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ON NONLINEAR, NONCONVEX EVOLUTION INCLUSIONS

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Abstract

We consider a nonlinear evolution inclusion driven by an m-accretive operator which generates an equicontinuous nonlinear semigroup of contractions. We establish the existence of extremal integral solutions and we show that they form a dense, G_δ-subset of the solution set of the original Cauchy problem. As an application, we obtain "bang-bang" type theorems for two nonlinear parabolic distributed parameter control systems.

1. Introduction

Let $T=[0, b]$ and X a separable reflexive Banach space, whose dual X^* is uniformly convex. We consider the following multivalued Cauchy problems:

$$(1)\qquad \left\{\begin{array}{c} -\dot{x}(t)\in Ax(t)+F(t, x(t)) \\ x(0)=x_0 \end{array}\right\},$$

$$(2)\qquad \text{and} \left\{\begin{array}{c} -\dot{x}(t)\in Ax(t)+\operatorname{ext} F(t, x(t)) \\ x(0)=x_0 \end{array}\right\}.$$

Here $A: D\subseteqq X\rightarrow 2^X\setminus\{\emptyset\}$ is an m-accretive operator, $F: T\times X\rightarrow 2^X\setminus\{\emptyset\}$ is a multifunction and $\operatorname{ext} F(t, x)$ denotes the extreme points of the set $F(t, x)$. By a solution of (1) (resp. of (2)), we mean a function $x(\cdot)\in C(T, X)$ which is an integral solution in the sense of Benilan (see section 2) of the Cauchy problem

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$-\dot{x}(t)\in Ax(t)+f(t)$, $x(0)=x_0$, with $f\in L^1(T, X)$, $f(t)\in F(t, x(t))$ a.e. (resp. $f(t)\in \operatorname{ext} F(t, x(t))$ a.e.). By $S(x_0)$ (resp. $S_e(x_0)$) we will denote the set of solutions of (1) (resp. (2)). The purpose of this paper is to establish the nonemptiness of $S_e(x_0)$ and show that it is dense in $S(x_0)$ for the $C(T, X)$-norm (strong relaxation theorem). Our work here extends that of DeBlasi-Pianigiani [8] (in particular, Theorems 4.1 and 5.1), where $A\equiv 0$. Also it extends the results of Avgerinos-Papageorgiou [1] and Mitiederi-Vrabie [13], who treated problem (1) with a nonconvex, closed valued perturbation term $F(t, x)$ which was assumed to be lower semicontinuous (l.s.c.) in $x\in X$. We remark that the multifunction $(t, x)\to \operatorname{ext} F(t, x)$ in general is not closed-valued and we can not say anything about its continuity properties with respect to x, even if $x\to F(t, x)$ is regular enough (say Hausdorff-continuous (h-continuous)).

2. Preliminaries

Let (Ω, Σ) be a measurable space and X a separable Banach space. By $P_{f(c)}(X)$ we will denote the collection of all nonempty, closed (and convex) subsets of X. A multifunction (set-valued function) $F:\Omega\to P_f(X)$ is said to be measurable, if for all $z\in X$ $\omega\to d(z, F(\omega))=\inf\{\|z-x\| : x\in F(\omega)\}$ is measurable. Let $\mu(\cdot)$ be a σ-finite measure on Σ. By S_F^1 we will denote the set of selectors of $F(\cdot)$ that belong in the Lebesgue-Bochner space $L^1(\Omega, X)$; i.e. $S_F^1=\{f\in L^1(\Omega, X): f(\omega)\in F(\omega)\ \mu\text{-a.e.}\}$. Using Aumann's selection theorem (see Wagner [16], Theorem 5.10), we can show that for a measurable multifunction $F:\Omega\to P_f(X)$, S_F^1 is nonempty if and only if $\omega\to\inf\{\|x\| : x\in F(\omega)\}\in L^1(\Omega)$. Note that the set S_F^1 is decomposable in the sense that, if $(f_1, f_2, A)\in S_F^1\times S_F^1\times\Sigma$, then $\chi_A f_1+\chi_{A^c} f_2\in S_F^1$ (here χ_A (resp. χ_{A^c}) denotes the characteristic function of A (resp. of A^c)).

On $P_f(X)$ we can define a generalized metric, known in the literature as the Hausdorff metric, by setting

$$h(A, B)=\max\left[\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(b, A)\right]$$

for all $A, B\in P_f(X)$. It is well-known that $(P_f(X), h)$ is a complete metric space (see Klein-Thompson [18]). A multifunction $G: X\to P_f(X)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from X into the metric space $(P_f(X), h)$.

