^* = \tau - \frac{1}{2}J, \quad \tau L^* = L\tau,$$
$$(d\theta)(X, Y) = -\frac{1}{2}\Omega_\theta(X, Y),$$

for any $X, Y \in T(M)$ (and is left as an exercise to the reader).

Let $X, Y, Z, W \in H(M)$. Take the inner product of (59) with W. This procedure furnishes

$$R^\theta(X, Y; Z, W) = R(X, Y; Z, W) - g_\theta((LX \wedge LY)Z, W) + \frac{1}{2}\Omega_\theta(X, Y)\Omega_\theta(Z, W), \tag{61}$$

for any $X, Y, Z, W \in H(M)$. Then we may use (61) twice so that to yield

$$R(X, Y; Z, W) = R(Z, W; X, Y) + g_\theta((LX \wedge LY)Z, W) - g_\theta((LZ \wedge LW)X, Y),$$

which in turn leads to (46) of Lemma 3. The general philosophy of this procedure is that one uses the known symmetries of the Riemann-Christoffel tensor R^{θ}_{jklm} of (M, g_{θ}) via (59), rather than establishing similar properties for R_{jklm}. Nevertheless, let us observe that $R_{jklm}+R_{kjlm}=0$ because R is a 2-form, and $R_{jklm}+R_{jkml}=0$ because $\nabla g_{\theta}=0$. The missing property is obviously $R_{jklm}=R_{lmjk}$. Any tentative to obtain a CR analogue of $R^{\theta}_{jklm}=R^{\theta}_{lmjk}$ passing through the Bianchi identities (of the Tanaka-Webster connection) would have to deal with the torsion terms there. As remarked in section 2, $R_{\alpha\bar{\beta}}$ is only a fragment of Ric and (as a consequence of (59)) we have

$$R^{\theta}_{\alpha\bar{\beta}}=h_{\alpha\bar{\beta}}-\frac{1}{2}R_{\alpha\bar{\beta}},$$

$$R_{\alpha\beta}=i(n-1)A_{\alpha\beta}, \tag{62}$$

$$R_{0\beta}=S^{\bar{\alpha}}_{\bar{\alpha}\beta}, \quad R_{\alpha 0}=R_{00}=0.$$

Here $S^{\bar{\alpha}}_{\bar{\beta}\gamma}$ are (among) the complex components of S (given by (60)). Also we set $R^{\theta}_{\alpha\bar{\beta}}=\text{trace}\{X\mapsto R^{\theta}(X, T_{\alpha})T_{\bar{\beta}}\}$. The proof of (62) is omitted. Finally, we wish to show that

$$2R=\text{trace(Ric)}. \tag{63}$$

Note that (63) was employed to derive the identity (54). By (59) the following identities hold

$$\text{Ric}(E_{\alpha}, E_{\beta})=i(n-1)(A_{\alpha\beta}-A_{\bar{\alpha}\bar{\beta}})+R_{\alpha\bar{\beta}}+R_{\bar{\alpha}\beta}, \tag{64}$$

$$\text{Ric}(JE_{\alpha}, E_{\beta})=-(n-1)(A_{\alpha\beta}+A_{\bar{\alpha}\bar{\beta}})+i(R_{\alpha\bar{\beta}}-R_{\bar{\alpha}\beta}), \tag{65}$$

$$\text{Ric}(E_{\alpha}, JE_{\beta})=-(n-1)(A_{\alpha\beta}+A_{\bar{\alpha}\bar{\beta}})+i({}_{\beta\bar{\alpha}}-R_{\bar{\beta}\alpha}), \tag{66}$$

$$\text{Ric}(JE_{\alpha}, JE_{\beta})=-i(n-1)(A_{\alpha\beta}-A_{\bar{\alpha}\bar{\beta}})+R_{\alpha\bar{\beta}}+R_{\bar{\alpha}\beta}. \tag{67}$$

Then $\text{trace(Ric)}=g^{ij}\,\text{Ric}(E_i, E_j)$, where

$$g^{\alpha+n,\beta+n}=g^{\alpha\beta}, \quad g^{\alpha,\beta+n}=-g^{\alpha+n,\beta},$$

$$g^{\alpha 0}=g^{0\alpha}=0, \quad g^{00}=1,$$

$$g^{\alpha\beta}=\frac{1}{4}(h^{\alpha\bar{\beta}}+h^{\bar{\alpha}\beta}), \quad g^{\alpha,\beta+n}=\frac{i}{4}(h^{\alpha\bar{\beta}}-h^{\bar{\alpha}\beta}),$$

and the identities (64)-(67) lead to (63). Q.E.D.

