Y. ADACHI
KODAI MATH. J.
18 (1995), 408—424

A GENERALIZATION OF THE BIG PICARD THEOREM

YUKINOBU ADACHI

Introduction

The classical big Picard theorem says that any holomorphic map f from the punctured disk Δ^* into $\boldsymbol{P}^1$ which omits three points can be extended to a holomorphic map $f: \Delta \to \boldsymbol{P}^1$. After Kobayashi's fundamental work [14, VI], Kiernan [9] generalized this theorem to the following result.

Let B an analytic subset of the complex manifold N whose singularities are normal crossings and let M be a hyperbolically imbedded subspace of the complex space X. Then any holomorphic map $f: N \setminus B \to M$ can be extended to a holomorphic map $f: N \to X$.

And Fujimoto [5] obtained the following another generalization of the big Picard theorem.

THEOREM. *Let B be a regular analytic subset of a complex manifold N and let M be the complementary domain of $n+2$ hyperplanes in general position in $\boldsymbol{P}^n$. Let $f: N \setminus B \to M$ be a holomorphic map. Then either the image $f(N \setminus B)$ lies in a diagonal hyperplane in $\boldsymbol{P}^n$ or f can be extended to a holomorphic map $f: N \to \boldsymbol{P}^n$.*

The purpose of this paper is to consider a generalization of the big Picard theorem of Fujimoto's type for any holomorphic map $f: \Delta^* \to \boldsymbol{P}^2 \setminus A$ where A is a curve in $\boldsymbol{P}^2$ with 4 or more irreducible components in general position in a certain sence (Theorem 10.1) and for any meromorphic map $f: N \setminus B \to \boldsymbol{P}^2 \setminus A$, where N is an arbitrary manifold, B is a proper analytic subset of N and A is the same of the former case (Theorem 12.1). To prove the former result, Kizuka's theorem in [11] (see Theorem 7.1 in this paper) as well as Fujimoto's theorem play an important role.

Chapter I. Preliminaries

1. Degeneracy locus of the Kobayashi pseudodistance

Throughout the sections 1~3, let X be a complex manifold of dimension n

Received January 14, 1994; revised October 21, 1994.

with a hermitian metric ds^2 and let M be a relatively compact subdomain of X. Denote by $d_M(p, q)$ the intrinsic pseudodistance of two points p and q of M introduced by Kobayashi [13]. In [3] we extended d_M onto the closure $\overline{M}$ of M in X as follows:

For $p, q\in\overline{M}$, we define

$$d_M(p, q)=\lim_{p'\to p, q'\to q} d_M(p', q'), \quad p', q'\in M.$$

It is clear that $0\leqq d_M(p, q)\leqq\infty$ and $d_M(p, r)\leqq d_M(p, q)+d_M(q, r)$ for $p, q, r\in\overline{M}$.

DEFINITION 1.1. We call $p\in\overline{M}$ a degeneracy point of d_M if there exists a point $q\in\overline{M}\setminus\{p\}$ such that $d_M(p, q)=0$. By $S_M(X)$ we denote the set of the degeneracy points of d_M on $\overline{M}$ and call it the degeneracy locus of d_M in X.

We studied properties of $S_M(X)$ in [3] and [1]. Let us recall and study some results.

We denote the disk $\{z\in\boldsymbol{C}; |z|<r\}$ by $\Delta(r)$ and $\Delta(1)$ by Δ. We have then, by Royden [18], the following criterion for the degeneracy points of d_M.

LEMMA 1.2. *p is a degeneracy point of d_M in $\overline{M}$ if and only if there exists a sequence of holomorphic maps $f_\nu: \Delta\to M (\nu=1, 2, \cdots)$ such that $\lim_{\nu\to\infty} f_\nu(0)=p$ and $\lim_{\nu\to\infty}\|f_\nu'(0)\|=\infty$, where $f_\nu'(0)=df_\nu(d/dw|_{w=0})$ and $\|\cdot\|$ is the norm with respect to the hermitian metric ds^2.*