Let Y, Z be Hausdorff topological spaces and $G: Y\to 2^Z\setminus\{\emptyset\}$. We say that $G(\cdot)$ is lower semicontinuous (l.s.c.), if for all $U\subseteq Z$ open, the set $G^-(U)=\{y\in Y: G(y)\cap U\neq\emptyset\}$ is open. If Y, Z are metric spaces, then lower semicontinuity of $G(\cdot)$ is equivalent to saying that if $y_n\to y$ in Y, then $G(y)\subseteq \underline{\lim}\, G(y_n)=\{z\in Z: z=\lim z_n,\ z_n\in G(y_n),\ n\geqq 1\}=\{z\in Z: \lim d(z, G(y_n))=0\}$.

Let $J: X\to 2^{X^*}$ be the duality map of X; i.e. $J(x)=\{x^*\in X^*: (x^*, x)=\|x\|^2=\|x^*\|^2\}$ for each $x\in X$. Clearly $J(x)$ is a closed, convex and bounded subset of X^* and because of the Hahn-Banach theorem is nonempty for every $x\in X$.

Recall that if X^* is strictly convex, then $J(\cdot)$ is single valued and if X^* is uniformly convex, then $J: X \to X^*$ is continuous and uniformly continuous on bounded subsets of X (see Zeidler [17]). If X is a pivot Hilbert space (i.e. X is identified with its dual), then $J=$Identity. Using the duality map we can define an upper semi-inner product on X (denoted by $(\cdot, \cdot)_+$) as follows:

$$(x, y)_+ = \sup\{(x^*, y): x^* \in J(x)\}$$

for all $x, y \in X$. An operator $A: D \subseteq X \to 2^X$ is said to be "accretive" if and only if $(x-x', y-y')_+ \geqq 0$ for each $x, x' \in D$ and $y \in Ax$, $y' \in Ax'$. We say that A is "m-accretive" if it is accretive and in addition $R(I+\lambda A)=X$ for each $\lambda>0$. It is well-known (cf. Barbu [3]) that an m-accretive operator generates a semigroup $\{K(t)\}_{t\in T}$ of nonlinear contractions via the Crandall-Liggett exponential formula $K(t)x = \lim_{n\to\infty}(I+(t/n)A)^{-n}x$, $t \in T$, $x \in \bar{D}$.

We will say that the semigroup of contractions $K(t)(\cdot)$, $t \in T$ is "equicontinuous", if for every bounded subset $B \subseteq D$, the family $\{K(\cdot)x : x \in B\}$ is equicontinuous at each $t \in (0, b]$. This is the case if $K(t)(\cdot)$ is compact for each $t \in (0, b]$ or if $A=\partial\varphi$ (the subdifferential of a proper, convex, l.s.c. function) or if A is locally bounded or finally if A is a homogeneous m-accretive operator (see Gutman [11]).

Now let $T=[0, b]$, $A: D \subseteq X \to 2^X$ is an m-accretive operator, $f \in L^1(T, X)$ and $x_0 \in \bar{D}$. We consider the following Cauchy problem

$$\left\{ \begin{array}{c} -\dot{x}(t) \in Ax(t)+f(t) \\ x(0)=x_0 \end{array} \right\}. \tag{3}$$

Following Benilan [5], we say that a function $x(\cdot) \in C(T, X)$ is an "integral solution" of (3), if $x(0)=x_0$ and

$$\|x(t)-y\|^2 \leqq \|x(s)-y\|^2 + \int_s^t (x(\tau)-y, f(\tau)-z)_+ d\tau$$

for each $(y, z) \in \mathrm{Gr}A$ and $0 \leqq s \leqq t \leqq b$. It is well-known that under the above hypotheses, the Cauchy problem (3) has a unique integral solution, which depends continuously on the data of the problem $(x_0, f) \in \bar{D} \times L^1(T, X)$; i.e., if $x_1(\cdot) \in C(T, X)$ is the integral solution corresponding to the pair (x_{01}, f_1) and $x_2(\cdot) \in C(T, X)$ is the integral solution corresponding to (x_{02}, f_2), then we have

$$\|x_1(t)-x_2(t)\|^2 \leqq \|x_1(s)-x_2(s)\|^2 + 2\int_s^t (x_1(\tau)-x_2(\tau), f_1(\tau)-f_2(\tau))_+ d\tau \quad \text{for } s, t \in T \text{ with } s \leqq t,$$

and so

$$\|x_1(t)-x_2(t)\| \leqq \|x_{01}-x_{02}\| + \int_0^t \|f_1(s)-f_2(s)\| ds, \quad t \in T,$$

(Benilan's inequalities). Note that if A is linear, closed and densely defined, then the notion of integral solution coincides with that of mild solution. Also recall that every strong solution is an integral solution. For further details we refer to Barbu [3].