8. Proof of Theorem 1

Let $\mathcal{U}=\{U_{\alpha}\}_{\alpha\in I}\in\text{Cov}(M)$ and $u_{\beta\alpha}\in\mathcal{P}(U_{\alpha}\cap U_{\beta})$ so that $i^{*}_{\alpha\beta}\theta_{\beta}=\exp(2u_{\beta\alpha})i^{*}_{\beta\alpha}\theta_{\alpha}$ where $i_{\alpha\beta}: U_{\alpha}\cap U_{\beta}\to U_{\beta}$ are inclusions. Then $\gamma(M)\in H^1(M, \mathcal{P})$ is the equivalence class of $[c]\in H^1(N(\mathcal{U}), \mathcal{P})$, where $c: \Delta(\alpha\beta)\mapsto u_{\beta\alpha}$. Let $\mathcal{W}\in\text{Cov}(M)$ so that $\mathcal{W}<\mathcal{U}$, $\mathcal{W}<\mathcal{V}$. Set $\mathcal{W}=\{W_{\alpha}\}_{\alpha\in J}$. There are maps $\phi: J\to I$ and $\psi: J\to\Sigma$ so that

$W_a \subset U_{\phi(a)} \cap V_{\psi(a)}$ for each $a \in J$. Set $\lambda_a = r_a^* \theta_{\phi(a)}$ and $\mu_a = s_a^* \varphi_{\psi(a)}$ where $r_a : W_a \to U_{\phi(a)}$ and $s_a : W_a \to V_{\psi(a)}$ are inclusions. Note that

$$k_{ab}^* \lambda_b = \exp(2h_{ba}) k_{ba}^* \lambda_a , \tag{68}$$

where $k_{ab}^* : W_a \cap W_b \to W_b$ are inclusions and $h_{ba} = u_{\beta\alpha} \circ r_{ab}$ with $\alpha = \phi(a)$ and $\beta = \phi(b)$, and $r_{ab} : W_a \cap W_b \subset U_\alpha \cap U_\beta$. In other words $h_{ba} = \rho_{\phi\sigma, \sigma}(u_{\beta\alpha})$ where $\rho_{\phi\sigma, \sigma} : \mathcal{P}(\cap \phi\sigma) \to \mathcal{P}(\cap \sigma)$ is the restriction map ($\sigma = \Delta(ab) \in N(\mathcal{W})$) and $\phi : N(\mathcal{W}) \to N(\mathcal{U})$ the natural simplicial map. If $\tilde{\phi} : C^1(N(\mathcal{U}), \mathcal{P}) \to C^1(N(\mathcal{W}), \mathcal{P})$ is the induced map on cochains, then $(\tilde{\phi} c)\sigma = h_{ba}$, and if $\phi^* : H^1(N(\mathcal{U}), \mathcal{P}) \to H^1(N(\mathcal{W}), \mathcal{P})$ is the induced map on cohomology then $\phi^* g = [\tilde{\phi} c]$ with $g = [c]$ so that

$$\gamma(M) = [\phi^* g] ,$$

(one checks that $g \sim \phi^* g$ by looking at $\mathcal{W}$ as a common refinement of itself and $\mathcal{U}$). Both (W_a, λ_a) and (W_a, μ_a) are strictly pseudoconvex CR manifolds so that

$$\mu_a = \exp(2v_a) \lambda_a , \tag{69}$$

for some $v_a \in \mathcal{E}(W_a)$. Let $v \in C^1(N(\mathcal{W}), \mathcal{E})$ be given by $v : \Delta(a) \mapsto v_a$. Similar to (68) we have

