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HARMONIC DIMENSION OF COVERING SURFACES, II

Dedicated to Professor Fumi-Yuki Maeda on his sixtieth birthday

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Introduction

Let F be an open Riemann surface of null boundary which has a single ideal boundary component in the sense of Kerékjártó-Stoïlow (cf. [3, p. 98]). A relatively noncompact subregion Ω of F is said to be an *end* of F if the relative boundary $\partial\Omega$ consists of finitely many analytic Jordan curves (cf. Heins [4]). We denote by $\mathcal{P}(\Omega)$ the class of all nonnegative harmonic functions on Ω with vanishing values on $\partial\Omega$. The *harmonic dimension* of Ω, $\dim\mathcal{P}(\Omega)$ in notation, is defined as the minimum number of elements of $\mathcal{P}(\Omega)$ generating $\mathcal{P}(\Omega)$ provided that such a finite set exists, otherwise as ∞. It is well-known that $\dim\mathcal{P}(\Omega)$ dose not depend on a choice of end of F: $\dim\mathcal{P}(\Omega)=\dim\mathcal{P}(\Omega')$ for any pair (Ω, Ω') of ends of F (cf. [4]). In terms of the Martin compactification $\dim\mathcal{P}(\Omega)$ coincides with the number of minimal points over the ideal boundary (cf. Constantinesc and Cornea [3]).

In this note we especially consider ends W which are subregion of p-sheeted unlimited covering surfaces of $\{0<|z|\leqq\infty\}$. For these W it is known that $1\leqq\dim\mathcal{P}(W)\leqq p$ (cf. [4]). Consider two positive sequences $\{a_n\}$ and $\{b_n\}$ satisfying $b_{n+1}<a_n<b_n<1$ and $\lim_{n\to\infty}a_n=0$. Set $G=\{0<|z|<1\}-I$ where $I=\bigcup_{n=1}^{\infty}I_n$ and $I_n=[a_n, b_n]$. We take p (>1) copies $G_1, \cdots, G_p$ of G. Joining the upper edge of I_n on G_j and the lower edge of I_n on G_{j+1} ($j \bmod p$) for every n, we obtain a p-sheeted covering surface $W=W_p^I$ of $\{0<|z|<1\}$ which is naturally considered as an end of a p-sheeted covering surface of $\{0<|z|\leqq\infty\}$. In the previous paper [6] we proved the following.

THEOREM A ([6, Theorem]). *Suppose that* $p=2^m$ $(m\in\boldsymbol{N})$. *Then*
(i) $\dim\mathcal{P}(W)=p$ *if and only if* I *is thin at* $z=0$;
(ii) $\dim\mathcal{P}(W)=1$ *if and only if* I *is not thin at* $z=0$.

The purpose of this note is to show that, in a bit more general setting for I, Theorem A is valid for every p (>1) (cf. § 1). Consequently we have the following.

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THEOREM. *For every integer p (>1), it holds that*
(i) $\dim \mathcal{P}(W)=p$ *if and only if* I *is thin at* $z=0$;
(ii) $\dim \mathcal{P}(W)=1$ *if and only if* I *is not thin at* $z=0$.

In § 1 we give preliminaries and state Main Theorem. The proof of Main Theorem is given in § 2.

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1. Preliminaries from potential theory and statement of Main Theorem

1.1. We begin with recalling the definition of balayage. Consider an open Riemann surface F possessing the Green's function. Denote by $\mathcal{S}=\mathcal{S}(F)$ the class of all nonnegative superharmonic functions on F. Let E be a subset of F and s belong to $\mathcal{S}$. Then the *balayage* $\hat{R}_s^E={}^F\hat{R}_s^E$ of s relative to E on F is defined by

$$\hat{R}_s^E(z)=\liminf_{x\to z}\ \inf\{u(x): u\in\mathcal{S},\ u\geqq s \text{ on } E\}$$

(cf. e.g. [1]). Let $G_\xi^F(\cdot)$ be the Green's function on F with pole at ξ. We here review fundamental properties of balayage (cf. [1], [2], [5], etc.).