Finally by $L^1_w(T, X)$ we will denote the space of equivalence classes of Bochner integrable functions $h: T \to X$, with the ("weak") norm $\|h\|_w = \sup\{\|\int_t^{t'} h(s)ds\| : 0 \leq t \leq t' \leq b\}$. The notation $\xrightarrow{\|\cdot\|_w}$ stands for convergence in $L^1_w(T, X)$.

3. Existence of extremal solutions

In this section we establish the nonemptiness of $S_e(x_0)$ (i.e. of the set of integral solutions for the multivalued Cauchy problem (2)).

We will need the following auxiliary result:

LEMMA 3.1. *If X is a Banach space, $\{x_n^*, x^*\}_{n\geq 1} \subseteq X^*$ and $x_n^* \xrightarrow{w^*} x^*$, then $(x_n^*, u) \to (x^*, u)$ uniformly for all $u \in K \subseteq X$ compact.*

Remark. In the above lemma $\xrightarrow{w^*}$ denotes convergence in the w^*-topology and $(\cdot, \cdot)$ denotes the duality brackets for the pair (X^*, X).

To establish the nonemptiness of $S_e(x_0)$, we will need the following hypotheses on the data. Here $T=[0, b]$ and X is a separable reflexive Banach space whose dual X^* is uniformly convex.

$H(A)$: $A: D \subseteq X \to 2^X$ is an m-accretive operator, which generates an equicontinuous semigroup of nonlinear contractions $\{K(t)(\cdot)\}_{t\in T}$.

$H(F)$: $F: T \times X \to P_{fc}(X)$ is a multifunction such that
(1) $t \to F(t, x)$ is measurable,
(2) $x \to F(t, x)$ is h-continuous,
(3) $|F(t, x)| = \sup\{\|v\| : v \in F(t, x)\} \leq a(t) + c(t)\|x\|$ a.e. with $a, c \in L^1(T)$,
(4) for every $B \subseteq X$ bounded, $\overline{F(t, B)} = \overline{\bigcup_{x\in B} F(t, x)}$ is compact.

THEOREM 3.2. *If hypotheses $H(A)$, $H(F)$ hold and $x_0 \in \bar{D}$, then $S_e(x_0) \neq \emptyset$.*

Proof. First let us obtain an a priori bound for the elements in $S(x_0)$ (hence for the elements of $S_e(x_0)$ too). So let $x \in S(x_0)$. From Benilan's inequality (see section 2), we have

$$\|x(t) - K(t)x_0\| \leq \int_0^t \|f(s)\| ds$$

with $f \in S^1_{F(\cdot, x(\cdot))}$ (recall that $y(t) = K(t)x_0$ is the unique integral solution of (3) when $f \equiv 0$). Since $\|K(t)x_0\| \leq M$, $M > 0$, for all $t \in T$, we have

$$\|x(t)\| \leqq M + \int_0^t (a(s)+c(s)\|x(s)\|)ds, \quad t \in T.$$

Invoking Gronwall's lemma, we deduce the existence of $M_1>0$ such that

$$\|x(\cdot)\|_{C(T,X)} \leqq M_1$$

for all $x \in S(x_0)$. Let $B_{M_1}=\{x \in X : \|x\| \leqq M_1\}$. Then because of hypothesis $H(F)$ (4), $\overline{F(t, B_{M_1})}=V(t)$ is compact in X for every $t \in T$. Also if $\overline{\{y_n\}}^{\|\cdot\|}_{n\geq 1}=B_{M_1}$, then because of hypothesis $H(F)$ (2) $V(t)=\overline{F(t, B_{M_1})}=\overline{\bigcup_{n\geq 1} F(t, y_n)}$ and so $t \rightarrow V(t)$ is measurable. Also observe that $|V(t)|=\sup\{\|v\| : v \in V(t)\} \leqq a(t)+c(t)m_1=\psi(t)$ a.e. with $\psi(\cdot) \in L^1(T)$. Now define $S_V=\{v \in L^1(T, X) : v(t) \in V(t) \text{ a.e.}\}$. This is a nonempty, weakly compact subset of $L^1(T, X)$ (Dunford-Pettis theorem). If $p : L^1(T, X) \rightarrow C(T, X)$ is the map which to each $f \in L^1(T, X)$ assigns the unique integral solution of (3), from hypothesis $H(A)$ and Theorem 2.3 of Gutman [11], we have that $p(S_V) \subseteqq C(T, X)$ is equicontinuous. In fact, we claim that the set $p(S_V)$ is compact in $C(T, X)$. Indeed let $\{x_n\}_{n\geq 1} \subseteqq p(S_V)$. Then for every $n \geqq 1$, $x_n=p(f_n)$ with $f_n \in S_V$. By passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^1(T, X)$, $f \in S_V$. Let $x=p(f)$. Then from Benilan's inequality, we have