$$k_{ab}^* \mu_b = \exp(2\tilde{h}_{ba}) k_{ba}^* \mu_a , \tag{70}$$

where $\tilde{h}_{ba} = \tilde{u}_{ji} \circ s_{ab}$ with $i = \psi(a)$, $j = \psi(b)$ and $s_{ab} : W_a \cap W_b \subset V_i \cap V_j$. Also $\tilde{u}_{ji} = U_{ji} \circ f_{ij}$ and $f_{ij} : V_i \cap V_j \to D_i \cap D_j$ is induced by f. Finally (68)-(70) lead to

$$\tilde{h}_{ba} = v_b \circ k_{ab} + h_{ba} - v_a \circ k_{ba} . \tag{71}$$

Let $j : C^1(N(\mathcal{W}), \mathcal{P}) \to C^1(N(\mathcal{W}), \mathcal{E})$ be induced by the natural sheaf morphism $\mathcal{P} \to \mathcal{E}$ (i.e. $\mathcal{P}(U) \to \mathcal{E}(U)$ is the inclusion, for each $U \subseteq M$ open). Then (71) may be written

$$j \tilde{\psi} f^* C = \delta_\varepsilon v + j \tilde{\phi} c ,$$

where

$$\delta_{\mathcal{E}} : C^1(N(\mathcal{W}), \mathcal{E}) \to C^2(N(\mathcal{W}), \mathcal{E}) ,$$

is the coboundary operator. Consequently

$$j \phi_{\mathcal{V}\mathcal{W}} f^* G = j \phi_{\mathcal{U}\mathcal{W}} g ,$$

where $j : H^1(N(\mathcal{W}), \mathcal{P}) \to H^1(N(\mathcal{W}), \mathcal{E})$. Finally, as j and ϕ^* (respectively j and ψ^*) commute it follows that $j(f^* \gamma(A) - \gamma(M)) = 0$. Note that in general $\mathrm{Ker}(j) \neq 0$ (because $B^1(N(\mathcal{W}), \mathcal{P}) \subset B^1(N(\mathcal{W}), \mathcal{E})$, strict inclusion). If each μ_a is pseudo-Einstein then $v_a \in \mathcal{P}(W_a)$ and (71) may be written

$$\tilde{\psi} f^* C = \delta v + \tilde{\phi} c ,$$

where $\delta : C^1(N(\mathcal{W}), \mathcal{P}) \to C^2(N(\mathcal{W}), \mathcal{P})$ is the coboundary operator. Thus

$$\phi_{\mathcal{V}\mathcal{W}} f^* G = \phi_{\mathcal{U}\mathcal{W}} g\,,$$

that is

$$f^* \gamma(A) = \gamma(M)\,,$$

and Theorem 1 is completely proved.

9. Examples

1) (*Heisenberg groups*)

Let H_n be the Heisenberg group endowed with the (strictly pseudoconvex) CR structure spanned by

$$T_\alpha = \frac{\partial}{\partial z_\alpha} + i \bar{z}_\alpha \frac{\partial}{\partial t}\,,$$

(the *Lewy operators*). Fix the contact 1-form θ_0 on H_n given by

$$\theta_0 = dt + i \sum_{\alpha=1}^{n} \{z_\alpha d\bar{z}_\alpha - \bar{z}_\alpha dz_\alpha\}\,.$$

The map $f : H_n \to H_N$, $N = n+k$, $k \geqq 1$, *induced by the natural inclusion* $\boldsymbol{C}^n \to \boldsymbol{C}^N$ (i.e. $f(z, t) = (z, 0, t)$, $0 \in \boldsymbol{C}^k$) *is a pseudohermitian immersion with a flat normal Tanaka-Webster connection.* Indeed, let $(w, s) = (w_1, \cdots, w_N, s)$ be the natural coordinates on H_N. Then

$$W_j = \frac{\partial}{\partial w_j} + i \bar{w}_j \frac{\partial}{\partial s}$$

span the CR structure of H_N while

$$\Theta_0 = ds + \sum_{j=1}^{N} \{w_j d\bar{w}_j - \bar{w}_j dw_j\}$$

is a contact 1-form on H_N (whose corresponding Levi form is positive definite). Next $f_* T_\alpha = W_\alpha$, $f^* \Theta_0 = \theta_0$ and $\mathrm{nor}(T_A) = 0$ (here $A = H_N$, $T_A = \partial/\partial s$) by straightforward calculation. Finally $R^\perp = 0$ as a consequence of (20) (the CR analogue of the Ricci equation) for $A = H_N$.

