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THE KÄHLER-EINSTEIN METRICS ON A K3 SURFACE CANNOT BE ALMOST KÄHLER WITH RESPECT TO AN OPPOSITE ALMOST COMPLEX STRUCTURE

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§ 1. Introduction

It is a fundamental fact that an almost complex structure on a manifold has its preferred orientation of the manifold. Even for an almost complex manifold, i.e., an oriented manifold with an almost complex structure J, there is some additional obstruction for the manifold to admit another almost complex structure whose preferred orientation is opposite to that of J. (Such an obstruction has been obtained in dimension four [9].)

It is then interesting to know whether or not the choice of orientation of a manifold affects an almost complex structure on the manifold to have *good properties* such as integrability or parallelizability for some metric connections. Of course, such a problem is valid for a manifold which admits two kinds of almost complex structures with different preferred orientations.

The purpose of the present note is to observe such interesting phenomena concerning almost complex structures on 4-dimensional manifolds and the choice of orientation of the manifolds.

By an *opposite almost complex structure* on an oriented smooth 4-manifold X, we mean an almost complex structure on $-\bar{X}$ (the 4-manifold X with orientation reversed) [9]. If X does not admit an almost complex structure but an opposite almost complex structure, then it is preferable to treat $-\bar{X}$ rather than X since it can be recognized as an almost complex structure. The notion of opposite almost complex structures on 4-manifolds is, therefore, meaningful for 4-manifolds already carrying almost complex structure (almost complex manifolds) or 4-manifolds with orientation chosen primarily.

The condition for a 4-manifold to admit a pair of an almost complex structure and an opposite almost complex structure is equivalent to the existence of a field of oriented tangent 2-planes on the 4-manifold [9] (see also [7]).

A 4-manifold X with an opposite almost complex structure is said to be

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opposite almost complex. If X is endowed with a metric g which is invariant by an opposite almost complex structure J', then the pair (g, J') is called an *opposite almost Hermitian structure*. Moreover, the pair is said to be *opposite Hermitian, opposite almost Kähler*, and *opposite Kähler*, if J' is integrable, the corresponding Kähler form is closed, and both conditions hold, respectively.

We now consider an almost Hermitian 4-manifold (X, g, J) which also carries an opposite almost complex structure. In spite of the fact that J is g-orthogonal, the opposite almost complex structure is not, in general, g-orthogonal. However, it is shown that there exists on X a g-orthogonal opposite almost complex structure, denoted by J', such that J and J' commute with each other at each point of X (see Lemma 4). We thus obtain a quadruple (X, g, J, J'), and call it a *double almost Hermitian* 4-manifold. As same as we are naturally interested in when the almost Hermitian structure (g, J) on X becomes almost Kähler structure or Kähler structure, we are led to consider for what double almost Hermitian 4-manifolds the opposite almost Hermitian structure (g, J') can be opposite almost Kähler or opposite Kähler. In these situations, we shall call a quadruple (X, g, J, J') *double Hermitian, double almost Kähler* and *double Kähler* if (g, J) and (g, J') are both Hermitian and opposite Hermitian, almost Kähler and opposite almost Kähler, and Kähler and opposite Kähler, respectively.

In the level of real category, there is no essential difference between J and J', except for the orientation. However, if we restrict our attention to complex surfaces, the choice of the orientation becomes crucial for the opposite almost Hermitian structure (g, J') to be opposite almost Kähler, or opposite Kähler. In the present paper we shall be studying such an interesting observation, under some additional condition: g is Einstein. Especially, we focus our attention to the Kähler-Einstein metrics on the K3 surfaces and the surfaces $\boldsymbol{CP}^2\#n\overline{\boldsymbol{CP}}^2$ for $n=3$, 5, or 7.

In section 2, we shall state our main results. In section 3, some basic arguments on opposite almost complex structure are given. We shall prove Theorems 1 and 2 in section 4. In section 5, Theorem 3 concerning double Kähler-Einstein surfaces is proved.

§ 2. Statement of results

It is well-known that a K3 surface admits a Kähler-Einstein metric. In [8], [9], it is shown that the classical K3 surface (the underlying real 4-manifold of the K3 surface) admits an opposite almost complex structure. Then natural questions arise:

(1) Does a K3 surface admit a Kähler-Einstein metric with respect to an opposite almost complex structure (an opposite Kähler-Einstein metric)?