$$\|x_n(t)-x(t)\|^2 \leqq \int_0^t (J(x_n(s)-x(s)), f_n(s)-f(s))ds.$$

Recall that since X^* is uniformly convex $\{u_n(\cdot)=J(x_n(\cdot)-x(\cdot))\}_{n\geq 1} \subseteqq C(T, X^*)$ and furthermore since the continuity of $J(\cdot)$ is uniform on bounded subsets of X and $\{x_n(\cdot)\}_{n\geq 1} \subseteqq C(T, X)$ is equicontinuous, we have that $\{u_n(\cdot)\}_{n\geq 1} \subseteqq C(T, X^*)$ is equicontinuous, and of course bounded. Since bounded subsets in X^* furnished with the relative weak topology, are compact, metrizable, from the Arzela-Ascoli theorem (see also Lakshmikantham-Leela [12], Theorem 1.1.6, p. 5), we deduce that $\{u_n\}_{n\geq 1}$ is relatively sequentially compact in $C(T, X^*_w)$. Here X^*_w denotes the dual space X^* equipped with the weak topology. So by passing to a subsequence if necessary, we may assume that $u_n \rightarrow u$ in $C(T, X^*_w)$. Then we have:

$$\begin{aligned}
&\int_0^t (J(x_n(s)-x(s)), f_n(s)-f(s))ds \\
&= \int_0^t (u_n(s), f_n(s)-f(s))ds \\
&= \int_0^t (u_n(s)-u(s), f_n(s)-f(s))ds + \int_0^t (u(s), f_n(s)-f(s))ds.
\end{aligned}$$

Since $u(\cdot) \in C(T, X^*_w) \Rightarrow u \in L^\infty(T, X^*)=L^1(T, X)^*$ and so

$$\int_0^t (u(s), f_n(s)-f(s))ds \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Also note that $f_n(s)-f(s)\in V(s)-V(s)=W(s)$, $s\in T$ and the latter is compact in X. Since $u_n(s)\xrightarrow{w}u(s)$ in X^*, from Lemma 3.1, we have

$$\sup_{w\in W(s)}|(u_n(s)-u(s),\ w)|\to 0 \quad \text{as } n\to\infty,$$

$$\Rightarrow\int_0^t(u_n(s)-u(s),\ f_n(s)-f(s))ds\to 0 \quad \text{as } n\to\infty,$$

$$\Rightarrow\int_0^t(J(x_n(s)-x(s)),\ f_n(s)-f(s))ds\to 0 \quad \text{as } n\to\infty,$$

$$\Rightarrow\|x_n(t)-x(t)\|\to 0 \quad \text{as } n\to\infty \text{ for all } t\in T.$$

Therefore for all $t\in T$, $\overline{\{x_n(t)\}}_{n\geqq 1}$ is compact in X and since $\{x_n(\cdot)\}_{n\geqq 1}$ is equicontinuous, from the Arzela-Ascoli theorem, we conclude that $x_n\to x$ in $C(T, X)$ and $x=p(f)$, $f\in S_V$; i.e. $p(S_V)$ is compact in $C(T, X)$ as claimed. Let $\Gamma=\overline{\text{conv}}\, p(S_V)$. Then $\Gamma\subseteqq C(T, X)$ is compact (Mazur's theorem).

Next let $R:\Gamma\to P_{fc}(L^1(T, X))$ be defined by $R(x)=S^1_{F(\cdot, x(\cdot))}$. Apply Tolstonogov's selection theorem [15], to get $r:\Gamma\to L^1_w(T, X)$ continuous, such that $r(x)\in\text{ext}\, R(x)$ for all $x\in\Gamma$. From Benamara [4], we know that $\text{ext}\, R(x)=\text{ext}\, S^1_{F(\cdot, x(\cdot))}=S^1_{\text{ext}\, F(\cdot, x(\cdot))}$. Then let $\gamma:\Gamma\to\Gamma$ be defined by $\gamma=p\circ r$. We claim that $\gamma(\cdot)$ is continuous. To this end let $x_n\to x$ in $C(T, X)$, $x_n\in\Gamma$, $n\geqq 1$. Note that for all $n\geqq 1$ and almost all $t\in T$, $r(x_n)(t)\in K(t)$ and the latter set is compact. Since $r(x_n)\xrightarrow{\|\cdot\|_w}r(x)$, we can apply the result of Gutman [10] and get that $r(x_n)\xrightarrow{w}r(x)$ in $L^1(T, X)$. Then exactly as before we can have that $\gamma(x_n)=p(r(x_n))\to\gamma(x)=p(r(x))\Rightarrow\gamma(\cdot)$ is continuous on Γ as claimed. Apply Schauder's fixed point theorem to get $x=\gamma(x)$. Then $x(\cdot)\in S_e(x_0)$. Q.E.D.