PROPOSITION 1.1. (i) *If* $E_1\subset E_2$ *then* $\hat{R}_s^{E_1}\leqq\hat{R}_s^{E_2}$;
(ii) $\hat{R}_s^{E_1\cup E_2}\leqq\hat{R}_s^{E_1}+\hat{R}_s^{E_2}$;
(iii) $\hat{R}_{u+v}^E=\hat{R}_u^E+\hat{R}_v^E$;
(iv) *if N is a polar set, then* $\hat{R}_s^{E\cup N}=\hat{R}_s^E$;
(v) *if E is a closed subset of F, then $\hat{R}_s^E(z)=s(z)$ on E except possibly for those $z\in\partial E$ which are irregular boundary points of $F-E$, and $\hat{R}_s^E=H_s^{F-E}$ on $F-E$, where $H_s^{F-E}={}^FH_s^{F-E}$ is the generalized Dirichlet solution for s on $F-E$.*

The following lemma gives us the relation between balayage on F and balayage on a covering surface of F.

LEMMA 1.1 (cf. [6, Lemma 3.1]). *Let $\tilde{F}$ be an unlimited covering surface of F, E a subset of F, s a positive superharmonic function on F and π the canonical projection from $\tilde{F}$ onto F. Then, it holds that*

$${}^F\hat{R}_s^E\circ\pi={}^{\tilde{F}}\hat{R}_{s\circ\pi}^{\pi^{-1}(E)}$$

on $\tilde{F}$.

Next we state the definition of thinness (cf. [2]).

DEFINITION 1.1. Let z be a point of F and E a subset of F. We say that E is thin at z if ${}^F\hat{R}_{G_z^F}^E\neq G_z^F$.

Assuming that E is closed and z belongs to ∂E in the above definition, it is well-known that E is thin at z if and only if z is an irregular point of $F-E$ with respect to Dirichlet problem (cf. e.g. [1, p. 348]).

1.2. In order to state Main Theorem, we begin with fixing the notations. Denote by D the open unit disc $\{|z|<1\}$. Let $\{J_n\}_{n=1}^{\infty}$ be a family of closed segments J_n in $(D-\{0\})\cap \boldsymbol{R}$ such that $J_n \cap J_m=\emptyset$ for every m and n with $m\neq n$ and that J_n accumulate only at $z=0$ in $D\cup\partial D$. Set $J=\cup_{n=1}^{\infty} J_n$ and $S=D-\{0\}-J$. We take p (>1) copies $S_1, \cdots, S_p$ of S. Joining the upper edge of J_n on S_j and the lower edge J_n on S_{j+1} ($j \bmod p$) for every n, we obtain a p-sheeted covering surface $W=W_p$ of $\{0<|z|<1\}$ which is naturally considered as an end of a p-sheeted covering surface of $\{0<|z|\leqq\infty\}$. Then, our previous paper [6] gives us the following results.

THEOREM B. *If J is thin at the origin, then* $\dim \mathcal{P}(W)=p$.

THEOREM C. *Suppose that $p=2^m$ $(m\in \boldsymbol{N})$. If neither of J and $\boldsymbol{R}-J$ is thin at the origin, then* $\dim \mathcal{P}(W)=1$.

We will prove that Theorem C holds for every integer p (>1).

THEOREM 1.1. *If neither of J and $\boldsymbol{R}-J$ is thin at the origin, then* $\dim \mathcal{P}(W)=1$.

By Theorems B and 1.1 we obtain Main Theorem.

MAIN THEOREM. *It holds that*
(i) $\dim \mathcal{P}(W)=p$ *if and only if J or $\boldsymbol{R}-J$ is thin at the origin*;
(ii) $\dim \mathcal{P}(W)=1$ *if and only if neither of J and $\boldsymbol{R}-J$ is thin at the origin.*

It is easily checked that Theorem in Introduction follows from Main Theorem.

2. Proof of Main Theorem

2.1. Here and hereafter, for simplicity, we denote by $G_\xi(\cdot)$ the Green's function on $\{|z|<1\}$ with pole at ξ. We first give the following lemma which is useful in the sequel:

LEMMA 2.1. *Let J and W_p be as in § 1, and K be the upper edge of J on S_1. Suppose that* $\dim \mathcal{P}(W)=1$. *If J is not thin at the origin, then, for every integer n $(1<n\leqq p)$,*

$${}^{W_n}\hat{R}^{K}_{G_0\circ\pi}=G_0\circ\pi$$

on W_n, where π is the canonical projection from W_n onto $D-\{0\}$.