(2) Does a K3 surface admit a double Kähler-Einstein metric?

Our first result is the following.

THEOREM 1. *A Kähler-Einstein metric g on a $K3$ surface cannot be almost Kähler with respect to a g-orthogonal opposite almost complex structure.*

It should be noted that Theorem 1 asserts that the Kähler-Einstein metric g on a K3 surface is *strictly* opposite almost Hermitian.

The K3 surface is an example of surfaces with the first Chern class $c_1=0$. We then turn our attention to surfaces with positive first Chern class. A surface X with the first Chern class $c_1(X)>0$ must be of the following (cf. [3], [13]): $\boldsymbol{CP}^1\times\boldsymbol{CP}^1$, or $\boldsymbol{CP}^2\#n\overline{\boldsymbol{CP}}^2$ $(0\leqq n\leqq 8)$, the surfaces obtained by blowing up $\boldsymbol{CP}^2$ at n *generic* points, where *generic* means that no three points are collinear, and no six points are in one quadratic curve in $\boldsymbol{CP}^2$. As symmetric spaces, $\boldsymbol{CP}^1\times\boldsymbol{CP}^1$ and $\boldsymbol{CP}^2$ admit Kähler-Einstein metrics. It is known that neither $\boldsymbol{CP}^2\#\overline{\boldsymbol{CP}}^2$ nor $\boldsymbol{CP}^2\#2\overline{\boldsymbol{CP}}^2$ has a Kähler-Einstein metric, since its automorphism group is not reductive (cf., e.g., [3]). Tian and Yau [13] proved that there is a Kähler-Einstein metric on one of the surfaces $\boldsymbol{CP}^2\#n\overline{\boldsymbol{CP}}^2$ $(3\leqq n\leqq 8)$, specified above.

Among these surfaces admitting Kähler-Einstein metrics, the following four surfaces: $\boldsymbol{CP}^1\times\boldsymbol{CP}^1$, $\boldsymbol{CP}^2\#3\overline{\boldsymbol{CP}}^2$, $\boldsymbol{CP}^2\#5\overline{\boldsymbol{CP}}^2$, and $\boldsymbol{CP}^2\#7\overline{\boldsymbol{CP}}^2$ can carry opposite almost complex structures [9]. We now state our second result as follows.

THEOREM 2. *A Kähler-Einstein metric g on one of the surfaces $\boldsymbol{CP}^2\#3\overline{\boldsymbol{CP}}^2$, $\boldsymbol{CP}^2\#5\overline{\boldsymbol{CP}}^2$, or $\boldsymbol{CP}^2\#7\overline{\boldsymbol{CP}}^2$ cannot be almost Kähler with respect to a g-orthogonal opposite almost complex structure.*

In the above two theorems, nonexistence of a *double* Kähler-Einstein metric is shown for some particular surfaces. On the other hand, for the existence we have the following.

THEOREM 3. *Let X be a double Kähler-Einstein surface with first Chern class $c_1(X)$.*

(i) *If $c_1(X)=0$ then X is a complex torus.*

(ii) *If $c_1(X)>0$, then X is $\boldsymbol{CP}^1\times\boldsymbol{CP}^1$.*

(iii) *If $c_1(X)<0$, then X is a surface of general type with the Chern numbers $c_1^2(X)$ and $c_2(X)$ satisfying $c_1^2(X)=2c_2(X)$ and $c_2\equiv 0 \bmod 2$.*

§ 3. Preliminary argument on opposite almost complex structures

Let X be a smooth oriented closed 4-manifold, with Hirzebruch index $\tau[X]$ and Euler characteristic $\chi[X]$. We may assume that the structure group of the tangent bundle TX has been reduced to $SO(4)$. We now suppose that X admits an almost complex structure J, i.e., there exists a characteristic element $w\in H^2(X, \boldsymbol{Z})$ such that $\mu(w, w)=3\tau[X]+2\chi[X]$, where μ is the intersection form on X and w is nothing but the first Chern class of the almost complex structureture [14]. In addition to the almost complex structure on X, we suppose that X admits an opposite almost complex structure J', i.e., there exists a characteristic element $w' \in H^2(X, \boldsymbol{Z})$ such that $\mu(w', w') = 3\tau[X] - 2\chi[X]$, where w' is considered as the first Chern class of the opposite almost complex structure [9].