4. Strong relaxation theorem

In this section, by strengthening our hypothesis on the orientor field, we show that $S_e(x_0)$ is a dense, G_δ-subset of $S(x_0)$ for the $C(T, X)$-topology. The new hypothesis on the multivalued perturbation $F(t, x)$ is the following:

$H(F)_1$: $F:T\times X\to P_{fc}(X)$ is a multifunction such that
- (1) $t\to F(t, x)$ is measurable,
- (2) $h(F(t, x), F(t, y))\leqq k(t)\|x-y\|$ a.e., with $k\in L^1(T)$,
- (3) $|F(t, x)|=\sup\{\|v\|: v\in F(t, x)\}\leqq a(t)+c(t)\|x\|$ a.e. with $a(\cdot)$, $c(\cdot)\in L^1(T)$,
- (4) for all $B\subseteqq X$ bounded, $\overline{F(t, B)}=\overline{\bigcup_{x\in B}F(t, x)}$ is compact in X.

THEOREM 4.1. *If hypotheses $H(A)$, $H(F)_1$ hold and $x_0\in\bar{D}$, then $S_e(x_0)$ is a dense, G_δ-subset of $S(x_0)$ for the $C(T, X)$-topology.*

Proof. Let $\Gamma \subseteq C(T, X)$ be the compact, convex set as in the proof of Theorem 3.2. Let $x \in S(x_0)$. Then by definition $x = p(f)$ with $f \in S^1_{F(\cdot, x(\cdot))}$. Given $y \in \Gamma$ and $\varepsilon > 0$ define $H: T \to 2^X \setminus \{\emptyset\}$ by

$$H(t) = \left\{ u \in X : \|f(t) - u\| < \frac{\varepsilon}{2M_1 b} + d(f(t), F(t, y(t))),\ u \in F(t, y(t)) \right\}$$

where $M_1 > 0$ is the a priori bound for the elements in $S(x_0)$ established in the proof of Theorem 3.2. Note that because of hypotheses $H(F)_1$ (1) and (2), $(t, x) \to F(t, x)$ is measurable and so $\mathrm{Gr}H = \{(t, u) \in T \times X : u \in H(t)\} \in B(T) \times B(X)$, with $B(T)$ (resp. $B(X)$) being the Borel σ-field of T (resp. of X). So we apply Aumann's selection theorem and get $u: T \to X$ measurable such that $u(t) \in H(t)$ a.e. Thus if we define $L: \Gamma \to 2^{L^1(T, X)}$ by

$$L(y) = \left\{ u \in S^1_{F(\ ,y(\cdot))} : \|f(t) - u(t)\| \leq \frac{\varepsilon}{2M_1 b} + d(f(t), F(t, y(t)))\ \text{a.e.} \right\}$$

we have just seen that this multifunction has nonempty values which are decomposable (see section 2) and furthermore from Proposition 2.3 of Fryszkowski [9], we also have that it is l.s.c. Hence $y \to \overline{L(y)}$ is l.s.c. with decomposable values. So we apply Theorem 3.1 of Fryszkowski [9] and get $u_\varepsilon: \Gamma \to L^1(T, X)$ a continuous map such that $u_\varepsilon(y) \in \overline{L(y)}$ for all $y \in \Gamma$. Therefore we have:

$$\begin{aligned} \|f(t) - u_\varepsilon(y)(t)\| &\leq \frac{\varepsilon}{2M_1 b} + d(f(t), F(t, y(t))) \\ &\leq \frac{\varepsilon}{2M_1 b} + k(t)\|x(t) - y(t)\| \ \text{a.e. (cf. hypothesis } H(F)\text{ (2)).} \end{aligned}$$