Proof. Suppose that $\dim \mathcal{P}(W_p)=1$ and $p \geqq 2$. First we prove the assertion of this lemma for $n=p$. Let θ be a covering transformation of W_p:

$$\theta(z_i)=z_{i+1} \ (i \bmod p,\ i=1, \cdots, p),$$

where $\pi^{-1}(z)=\{z_1, \cdots, z_p\}$ and $z_i \in S_i$ for $z \in D-\{0\}$. We note that θ^p is the identity mapping on W_p. We set $K_i=\theta^{i-1}(K)$ $(i=1, \cdots, p)$. Since J is not thin at the origin, by Lemma 1.1 and (ii) of Proposition 1.1, we have,

$$(1) \qquad G_0 \circ \pi = {}^D\hat{R}^J_{G_0} \circ \pi = {}^{W_p}\hat{R}^{\pi^{-1}(J)}_{G_0 \circ \pi} \leqq \sum_{j=1}^{p} {}^{W_p}\hat{R}^{K_j}_{G_0 \circ \pi}$$

on W_p. By the fact $\dim \mathcal{P}(W_p)=1$ and by Naïm's theorem (cf. [3, Lemma 11.2]), ${}^{W_p}\hat{R}^{K_j}_{G_0 \circ \pi}$ $(j=1, \cdots, p)$ is equal to $G_0 \circ \pi$ or a Green potential on W_p, and hence, by (1), we can find an integer λ $(\leqq p)$ such that

$$(2) \qquad {}^{W_p}\hat{R}^{K_\lambda}_{G_0 \circ \pi} = G_0 \circ \pi$$

on W_p. By definition of balayage, we have, for every $z \in W_p$,

$$\begin{aligned}(3) \qquad {}^{W_p}\hat{R}^{K_\lambda}_{G_0 \circ \pi}(z) &= \liminf_{y \to z} \inf\{s(y) \mid s \in \mathcal{S}(W_p),\ s \geqq G_0 \circ \pi \text{ on } K_\lambda\} \\ &= \liminf_{y \to z} \inf\{(s \circ \theta^{\lambda-1})(\theta^{1-\lambda})(y) \mid s \circ \theta^{\lambda-1} \in \mathcal{S}(W_p),\ s \circ \theta^{\lambda-1} \geqq G_0 \circ \pi \text{ on } K\} \\ &= {}^{W_p}\hat{R}^{K}_{G_0 \circ \pi}(\theta^{1-\lambda}(z))\end{aligned}$$

on W_p. Therefore, by (2) and (3), we have

$$(4) \qquad {}^{W_p}\hat{R}^{K_j}_{G_0 \circ \pi} = G_0 \circ \pi$$

on W_p $(j=1, \cdots, p)$.

Next we prove the assertion of this lemma for $n=p-1$ $(p>2)$. By (4) and (i) of Proposition 1.1, we have

$$G_0 \circ \pi = {}^{W_p}\hat{R}^{K_p}_{G_0 \circ \pi} \leqq {}^{W_p}\hat{R}^{K_{p-1} \cup K_p}_{G_0 \circ \pi} \leqq G_0 \circ \pi$$

on W_p, and hence,

$$(5) \qquad {}^{W_p}\hat{R}^{K_{p-1} \cup K_p}_{G_0 \circ \pi} = G_0 \circ \pi$$

on W_p. Thus, by (v) of Proposition 1.1 and definition of generalized Dirichlet solution (cf. [1]), we have, for every $z \in W_p-(S_p \cup K_{p-1} \cup K_p)$,

$$\begin{aligned}(G_0 \circ \pi)(z) &= {}^{W_p}\hat{R}^{K_{p-1} \cup K_p}_{G_0 \circ \pi}(z) = {}^{W_p}H^{W_p-(S_p \cup K_{p-1} \cup K_p)}_{G_0 \circ \pi}(z) \\ &= {}^{W_{p-1}}H^{W_{p-1}-K_{p-1}}_{G_0 \circ \pi}(z) = {}^{W_{p-1}}\hat{R}^{K_{p-1}}_{G_0 \circ \pi}(z) = {}^{W_{p-1}}\hat{R}^{K}_{G_0 \circ \pi}(\theta^{2-p}(z)),\end{aligned}$$

where we consider a point of $W_{p-1}-K_{p-1}$ as a point of $W_p-(S_p \cup K_{p-1} \cup K_p)$. Hence we have

$${}^{W_{p-1}}\hat{R}^{K}_{G_0 \circ \pi} = G_0 \circ \pi$$

on W_{p-1}.