Proof. Assume that there exists a sequence of such holomorphic maps f_ν. Let $\overline{U}$ be any closed neighborhood of p in X which is biholomorphic to the closed unit ball $\{(z_1, \cdots, z_n)\in\boldsymbol{C}^n; \sum_{i=1}^n|z_i|^2\leqq 1\}$. From Schwarz lemma, for any positive number $r<1$, there exists a ν such that $f_\nu(\Delta(r))\not\subset U$. Therefore, there exists a sequence of $\{w_\lambda\}_{\lambda=1,2,\dots}$ such that $\lim_{\lambda\to\infty} w_\lambda=0$ and $f_{\nu_\lambda}(w_\lambda)\in\partial U$. By taking a subsequence we may assume $f_{\nu_\lambda}(w_\lambda)\to q\in\partial U$. Then $d_M(p, q)\leqq\lim_{\lambda\to\infty} d_M(f_{\nu_\lambda}(0), f_{\nu_\lambda}(w_\lambda))\leqq\lim_{\lambda\to\infty} d_\Delta(0, w_\lambda)=0$. So, p is a degeneracy point of d_M in $\overline{M}$.

Next, we assume that there exists a point $q\in\overline{M}\setminus\{p\}$ such that $d_M(p, q)=0$ and there is no such a sequence of holomorphic maps f_ν. Then there exist a neighborhood U of p in X and positive constant c such that $q\notin\overline{U}$ and for every point $r\in U\cap M$ and $v_r\in T_r(M)$, $F_M(r, v_r)\geqq c\|v_r\|$, where $T_r(M)$ is the tangent space of M at r and $F_M(r, v_r)$ is the Royden function. (cf. [18]). We take a neighborhood V of p such that $V\Subset U$ and $r\in V\cap M$ and $s\in\overline{U}^c\cap M$. Let $\gamma(t)$ be any piecewisesmooth curve on M such that $\gamma(0)=r$ and $\gamma(1)=s$. From [18]

$$\begin{aligned} d_M(r, s)&=\inf_\gamma\int_0^1 F_M(\gamma(t), \gamma'(t))dt \\ &\geqq\inf_\gamma c\cdot\int_{t\in E}\|\gamma'(t)\|dt \\ &\geqq c\cdot\mathrm{dist}(\partial V, \partial U)>0, \end{aligned}$$

where $E=\{t\in[0, 1]\,;\ \gamma(t)\in U\}$. This contradicts to $d_M(p, q)=0$. □

PROPOSITION 1.3. *$S_M(X)$ is a closed subset of X.*

Proof. Let $p_\nu\in S_M(X)$ such that $\lim_{\nu\to\infty} p_\nu=p$ and let $\bar{U}$ be any closed neighborhood of p in X which is biholomorphic to the closed unit ball. We may assume that $p_\nu\in U$ for every ν. From the proof of Lemma 1.2, there exists $q_\nu\in\partial U\cap\overline{M}$ such that $d_M(p_\nu, q_\nu)=0$ for every ν. By taking a subsequence we may assume $q_\nu\to q\in\partial U$. Then $p\in S_M(X)$ by the definition of d_M. □

DEFINITION 1.4. (cf. [6], [19] and [21]). A closed subset E of X will be called a pseudoconcave subset of order 1, if for any coordinate neighborhood

$$U:\ |z_1|<1,\ \cdots,\ |z_n|<1$$

of X and positive numbers r, s with $0<r<1$, $0<s<1$ such that $U^*\cap E=\emptyset$, one obtains $U\cap E=\emptyset$, where

$$U^*=\{p\in U\,;\ |z_1(p)|\leqq r\}\cup\{p\in U\,;\ s\leqq \max_{2\leqq i\leqq n}|z_i(p)|\}.$$

In [3], we proved the following theorems.

THEOREM 1.5. *$S_M(X)$ is a pseudoconcave subset of order 1 in X.*

THEOREM 1.6. *If $S_M(X)$ is an analytic subset of dimension 1 of X, then each irreducible component of $S_M(X)$ is of genus $\leqq 1$.*

Let S be an analytic subset of X. The following definition is due to Kiernan-Kobayashi [10] (cf. also Lang [15], p 37).

DEFINITION 1.7. M is hyperbolically imbedded modulo S in X if, for every pair of distinct points p, $q\in\overline{M}$ not both contained in S, there exist neighborhoods V_p and V_q of p and q in X such that $d_M(V_p\cap M, V_q\cap M)>0$.

It is easy to see the following propositions.