Also from Tolstonogov's selection theorem [15], we can get $v_\varepsilon: \Gamma \to L^1(T, X)$ continuous such that for all $y \in \Gamma$, $v_\varepsilon(y) \in \mathrm{ext}\, \Gamma(y)$ (hence $v_\varepsilon(y) \in S^1_{\mathrm{ext}\, F(\cdot, y(\cdot))}$ and

$$\|u_\varepsilon(y) - v_\varepsilon(y)\|_w < \varepsilon .$$

Next let $\varepsilon_n \downarrow 0$ and set $u_n = u_{\varepsilon_n}$ and $v_n = v_{\varepsilon_n}$. As in the proof of Theorem 3.2, via Schauder's fixed point theorem, we can find $x_n \in \Gamma$ such that $x_n = p(v_n(x_n))$. Note that $x_n \in S_e(x_0)$ $n \geq 1$. Since $\{x_n\}_{n \geq 1} \subseteq \Gamma$ and the latter is compact in $C(T, X)$, by passing to a subsequence if necessary, we may assume that $x_n \to \hat{x}$ in $C(T, X)$. Using Benilan's inequality, we get

$$\begin{aligned} \|x_n(t) - x(t)\|^2 &\leq 2\int_0^t (J(x_n(s) - x(s)), v_n(s) - f(s))ds \\ &\leq 2\int_0^t (J(x_n(s) - x(s)), v_n(s) - u_n(s))ds \\ &\quad + 2\int_0^t (J(x_n(x) - x(s)), u_n(s) - f(s))ds . \end{aligned}$$

Recall that $J(\cdot)$ is uniformly continuous on bounded sets in X, while by construction $v_n(x_n)-u_n(x_n)\xrightarrow{\|\cdot\|_w}0$ and so $v_n(x_n)-u_n(x_n)\xrightarrow{w}0$ in $L^1(T, X)$ (cf. Gutman [10]). So we get

$$\int_0^t (J(x_n(s)-x(s)),\ v_n(x_n)(s)-u_n(x_n)(s))ds \to 0 \quad \text{as } n\to\infty .$$

Furthermore, note that

$$\begin{aligned}&\int_0^t (J(x_n(s)-x(s)),\ u_n(x_n)(s)-f(s))ds\\ &\leqq \int_0^t \|x_n(s)-x(s)\|\Big(\frac{\varepsilon_n}{2M_1 b}+k(s)\|x_n(s)-x(s)\|\Big)ds\\ &\leqq \varepsilon_n+\int_0^t k(s)\|x_n(s)-x(s)\|^2 ds .\end{aligned}$$

Thus in the limit as $n\to\infty$, we get

$$\begin{aligned}&\|\hat{x}(t)-x(t)\|^2\leqq\int_0^t k(s)\|\hat{x}(s)-x(s)\|^2 ds\\ &\qquad\Rightarrow \hat{x}=x \text{ (via Gronwall's lemma).}\end{aligned}$$

Since $x_n\in S_e(x_0)$ and $x_n\to x$ in $C(T, X)$, we conclude that

$$\overline{S_e(x_0)}^{C(T,H)}=S(x_0)$$

(recall that $S(x_0)$ is compact in $C(T, X)$; see Avgerinos-Papageorgiou [1]).

Next let $\delta_F(t, x, v)$ be the Choquet function corresponding to $F(t, x)$ (for the definition and properties of $\delta_F(\cdot,\cdot,\cdot)$, we refer to DeBlasi-Pianigiani [8] and Pianigiani [14]). Let $Z_\lambda=\{x\in S(x_0): \int_0^b \delta_F(t, x(t), f(t))dt<\lambda\}$, where $x=p(f)$ with $f\in S^1_{F(\cdot, x(\cdot))}$. Using the fact that $\delta_F(t,\cdot,\cdot)$ is *u.s.c.*, $\delta_F(t, x,\cdot)$ is concave and invoking Theorem 2.1 of Balder [2], we can easily check that Z_λ is open in $C(T, X)$. Let $\lambda_n\downarrow 0$ and set $Z_{\lambda_n}=Z_n$. Then since $0\leqq\delta_F(t, x, v)$ for all $(t, x, v)\in T\times X\times X$ and $\delta_F(t, x, v)=0$ if and only if $v\in \operatorname{ext} F(t, x)$, we readily see that $S_e(x_0)=\bigcap_{n\geqq 1} Z_n$. Hence $S_e(x_0)$ is also a G_δ-subset of $S(x_0)$. Q.E.D.

5. Examples

In this section we present two examples illustrating the applicability of the abstract results established in this paper.