2) (*Quotients of Heisenberg groups by discrete groups of dilations*)

Let $H_n(s)$, $0 < s < 1$, carry the CR structure induced by the covering map $\pi : H_n - \{0\} \to H_n(s)$, and the contact 1-form θ given by

$$\theta_{\pi(x)} = |x|^{-2} \theta_{0,x} \circ (d_x \pi)^{-1}, \tag{72}$$

for any $x \in H_n - \{0\}$. *The map* $F : H_n(s) \to H_N(s)$ *induced by* $f : H_n - \{0\} \to H_N - \{0\}$ (i.e. $F \circ \pi = \Pi \circ f$, where $\Pi : H_N - \{0\} \to H_N(s)$ is the natural covering map) *is a pseudohermitian immersion.* Indeed, if $H_N(s)$ is endowed with the contact 1-form Θ given by

$$\Theta_{\Pi(X)} = |X|^{-2} \Theta_{0,X} \circ (d_X \Pi)^{-1}, \tag{73}$$

for any $X\in H_N-\{0\}$ then $|f(x)|=|x|$, $x\in H_n$, yields $F^*\Theta=\theta$ (i.e. F is isopseudohermitian). Moreover, we may write (72)-(73) as

$$\theta=e^{2u}\theta_0, \quad \Theta=e^{2U}\Theta_0,$$

(with $U=\log|X|^{-1}$ and $u=U\circ f$). Therefore, the characteristic directions T and T_A of $(H_n(s), \theta)$ and $(H_N(s), \Theta)$ are respectively given by

$$T=e^{-2u}\left\{\frac{\partial}{\partial t}-2iu^{\beta}T_{\beta}+2iu^{\bar\beta}T_{\bar\beta}\right\},$$

$$T_A=e^{-2U}\left\{\frac{\partial}{\partial s}-2iU^{j}W_j+2iU^{\bar j}W_{\bar j}\right\}.$$

Note that

$$U^j(f(x))=|x|^2W_{\bar j}(U)_{f(x)},$$

and

$$W_{\bar j}(U)=-\frac{1}{2}|X|^{-4}w_j\bar{\Phi},$$

where $\Phi(w, s)=|w|^2+is$ (note that $\bar{\Phi}$ is CR-holomorphic). Finally

$$U^{\alpha}=|x|^2U_{\bar\alpha},$$

$$U_{\bar\alpha}\circ f=u_{\bar\alpha}, \quad U^{\alpha}\circ f=u^{\alpha},$$

and

$$T_{\bar\alpha}(u)=-\frac{1}{2}|x|^{-4}z_{\alpha}\bar{\phi},$$

(where $\phi=\Phi\circ f$) yield $f_*T=T_A$. Next, let us compute the curvature of the normal Tanaka-Webster connection $\nabla^{\perp}$ of F. We perform our task in a more general setting, as follows. Let $f: M\to A$ be a pseudohermitian immersion between (M, θ) and (A, Θ) and set $\hat\theta=e^{2u}\theta$, $\hat\Theta=e^{2U}\Theta$ with $U\in C^{\infty}(A)$, $u=U\circ f$. Readily $f^*\hat\Theta=\hat\theta$. Set

$$\hat T=e^{-2u}\{T-2iu^{\beta}T_{\beta}+2iu^{\bar\beta}T_{\bar\beta}\},$$

where T is the characteristic direction of (M, θ). As $U^{\alpha}\circ f=u^{\alpha}$ we obtain