For a general integer n $(1<n\leqq p)$, repeating the same argument successively as in the case: $n=p-1$, we obtain the desired result.

2.2. *Proof of Theorem* 1.1. For a point $z\in W=W_p$ which belongs to S_i $(i=1, \cdots, p)$, we denote by $\bar{z}$ the point in S_i whose projection coincides with $\overline{\pi(z)}$. Let f be a mapping on W_p defined by the following fashion; for $z_i\in S_i$ $(i=1, \cdots, p)$ with $\pi(z_i)=z\in D-\{0\}$,

$$f(z_j)=\overline{z_{p+2-j}} \quad (j \bmod p,\ j=1, \cdots, p). \tag{6}$$

Observe that f is an anti-conformal automorphism of W and that $f\circ f=\mathrm{id}$.

First, we prove that, if h is an element of $\mathcal{P}(W)$ such that $h\circ f=h$ on W, there exists a positive constant α such that

$$h=\alpha G_0\circ\pi \tag{7}$$

on W. Letting θ be the covering transformation of W as in the proof of Lemma 2.1, we can find a positive constant β such that

$$\beta G_0\circ\pi=\sum_{j=1}^{p} h\circ\theta^j \tag{8}$$

on W. Let K be the upper edge of J on S_1. Since neither of J and $\boldsymbol{R}-J$ is thin at the origin, by Theorem C and Lemma 2.1 we have

$$\beta G_0\circ\pi=\hat{R}^K_{\beta G_0\circ\pi}\ (={}^W\hat{R}^K_{\beta G_0\circ\pi}) \tag{9}$$

on W. By (8), (9) and (iii) of Proposition 1.1, we have

$$\beta G_0\circ\pi=\hat{R}^K_{\Sigma_{j=1}^p h\circ\theta^j}=\sum_{j=1}^{p}\hat{R}^K_{h\circ\theta^j}\leqq\sum_{j=1}^{p} h\circ\theta^j=\beta G_0\circ\pi \tag{10}$$

on W, and hence,

$$h\circ\theta^j=\hat{R}^K_{h\circ\theta^j} \tag{11}$$

on W $(j=1, \cdots, p)$. On the other hand, we find that

$$h\circ\theta^{p-1}=h \tag{12}$$

on K, because $h\circ f=h$ on W. By (11) and (12), we have

$$h\circ\theta^{p-1}=\hat{R}^K_{h\circ\theta^{p-1}}=\hat{R}^K_h=h \tag{13}$$

on W, and hence,

$$h\circ\theta=h \tag{14}$$

on W. By (14) we can consider h as an element of $\mathcal{P}(D-\{0\})$ and hence, there exists a positive constant α such that the equation (7) holds.

Next, let $h\in\mathcal{P}(W)$ be a minimal function on W. Setting

$$K'=(\pi^{-1}(\boldsymbol{R}-J-\{0\}))\cap S_1,$$

we prove that there exists an integer μ $(1\leqq\mu\leqq p)$ such that

$$h\circ\theta^{\mu}=\hat{R}^{K'}_{h\circ\theta^{\mu}} \tag{15}$$

on W. The assumption that $\boldsymbol{R}-J$ is not thin at the origin implies that $(\boldsymbol{R}-J)\cap D$ is not thin at the origin (cf. e. g. [2]) and hence, by (iv) of Proposition 1.1, $J'=(\boldsymbol{R}-J-\{0\})\cap D$ is not thin at the origin. Thus, by Lemma 1.1, (8) and (iii) of Proposition 1.1, we have