(A) Let Z be a bounded domain in $\boldsymbol{R}^N$, with smooth boundary. Let $\beta:\boldsymbol{R}\to 2^{\boldsymbol{R}}$ is a set-valued map. We consider the following distributed parameter system:

$$(4)\qquad \left\{\begin{array}{l} \dfrac{\partial x}{\partial z}-\Delta x+\beta(x)\ni g(t,\ z,\ x(t,\ z))+\displaystyle\int_Z l(z,\ z')u(z')dz \\ x|_{T\times\Gamma}=0, \quad x(0,\ z)=x_0(z) \\ u(t,\cdot)\in U(t,\ x(t,\cdot)) \text{ a.e. on } T. \end{array}\right\}$$

We will need the following hypotheses:

$H(\beta)$: $\beta(\cdot)$ is maximal monotone with $0\in D(\beta)$. So $\beta=\partial j$ with $j:\boldsymbol{R}\to\boldsymbol{R}$ proper, convex and l.s.c.

$H(g)$: $g:T\times Z\times\boldsymbol{R}\to\boldsymbol{R}$ is a function such that
(1) $(t,\ z)\to g(t,\ z,\ x)$ is measurable,
(2) $|g(t,\ z,\ x)-g(t,\ z,\ x')|\leqq k(t,\ z)|x-x'|$ a.e., with $k(\cdot,\cdot)\in L^1(T\times Z)$,
(3) $|g(t,\ z,\ x)|\leqq a(t,\ z)+c(t,\ z)|x|$ a.e. with $a(\cdot,\cdot)\in L^1(T,\ L^p(Z))$, $c(\cdot,\cdot)\in L^1(T,\ L^\infty(Z))$, $1<p<\infty$.

$H(l)$: $l\in L^p(Z\times Z)$.

$H(U)$: $U:T\times L^p(Z)\to P_{fc}(L^p(Z))$ is a multifunction
(1) $t\to U(t,\ x)$ is measurable,
(2) $h(U(t,\ x),\ U(t,\ x'))\leqq k'(t)\|x-x'\|_p$ a.e., with $k'(\cdot)\in L^1(T)$,
(3) $|U(t,\ x)|=\sup\{\|u\|_p: u\in U(t,\ x)\}\leqq a'(t)+c'(t)\|x\|_p$ a.e., with $a'(\cdot)$, $c'(\cdot)\in L^1(T)$.

In this case $X=L^p(Z)$, which is uniformly convex together with its dual $L^q(Z)(1/p+1/q=1)$. Let $\hat{\beta}(\cdot)$ be the lifting of $\beta(\cdot)$ on $L^p(Z)$; i.e. $\hat{\beta}(x)=\{y\in L^p(Z): y(z)\in\beta(x(z))$ a.e. on $Z\}$ with $D(\hat{\beta})=\{x\in L^p(Z)$: there exists $y\in L^p(Z)$ such that $y(z)\in\beta(x(z))$ a.e.$\}$. It is easy to check that $\hat{\beta}$ is m-accretive on $X=L^p(Z)$. Define $A:D\subseteqq X\to X$ by $Ax=-\Delta x+\hat{\beta}(x)$ for all $x\in D=W_0^{1,p}(Z)\cap W^{2,p}(Z)\cap D(\hat{\beta})$. It is known (cf. Brezis [7] and Barbu [3]) that this is an m-accretive operator which generates an equicontinuous semigroup of nonlinear contractions.

Let $\hat{g}:T\times X\to X$ and $B\in\mathcal{L}(X)$ be defined by

$\hat{g}(t,\ x)(\cdot)=g(t,\cdot,\ x(\cdot))$ (the Nemitsky operator corresponding to $g(\cdot,\cdot,\cdot)$)

and $Bu(\cdot)=\displaystyle\int_Z l(\cdot,\ z')u(z')dz'$.

Recall that $B(\cdot)$ is a compact linear operator on $L^p(Z)$ (cf. hypothesis $H(l)$). So if we set $F(t,\ x)=\hat{g}(t,\ x)+BU(t,\ x)$, we readily see that it satisfies hypothesis $H(F)_1$. Rewrite (4) in the following equivalent deparametrized (i.e. control-free) form:

$$\left\{\begin{array}{c} -\dot{x}(t)\in Ax(t)+F(t,\ x(t)) \\ x(0)=\hat{x}_0 \end{array}\right\}$$

with $\hat{x}_0=x_0(\cdot)\in\bar{D}$. Invoking Theorem 4.1, we get the following result:

THEOREM 5.1. *If hypotheses $H(\beta)$, $H(g)$, $H(l)$, $H(U)$ hold, $x_0(\cdot)\in\bar{D}$ and $x\in C(T,\ L^p(Z))$ is a solution for (4) then given $\varepsilon>0$, we can find $y\in C(T,\ L^p(Z))$ a solution of (4) generated by a control $u(t,\cdot)\in\operatorname{ext} U(t,\ x(t.\cdot))$ a.e. such that* $\sup_{t\in T}\int_Z |x(t,\ z)-y(t,\ z)|^2 dz<\varepsilon$.