The almost complex structures on $T_pX (\cong \boldsymbol{R}^4)$ at each point $p \in X$ are parametrized by a quotient space $SO(4)/U(2) \cong S^2$. Similarly, opposite almost complex structures on T_pX are parametrized by a quotient space $SO(4)/U'(2) \cong S^2$, where $U'(2)$ is an isotropy subgroup of a typical opposite almost complex structure at p (cf. [10]). (We are thus restricting our attention to orthogonal almost complex structures and orthogonal opposite almost complex structures for some Riemannian metric on X.) Note that the total spaces of the $SO(4)/U(2)$-bundle and the $SO(4)/U'(2)$-bundle over X are the twistor spaces $Z^+(X)$ and $Z^-(X)$ over X, respectively [9]. Both almost complex structure J and opposite almost complex structure J' are the linear transformations on the tangent space at each point of X with $J^2 = J'^2 = -1$. Such linear transformations can be written as anti-symmetric 4×4 matrices in $SO(4)$. The space of all anti-symmetric 4×4 matrices is identified with the Lie algebra $so(4)$. Due to the isomorphism $so(4) = so(3) \oplus so(3)$, we have

$$SO(4) \cap so(4) = \{SO(4) \cap (so(3) \oplus \{0\})\} \oplus \{SO(4) \cap (\{0\} \oplus so(3))\}.$$

Then, we can choose a basis K_1, K_2, K_3 for the former component $SO(4) \cap (so(3) \oplus \{0\})$, and K_1', K_2', K_3' for the latter $SO(4) \cap (\{0\} \oplus so(3))$ such that $[K_i, K_j] = 2K_k$ and $[K_i', K_j'] = 2K_k'$ for cyclic $i, j, k = 1, 2, 3$ and $[K_i, K_j'] = 0$ $(i, j = 1, 2, 3)$, with

$$K_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad K_1' = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

In terms of K_i and K_j', an almost complex structure J and an opposite almost complex structure J' at each point of X are written as follows:

$$J = aK_1 + bK_2 + cK_3, \qquad a^2 + b^2 + c^2 = 1$$

$$J' = a'K_1' + b'K_2' + c'K_3', \qquad a'^2 + b'^2 + c'^2 = 1.$$

From the above argument, we see the following basic fact (cf. Lemma 4.3 in [10]).

LEMMA 4. *If a 4-manifold X admits an almost complex structure and an opposite almost complex structure, then there exist a Riemannian metric g, a g-orthogonal almost complex structure J, and a g-orthogonal opposite almost complex structure J' such that J and J' commute with each other.*

From now on, let X be a compact complex surface with Chern numbers

$c_1^2(X)$ and $c_2(X)$. Concerning the existence of an opposite almost complex structure on a surface X, we have

PROPOSITION 5. *A complex surface X admits an opposite almost complex structure if and only if the second Chern number c_2 $(=\chi[X]$ the Euler characteristic) of X is even.*

Proof. The condition for X to admit an opposite almost complex structure is given in the form: $c_1^2-5c_2\equiv 0 \bmod 12$ [9]. It is known that the Chern number of any surface X must satisfy $c_1^2+c_2\equiv 0 \bmod 12$. Given the constraint $c_1^2+c_2\equiv 0 \bmod 12$, we see that the condition $c_1^2-5c_2\equiv 0 \bmod 12$ is equivalent to $c_2\equiv 0 \bmod 2$. □

COROLLARY 6. *Let X be one of the following surfaces: A K3 surface, or $CP^2\#n\overline{CP}^2$ with n odd, or $CP^1\times CP^1$, or a torus. Then, M admits an opposite almost complex structure.*

Proof. The second Chern numbers c_2 of a K3 surface, $CP^2\#n\overline{CP}^2$, $CP^1\times CP^1$, and a torus are 24, $n+3$, 4, and 0, respectively. The assertion follows from Proposition 5. □

§ 4. Proof of Theorems 1 and 2

In this section, we devote ourselves to the proof of Theorems 1 and 2.