PROPOSITION 1.8. *M is hyperbolically imbedded modulo S in X if and only if, for every pair of points p, $q\in\overline{M}$ such that $d_M(p, q)=0$ not both contained in S we conclude $p=q$.*

PROPOSITION 1.9. *If M is hyperbolically imbedded modulo S in X, then $S_M(X)\subset S$.*

DEFINITION 1.10. (cf. Lang [15], p 32). Let $\{p_\nu\}$ and $\{q_\nu\}$ be two sequences in M converging to points p, q in $\overline{M}$ respectively. M is hyperbolically imbedded in X if $\lim_{\nu\to\infty} d_M(p_\nu, q_\nu)=0$ then $p=q$.

It is easy to see the following

PROPOSITION 1.11. *M is hyperbolically imbedded in X if and only if $S_M(X)=\emptyset$.*

2. Basic theorem for an extension of a holomorphic map

LEMMA 2.1. *Let f_ν: $\Delta^* \rightarrow M$ be a sequence of holomorphic maps and let $\{z_\nu\}$ be a sequence in Δ^* converging to 0 such that $f_\nu(z_\nu) \rightarrow p \notin S_M(X)$. Then $f_\nu(\rho_\nu) \rightarrow p(\nu \rightarrow \infty)$, where $\rho_\nu = \{z \in \boldsymbol{C}$; $|z| = |z_\nu|\}$.*

Proof. For every $\tilde{z}_\nu \in \rho_\nu$, we have the following inequality:

$$\lim_{\nu \to \infty} d_M(f_\nu(\tilde{z}_\nu), p) \leqq \lim_{\nu \to \infty} d_M(f_\nu(\tilde{z}_\nu), f_\nu(z_\nu))$$
$$\leqq \lim_{\nu \to \infty} d_{\Delta^*}(\tilde{z}_\nu, z_\nu).$$

Since $d_{\Delta^*}(\tilde{z}_\nu, z_\nu) \sim O(1/\log|z_\nu|)$ (cf. [14], p 81), and $p \notin S_M(X)$, then $f_\nu(\tilde{z}_\nu) \rightarrow p(\nu \rightarrow \infty)$. □

The following theorem is basic for an extension of a holomorphic map. The proof is essentially same as Kiernan's proof (cf. Theorem 1 in [9]).

THEOREM 2.2. *Let f_ν: $\Delta^* \rightarrow M$ be a sequence of holomorphic maps. If there is a sequence $\{z_\nu\}$ in Δ^* converging to 0 such that $f_\nu(z_\nu) \rightarrow p \notin S_M(X)$, then $f_\nu(z_\nu') \rightarrow p$ for every sequence $\{z_\nu'\}$ in Δ^* converging to 0.*

Proof. We show that it is absurd if we assume that there is a sequence $\{z_\nu'\}$ converging to 0 such that $f_\nu(z_\nu') \rightarrow q \neq p$.

(i) Assume that $|z_\nu| \leqq |z_\nu'|$ by taking a subsequence and relabelling. There exists the closed neighborhood $\bar{U}$ of p in X which is biholomorphic to the closed unit ball $\bar{B} = \{(w_1, \cdots, w_n) \in \boldsymbol{C}^n$; $\sum_{i=1}^n |w_i|^2 \leqq 1\}$ such that $\bar{U} \cap S_M(x) = \emptyset$ and $q \notin \bar{U}$ from Proposition 1.3. Let ν be sufficiently large such that $f_\nu(\rho_\nu) \subset U$ and let R_ν be the largest annulus such that $\rho_\nu \subset R_\nu$ and $f_\nu(R_\nu) \subset U$ where $\rho_\nu = \{z \in \boldsymbol{C}$; $|z| = |z_\nu|\}$. Then there exist $a_\nu \geqq 0$ and $b_\nu < 1$ such that $R_\nu = \{z \in \Delta^*$; $a_\nu < |z| < b_\nu\}$. We can assume that either $a_\nu \neq 0$ or $a_\nu = 0$ for every ν by taking a subsequence and relabelling. We consider the former case first. Let $\sigma_\nu = \{z \in \boldsymbol{C}$; $|z| = a_\nu\}$ and $\tau_\nu = \{z \in \boldsymbol{C}$; $|z| = b_\nu\}$. Then there exist $\alpha_\nu \in \sigma_\nu$ and $\beta_\nu \in \tau_\nu$ such that $f_\nu(\alpha_\nu)$, $f_\nu(\beta_\nu) \in \partial U$. By taking a subsequence and relabelling, we may assume $f_\nu(\alpha_\nu) \rightarrow q' \in \partial U$ and $f_\nu(\beta_\nu) \rightarrow q'' \in \partial U$. Since $a_\nu, b_\nu \rightarrow 0$ and $q', q'' \notin S_M(X)$, $f_\nu(\sigma_\nu) \rightarrow q'$ and $f_\nu(\tau_\nu) \rightarrow q''$ from Lemma 2.1. By rotating $\bar{B}$ if necessary, we can assume that $|w_1(q')| = \delta' > 0$ and $|w_1(q'')| = \delta'' > 0$. By the argument principle, for all sufficiently large ν we have