$$\hat T_A(f(x))=(f_*\hat T)(f(x))$$
$$+2ie^{-2u(x)}\{U^{\overline{a+n}}(f(x))W_{\overline{a+n}}(f(x))-U^{a+n}(f(x))W_{a+n}(f(x))\},$$

for any $x\in M$. *Thus* $f_*\hat T=\hat T_A$ (i.e. f *is a pseudohermitian immersion from* $(M, \hat\theta)$ *into* $(A, \hat\Theta)$) *if and only if* $\zeta_{\bar a}(U)=0$. Let us look now at the relation between $\nabla^{\perp}$ and $\hat\nabla^{\perp}$ (the normal Tanaka-Webster connection of $(M, \hat\theta)$ in $(A, \hat\Theta)$). Let $\hat\nu^{2k}(f)_x$ be the orthogonal complement (with respect to $g_{\hat\Theta, f(x)}$) of $(d_xf)T_x(M)$ in $T_{f(x)}(A)$, for any $x\in M$. Then $\hat\nu^{2k}(f)_x=\nu^{2k}(f)_x$, although the Webster metrics $g_{\hat\Theta}$, g_{Θ} are not conformally related.

Assume from now on that f is a pseudohermitian immersion both as a map of (M, θ) into (A, Θ), respectively of $(M, \hat\theta)$ into $(A, \hat\Theta)$. We need to recall

LEMMA 5 (cf. [D2], p. 39). *Let $(M, T_{1,0}(M), \theta, T)$ be a non-degenerate CR manifold. Then, under a transformation $\hat{\theta}=e^{2u}\theta$, the Christoffel symbols of the Tanaka-Webster connection of $(T_{1,0}(M), \theta)$ and $(T_{1,0}(M), \hat{\theta})$ are related by*

$$\hat{\Gamma}^{\sigma}_{\beta\alpha}=\Gamma^{\sigma}_{\beta\alpha}+2u_{\beta}\delta^{\sigma}_{\alpha}+2u_{\alpha}\delta^{\sigma}_{\beta},$$

$$\hat{\Gamma}^{\sigma}_{\bar{\beta}\alpha}=\Gamma^{\sigma}_{\bar{\beta}\alpha}-2u^{\sigma}h_{\bar{\beta}\alpha},$$

$$e^{2u}\hat{\Gamma}^{\sigma}_{0\alpha}=\Gamma^{\sigma}_{0\alpha}+2u_0\delta^{\sigma}_{\alpha}+iu_{\alpha,}{}^{\sigma}+2i\Gamma^{\sigma}_{\bar{\mu}\alpha}u^{\bar{\mu}}-2i\Gamma^{\sigma}_{\mu\alpha}u^{\mu},$$

where $u_{\alpha,}{}^{\sigma}=u_{\alpha,\bar{\beta}}h^{\sigma\bar{\beta}}$.

Using Lemma 5, the identity (15) and

$$\hat{\nabla}^{A}_{f_*X}\xi=-f_*\hat{a}_{\xi}X+\hat{\nabla}^{\perp}_{X}\xi,$$

for any $X\in\mathfrak{X}(M)$, $\xi\in\Gamma^{\infty}(\nu^{2k}(f))$, we find

$$\hat{\nabla}^{\perp}_{T_\beta}\xi_a=\nabla^{\perp}_{T_\beta}\xi_a+2u_\beta\zeta_a,$$

(74) $$\hat{\nabla}^{\perp}_{T_{\bar{\beta}}}\zeta_a=\nabla^{\perp}_{T_{\bar{\beta}}}\zeta_a,$$

$$\hat{\nabla}^{\perp}_{\hat{T}}\zeta_a=\nabla^{\perp}_{\hat{T}}\zeta_a+2u_0e^{-2u}\zeta_a.$$