Proof of Theorem 1. Let M be a K3 surface, endowed with a Kähler-Einstein metric g. We denote by J the complex structure on M such that (g, J) is the Kähler structure. From Corollary 6, M has an opposite almost complex structure. Moreover from Lemma 4, M has a g-orthogonal opposite almost complex structure, denoted by J'. Hence, the pair (g, J') is, at this stage, an opposite almost Hermitian structure. Concerning the integrability of J', we must recall Beauville's result.

LEMMA 7 ([1]). *Let X be a compact complex surface which admits an integrable opposite almost complex structure (opposite complex structure). Then, X must be*

(i) *a minimal rational surface $CP^1\times CP^1$, or*
(ii) *a minimal ruled surface of genus $\geqq 2$, or*
(iii) *a surface with $c_1^2=c_2=0$, or*
(iv) *a surface of general type with $c_2\equiv 0 \bmod 2$ and $c_1^2\geqq c_2$.*

Since $c_1^2(M)=0$ $(c_1(M)=0)$ and $c_2(M)=24$, no opposite almost complex structure on M is integrable. Thus, (g, J') cannot be opposite Hermitian, and of course it cannot be opposite Kähler. Then, we must investigate if (g, J') can be an opposite *almost* Kähler.

To this end, we now recall Sekigawa's Theorem [12] stating that any almost Kähler metric on a 4-manifold is Kähler if it is Einstein and if the scalar curvature of the metric is nonnegative. It is crucial to recognize that the *opposite* version of Sekigawa's Theorem holds, that is, an opposite almost Kähler structure on a 4-manifold must be opposite Kähler if the metric is Einstein and if the scalar curvature of the metric is nonnegative. Now, suppose that (g, J') on X is opposite almost Kähler. Since the scalar curvature of g is zero, the opposite version of Sekigawa's Theorem can be applied to the present case. Then, (g, J') must be opposite Kähler, which contradicts the fact that J' is not integrable. This proves that (g, J') is *strictly* almost Hermitian. □

Proof of Theorem 2. The surface $CP^2\#n\overline{CP}^2$ has the Chern numbers $c_1^2=9-n$ and $c_2=n+3$. Thus, we have $(c_1^2, c_2)=(6, 6)$, $(4, 8)$, and $(2, 10)$, for $CP^2\#3\overline{CP}^2$, $CP^2\#5\overline{CP}^2$, and $CP^2\#7\overline{CP}^2$, respectively.

Even though the Chern numbers satisfy the equality in the inequality in Lemma 7 (iv), the surface $CP^2\#3\overline{CP}^2$ is not of general type, and hence it is not a surface listed in Lemma 7. Thus, it does not admit an integrable opposite almost complex structure.

It is clear from the Chern numbers that the other two surfaces $CP^2\#5\overline{CP}^2$ and $CP^2\#7\overline{CP}^2$ are also not of the forms in Lemma 7. Therefore, no opposite almost complex structure on these surfaces is integrable.

We now consider a Kähler-Einstein metric g on one of the surfaces $CP^2\#n\overline{CP}^2$ for $n=3$, 5, or 7, together with a g-orthogonal opposite almost complex structure J' on it. From the above argument, (g, J') cannot be an opposite Kähler.

We then suppose that (g, J') is an opposite almost Kähler. The scalar curvature of the Einstein metric g is, in this case, positive, since $c_1>0$. Thus, the opposite version of Sekigawa's Theorem can be applied to the present case, too. This asserts that (g, J') must be opposite Kähler, which contradicts the fact that J' cannot be integrable. This proves the assertion. Finally, note that (g, J') is *strictly* opposite almost Hermitian. □

§ 5. Double Kähler-Einstein surfaces

We must mention about examples of double Kähler-Einstein surfaces, i.e., surfaces with Kähler-Einstein metrics which are also Kähler with respect to some opposite almost complex structures. Such examples are very few if the first Chern classes are positive or zero as shown in Theorem 3.