$$\begin{aligned}\beta G_0\circ\pi&=\hat{R}^{J'}_{G_0}\circ\pi=\hat{R}^{\pi^{-1}(J')}_{\beta G_0\circ\pi}=\hat{R}^{\pi^{-1}(J')}_{\Sigma_{j=1}^{p}h\circ\theta^j}\\&=\sum_{j=1}^{p}\hat{R}^{\pi^{-1}(J')}_{h\circ\theta^j}\leqq\sum_{j=1}^{p}h\circ\theta^j=\beta G_0\circ\pi\end{aligned} \tag{16}$$

on W and hence,

$$h\circ\theta^j=\hat{R}^{\pi^{-1}(J')}_{h\circ\theta^j} \tag{17}$$

on W $(j=1, \cdots, p)$. Setting

$$K'_j=\theta^j(K') \quad (j=1, \cdots, p),$$

by (17) and (ii) of Proposition 1.1, we have

$$h=\hat{R}^{\pi^{-1}(J')}_{h}=\hat{R}^{\cup_{j=1}^{p}K'_j}_{h}\leqq\sum_{j=1}^{p}\hat{R}^{K'_j}_{h} \tag{18}$$

on W. Since h is a minimal harmonic function on W, by Naïm's theorem (cf. [3, Lemma 11.2]), $\hat{R}^{K'_j}_{h}$ $(j=1, \cdots, p)$ is equal to h or a Green potential on W, and hence, by (18), we find an integer μ $(1\leqq\mu\leqq p)$ such that

$$\hat{R}^{K'_\mu}_{h}=h \tag{19}$$

on W. By (19) and definition of balayage, we have

$$h=\hat{R}^{K'_\mu}_{h}=\hat{R}^{\theta^{\mu}(K')}_{h}=(\hat{R}^{K'}_{h\circ\theta^{\mu}})\circ\theta^{-\mu} \tag{20}$$

on W and hence, we obtain the equation (15).

Finally, we prove that, if $h\in\mathcal{P}(W)$ be a minimal function on W, there exists a positive constant γ such that $h=\gamma G_0\circ\pi$ on W. We set

$$H=h\circ\theta^{\mu}+h\circ\theta^{\mu}\circ f, \tag{21}$$

where λ is the same integer as in (15). Since $H\circ f=H$, we see from the first observation that there exists a positive constant δ such that

$$h\circ\theta^{\mu}+h\circ\theta^{\mu}\circ f=\delta G_0\circ\pi \tag{22}$$

on W and hence, we have

(23) $$h\circ\theta^{\mu}=h\circ\theta^{\mu}\circ f=\frac{\delta}{2}G_0\circ\pi$$

on K'. By (15) and (20) we have

(24) $$h\circ\theta^{\mu}=\hat{R}^{K'}_{h\circ\theta^{\mu}}=\hat{R}^{K'}_{h\circ\theta^{\mu}\circ f}\leqq h\circ\theta^{\mu}\circ f$$

on W. Since $h\circ\theta^{\mu}\circ f$ is a minimal harmonic function on W, by (23), we find that

(25) $$h\circ\theta^{\mu}=h\circ\theta^{\mu}\circ f$$

on W and hence, by (22), we have

(26) $$h\circ\theta^{\mu}=\frac{\delta}{2}G_0\circ\pi$$

on W. Therefore we have the desired result.

2.3. *Proof of Theorem.* In order to prove Main Theorem, by Theorem B and 1.1, we have only to prove that, if $\boldsymbol{R}-J$ is thin at the origin, $\dim\mathcal{P}(W)=p$. Denote by $\{J'_n\}_{n=1}^{\infty}$ the family of connected components of $(\boldsymbol{R}-J-\{0\})\cap D$ and by $\bar{J}'_n$ the closure of J'_n for each n. By replacing $\{J_n\}_{n=1}^{\infty}$ in 1.2 with $\{J'_n\}_{n=1}^{\infty}$, we construct a p-sheeted covering surface W' of $\{0<|z|<1\}$ in the same way as in 1.2. Then $\cup_{n=1}^{\infty}\bar{J}'_n$ is thin at the origin and hence Theorem B yields that $\dim\mathcal{P}(W')=p$. Therefore we find that $\dim\mathcal{P}(W)=p$ since W is conformally equivalent to W'.

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