If $M=H_n$ and $A=H_N$ we have $\nabla^A\zeta_a=0$ and thus $\nabla^{\perp}\zeta_a=0$. Thus (by (74)) if $M=H_n(s)$ and $A=H_N(s)$ the normal Tanaka-Webster connection of F is given by

$$\nabla^{\perp}_{T_\beta}\zeta_a=2u_\beta\zeta_a,$$

(75) $$\nabla^{\perp}_{T_{\bar{\beta}}}\zeta_a=0,$$

$$\nabla^{\perp}_{T}\zeta_a=2u_0e^{-2u}\zeta_a,$$

with $u=\log|x|^{-1}$. Next (as a consequence of (75)) we may use the identities

$$[T_\alpha, T_\beta]=0,$$

$$[T_\alpha, T_{\bar{\beta}}]=-2i\delta_{\alpha\beta}\frac{\partial}{\partial t},$$

and

$$\nabla^{\perp}_{\partial/\partial t}\zeta_a=2(u_0+2iu_\beta u^\beta)\zeta_a,$$

so that to yield

(76) $$R^{\perp}(T_\alpha, T_\beta)\zeta_a=0,\quad R^{\perp}(T_{\bar{\alpha}}, T_{\bar{\beta}})\zeta_a=0,$$

and

$$R^{\perp}(T_\alpha, T_{\bar{\beta}})\zeta_a=\{-2T_{\bar{\beta}}(u_\alpha)+4i\delta_{\alpha\beta}(u_0+2iu_\sigma u^\sigma)\}\zeta_a.$$

Finally, taking into account the identities

$$u_\alpha=-\frac{1}{2}|x|^{-4}\bar{z}_\alpha\phi, \quad T_{\bar\beta}(u_\alpha)=-\frac{1}{2}|x|^{-4}\delta_{\alpha\beta}\phi,$$

$$u_0=-\frac{1}{2}|x|^{-4}t, \quad \phi\bar\phi=|x|^4,$$

$$u_\sigma u^\sigma=\frac{1}{4}|x|^{-4}|z|^2,$$

it follows that

$$R^\perp(T_\alpha, T_{\bar\beta})\zeta_a=-|x|^{-4}\phi\delta_{\alpha\beta}\zeta_a. \tag{77}$$

Summing up, the pseudohermitian immersion $F: H_n(s)\to H_N(s)$ has (by (77)) $R^\perp\neq 0$. However (55) yields $K_{\alpha\bar\beta}=\lambda R_{\alpha\bar\beta}$ with $\lambda=(N+1)/(n+1)$.

3) (*Pseudo-Siegel domains*)

Let $(\alpha, \beta)=(\alpha_1, \cdots, \alpha_n, \beta)\in \boldsymbol{Z}_+^{n+1}$ be a fixed multiindex and $D_{\alpha,\beta}=\{(z_1, \cdots, z_n, w)\in \boldsymbol{C}^{n+1}: \sum_{j=1}^n |z_j|^{2\alpha_j}+\mathrm{Im}(w^\beta)-1<0\}$ (cf. [BP]). Then $D_{1,1}$ is the Siegel domain in $\boldsymbol{C}^{n+1}$ (and $\partial D_{1,1}\approx H_n$). Assume $\beta>1$ from now on. The boundary $\partial D_{\alpha,\beta}$ of $D_{\alpha,\beta}$ inherits a CR structure (as a real hypersurface of $\boldsymbol{C}^{n+1}$) spanned by

$$T_j=\frac{\partial}{\partial z_j}-2if_j\frac{\partial}{\partial w}, \tag{78}$$

in some neighborhood of $w\neq 0$ where

$$f_j=\frac{\alpha_j}{\beta}w^{1-\beta}z_j^{\alpha_j-1}\bar{z}_j^{\alpha_j}.$$

Hence we have the commutation relations

$$[T_j, T_k]=0,$$

$$[T_j, T_{\bar k}]=\frac{2i}{\beta}\left\{\frac{\alpha_j^2|z_j|^{2(\alpha_j-1)}}{w^{\beta-1}}\frac{\partial}{\partial w}+\frac{\alpha_k^2|z_k|^{2(\alpha_k-1)}}{\bar{w}^{\beta-1}}\frac{\partial}{\partial\bar{w}}\right\}\delta_{jk}. \tag{79}$$