First, we shall prove the following

PROPOSITION 8. *Let X be a double Kähler-Einstein surface with Chern numbers c_1^2 and c_2. Then,*

$$c_1^2=2c_2\geqq 0\,.$$

Proof. Let g be a Kähler-Einstein metric on X. By the assumption, X admits an integrable g-orthogonal opposite almost complex structure J'. Let ρ be the Ricci curvature of g. Then, ρ defines two kinds of Ricci forms: $(1/2\pi)\rho(Jx, y)$ and $(1/2\pi)\rho(J'x, y)$ for any $x, y\in T_pX$, $p\in X$. These 2-forms represent the first Chern class c_1 of X (in fact, of J), and the first Chern class $c_1^{J'}$ determined by J', respectively. Moreover, there are two kinds of Kähler forms Ω and Ω', defined by $\Omega(x, y)=g(Jx, y)$ and $\Omega'(x, y)=g(J'x, y)$, respectively. Since g is Einstein, two kinds of first Chern classes are represented by the 2-forms $(S/8\pi)\Omega$ and $(S/8\pi)\Omega'$, where S is the scalar curvature. Since Ω and Ω' are self-dual and anti-self-dual, respectively, we have that $\Omega\wedge\Omega=-\Omega'\wedge\Omega'=2\omega$ (ω: the volume form on X). Therefore, we have

$$c_1^2=\int_X\left(\frac{S}{8\pi}\right)^2\Omega\wedge\Omega=\frac{S^2}{32\pi^2}\int_X\omega$$

$$(c_1^{J'})^2=c_1^2(-\bar{X})=\int_{-\bar{X}}\left(\frac{S}{8\pi}\right)^2\Omega'\wedge\Omega'=\frac{S^2}{32\pi}\int_X\omega\,.$$

Thus, $c_1^2(-\bar{X})=c_1^2$. On the other hand, the equality $c_1^2(-\bar{X})=4c_2-c_1^2$ holds [1], [9]. Therefore, we have the desired relation. □

We are now at a good position to prove Theorem 3.

Proof of Theorem 3. We must recall the fact that an Einstein 4-manifold with $\chi=0$ is flat [2], [5]. Therefore, if a surface X has an Einstein metric and $c_1^2=c_2=0$, then X must be a complex torus. In fact, the flat torus admits a double Kähler-Einstein structure. This leads to the first case (i).

Taking account of Theorem 2, we see that $\boldsymbol{CP}^1\times\boldsymbol{CP}^1$ is the only one candidate of double Kähler-Einstein surfaces with $c_1>0$. We must see that the surface, in fact, admits a desired structure. It is easy to see that the standard Kähler-Enstein metric on the surface $\boldsymbol{CP}^1\times\boldsymbol{CP}^1$ is also Kähler with respect to a canonical opposite almost complex structure on it, which is defined as the *sum* of the complex structure on the first component $\boldsymbol{CP}^1$ and the minus of the complex structure on the second $\boldsymbol{CP}^1$. Thus, we see the case (ii).

We consider the third case $c_1<0$. A surface with $c_1<0$ which admits an opposite complex structure must be a surface of general type with $c_2\equiv 0 \bmod 2$ and $c_1^2\geqq c_2$, listed in Lemma 7(iv). From Proposition 8, we see that the Chern numbers of a double Kähler-Einstein surface with $c_1<0$ must satisfy $c_1^2=2c_2$ and $c_2\equiv 0 \bmod 2$. This completes the proof. □

It is interesting to know if any surface of general type, which are not product of two curves of higher genus, with $c_1^2=2c_2$, can be double Kähler-

Einstein. It is still open whether or not there are any nonexistence theorems, similar to Theorems 1 and 2, for a double Kähler-Einstein structure on a surface of general type with negative first Chern class.

Moreover, it is recently reported in our paper [4], with focus on almost complex manifolds which carry holomorphic distributions, that a 4-manifold with two kinds of almost complex structures is double almost Kähler if and only if the manifold admits two complementary minimal foliations.

Sekigawa's Theorem is a partial answer to Goldberg's conjecture [6] for the case of *nonnegative* scalar curvature in dimension 4. Recently, as a step into the case of negative scalar curvature, Oguro and Sekigawa [11] reported that the hyperbolic spaces $\boldsymbol{H}^{2n}$ of dimension $2n(\geqq 4)$ can not admit an almost Kähler structure.

In this connection, an interesting problem will be: are there any Kähler-Einstein 4-manifolds of negative scalar curvature whose Kähler-Einstein metrics are strictly almost Kähler with respect to opposite almost complex structure?

Finally, we must note that it is still open whether or not any of a K3 surface and $CP^2\# n\overline{CP}^2$ ($n=3$, 5, and 7) admits a *strictly* opposite almost Kähler metric.

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