(*B*) Again, let Z be a bounded domain in $\boldsymbol{R}^N$, $2\leqq r<\infty$, $\theta>0$, $D_k=\partial/\partial z_k$, $k=\{1,\ \cdots,\ N\}$ and $D=\text{grad}$. We consider the following optimal control problem:

$$(5)\left\{\begin{array}{l} \eta(x(b,\cdot))\to\inf=m \\ \text{s.t.}\ \dfrac{\partial x}{\partial t}-\sum_{k=1}^{N} D_k(|D_k x|^{r-2}D_k x)+\theta x|x|^{r-2}=g(t,\ z,\ x(t,\ z))+\int_Z l(z,\ z')u(t,\ z')dz' \\ -\dfrac{\partial x}{\partial \nu_r}=-\sum_{k=1}^{N} |D_k x|^{p-2}D_k x\ cox(n,\ e_k)\in\beta(x(t,\ z))\ \text{a.e.} \\ x(0,\ z)=x_0(z)\ \text{a.e. and}\ u(t,\cdot)\in U(t,\ x(t,\cdot))\ \text{a.e.} \end{array}\right\}$$

Assume that hypotheses $H(\beta)$, $H(g)$, $H(l)$ and $H(U)$ hold with $p=2$, $j\geqq 0$. Let $X=L^2(Z)$ and define $A: D\subseteqq X\to X$ by

$$Ax=-\sum_{k=1}^{N} D_k(|D_k x|^{r-2}D_k x)+\theta x|x|^{r-2}=-\Delta_r^\theta x\ \text{(the pseudo-Laplacian)}$$

$$\text{and } D=\left\{x\in W^{1,r}(Z):\ \Delta_r^\theta x\in L^2(Z),\ -\frac{\partial x}{\partial \nu_r}\in\beta(x(z))\ \text{a.e. on } \Gamma\right\}.$$

We know (cf. Brezis [7], p. 43) that $A(\cdot)$ is m-accretive. In fact $A=\partial\varphi$ where $\varphi: X\to\bar{\boldsymbol{R}}=\boldsymbol{R}\cup\{+\infty\}$ is defined by

$$\varphi(x)=\left\{\begin{array}{ll} \dfrac{1}{r}\int_Z \|Dx\|_N^r dz+\dfrac{\theta}{r}\int_Z |x(z)|^r dz+\int_\Gamma j(x(z))dz & \text{if } x\in W^1(Z),\ j(x(\cdot))\in L^1(\Gamma) \\ +\infty & \text{otherwise.} \end{array}\right.$$

Clearly $\varphi(\cdot)$ is proper, convex and l.s.c. Furthermore it is of compact type; i.e. for every $\lambda\geqq 0\{x\in X: \varphi(x)+\|x\|^2\leqq\lambda\}$ is compact in X. This follows immediately from the compact embedding of $W^{1,r}(Z)$ into $L^2(Z)$ (recall $2\leqq r<\infty$). So by the Brezis-Konishi theorem (cf. Brezis [7], $A(\cdot)$ generates a compact semigroup, which in particular is equicontinuous.

Rewrite (5) in the following equivalent abstract form:

$$\left\{\begin{array}{c} \eta(x(b))\to\inf=m \\ \text{s.t.}\ \ -\dot{x}(t)\in Ax(t)+F(t,\ x(t)) \\ x(0)=\hat{x}_0 \end{array}\right\}.$$

We make the following hypotheses concerning the cost functional η:

$H(\eta)$: $\eta: X\to \boldsymbol{R}$ is continuous.

Then invoking Theorem 4.1 and regularity results concerning subdifferential evolution inclusions (see Brezis [6]), we get the following "bang-bang" principle of (5):

THEOREM 5.2. *If hypotheses $H(\beta)$ (with $j\geqq 0$), $H(g)$, $H(U)$ (with $p=2$), $H(l)$, $H(\eta)$ hold and $x_0\in W^{1,r}(Z)$ with $j(x_0(\cdot))\in L^1(Z)$, then problem (5) admits a solution $x\in C(T,\ L^2(Z))$ with $\partial x/\partial t\in L^2(T\times Z)$ and given $\varepsilon>0$ we can find a trajectory $y\in C(T,\ L^2(X))$ with $\partial y/\partial t\in L^2(T\times Z)$ generated by a control $u\in L^2(T\times Z)$ such that $u(t,\cdot)\in \operatorname{ext} U(t,\ x(t,\cdot))$ a.e. ("bang-bang" control) and $\eta(y(b))-m<\varepsilon$ (ε-optimality of $y(\cdot,\cdot)$).*

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