Endow $\partial D_{\alpha,\beta}$ with the pseudohermitian structure $\theta=\theta_{\alpha,\beta}$ given by

$$\theta=\beta w^{\beta-1}dw+\beta\bar{w}^{\beta-1}d\bar{w}+2i\sum_{j=1}^n(g_j dz_j-\bar{g}_j d\bar{z}_j), \tag{80}$$

where

$$g_j=\alpha_j z_j^{\alpha_j-1}\bar{z}_j^{\alpha_j}.$$

Therefore the Levi form of $(\partial D_{\alpha,\beta}, \theta)$ is $\mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ where

$$\lambda_j=4\alpha_j^2|z_j|^{2(\alpha_j-1)}.$$

Therefore, if $\alpha_j>1$, $1\leqq j\leqq n$, then G_θ is degenerate at each point of $\bigcup_{j=1}^n M_j$, where M_j is the trace of the complex hyperplane $L_j=\{(z, w): z_j=0\}$ on the boundary of $D_{\alpha,\beta}$. Next $U_{\alpha,\beta}=\partial D_{\alpha,\beta}-(\bigcup_{j=1}^n M_j)$ (an open subset of $\partial D_{\alpha,\beta}$) is a strictly pseudoconvex CR manifold. The characteristic direction T of

$$d\theta = -4i\alpha_j^2 |z_j|^{2(\alpha_j - 1)} dz_j \wedge d\bar{z}^j$$

is given by

$$T = \frac{1}{4\beta |w|^{2(\beta-1)}} \left\{ \bar{w}^{\beta-1} \frac{\partial}{\partial w} + w^{\beta-1} \frac{\partial}{\partial \bar{w}} \right\}.$$

Note that (79) may be written

$$j \neq k \Longrightarrow [T_j, T_{\bar{k}}] = 0,$$

$$[T_j, T_{\bar{j}}] = i\lambda_j T.$$

Also

$$[T_j, T] = 0.$$

Using (A.3)-(A.5) in [D2], p. 48, we derive the (Christoffel symbols of the) Tanaka-Webster connection of $(U_{\alpha,\beta}, \theta)$

$$\Gamma^s_{jk} = \frac{\alpha_j - 1}{z_j} \delta_{jk} \delta_{js}, \quad \Gamma^s_{\bar{j}k} = 0, \quad \Gamma^s_{0k} = 0. \tag{81}$$

Therefore $(U_{\alpha,\beta}, \theta)$ has a vanishing pseudohermitian torsion ($\tau = 0$). As a straightforward consequence of (81) *the Tanaka-Webster connection of* $(U_{\alpha,\beta}, \theta)$ *is flat* ($R = 0$).

Finally we look at the structure of the points of weak pseudoconvexity of $\partial D_{\alpha,\beta}$. Let $1 \leqq p \leqq n$ and set $M_{j_1 \cdots j_p} = \partial D_{\alpha,\beta} \cap L_{j_1} \cap \cdots \cap L_{j_p}$. Then

$$M_{j_1 \cdots j_p} \approx \partial D_{\alpha_{j_1 \cdots j_p}, \beta} \subset \boldsymbol{C}^{n+1-p},$$

(a diffeomorphism), where $\alpha_{j_1 \cdots j_p} = (\alpha_1, \cdots, \hat{\alpha}_{j_1}, \cdots, \hat{\alpha}_{j_p}, \cdots, \alpha_n)$. A natural question is how does $M_{j_1 \cdots j_p}$ sit in $\partial D_{\alpha,\beta}$ i.e. equivalently study the geometry of the immersion $f : \partial D_{(\alpha_1, \cdots, \alpha_k), \beta} \to \partial D_{\alpha,\beta}$ induced by the natural map $\boldsymbol{C}^k \times \boldsymbol{C} \to \boldsymbol{C}^n \times \boldsymbol{C}$, $(z, w) \mapsto (z, 0, w)$, $0 \in \boldsymbol{C}^k$, $0 < k < n$. Using (78) one may show that f is a CR immersion. Finally (80) yields $f^* \theta_{\alpha,\beta} = \theta_{(\alpha_1, \cdots, \alpha_k), \beta}$ i.e. f is isopseudohermitian.

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