$$\int_{\sigma_\nu} d\log(w_1\circ f_\nu(z)-w_1\circ f_\nu(z_\nu))$$

$$=\int_{\tau_\nu} d\log(w_1\circ f_\nu(z)-w_1\circ f_\nu(z_\nu))=0.$$

On the other hand, we have

$$\int_{\tau_\nu} d\log(w_1\circ f_\nu(z)-w_1\circ f_\nu(z_\nu))$$

$$-\int_{\sigma_\nu} d\log(w_1\circ f_\nu(z)-w_1\circ f_\nu(z_\nu))$$

$$=2\pi i(N-P),$$

where N and P are the number of zeros and poles of the function $w_1\circ f_\nu(z)-w_1\circ f_\nu(z_\nu)$ on the annulus R_ν. This is a contradiction since $N>0$ and $P=0$.

If $a_\nu=0$ for every ν, f_ν extends holomorphically to $\Delta(b_\nu)$ with $f_\nu(0)$ in B by the Riemann extension theorem since $f_\nu(R_\nu)\subset U$. Setting $\sigma_\nu=\emptyset$, the argument used in the preceding paragraph leads to a contradiction. This proves the theorem in case (i).

(ii) Assume that $|z_\nu'|\leqq|z_\nu|$ by taking a subsequence and relabelling. There exists the closed neighborhood $\bar{U}$ of p in X which is biholomorphic to the closed unit ball $\bar{B}$ such that $\bar{U}\cap S_M(X)=\emptyset$ and $q\notin\bar{U}$. Since $f_\nu(z_\nu')\to q$ $(\nu\to\infty)$ and $f_\nu(\rho_\nu)\subset U$ for sufficiently large ν where $\rho_\nu=\{z\in\boldsymbol{C}\,;\,|z|=|z_\nu|\}$, there exists z_ν'' such that $|z_\nu''|<|z_\nu|$ and $f_\nu(z_\nu'')\in\partial U$. By taking a subsequence and relabelling, we may assume that $f_\nu(z_\nu'')\to r\in\partial U$. Since $r\notin S_M(X)$, there exists a closed neighborhood $\bar{U}'$ of r in X which is biholomorphic to the closed unit ball $\bar{B}$ such that $\bar{U}'\cap S_M(X)=\emptyset$ and $\bar{U}'\not\ni p$ from Proposition 1.3. By considering r, p, z_ν'' and z_ν in place of p, q, z_ν and z_ν', we can reduce to case (i). □

We obtain

COROLLARY 2.3. *Let $f:\Delta^*\to M$ be a holomorphic map. If there is a sequence $\{z_\nu\}$ in Δ^* converging to 0 such that $f(z_\nu)\to p\notin S_M(X)$, then f can be extended to a holomorphic map $f:\Delta\to X$.*

COROLLARY 2.4. *Let $f_\nu:\Delta^*\to M$ be a sequence of holomorphic maps. Assume that each f_ν can be extended to a holomorphic map $f_\nu:\Delta\to X$. If there exists a sequence $\{z_\nu\}$ in Δ^* converging to 0 such that $f_\nu(z_\nu)\to p\notin S_M(X)$, then $f_\nu(0)\to p$ $(\nu\to\infty)$.*

Proof. If there is a subsequence $\{\nu_\lambda\}$ of $\{\nu\}$ such that $f_{\nu_\lambda}(0)\to q\neq p$, then there exists a sequence $\{z_\lambda''\}$ in Δ^* converging to 0 such that $f_{\nu_\lambda}(z_\lambda'')\to q$. This is a contradiction since for every sequence $\{z_\nu'\}$ in Δ^* converging to 0, $f_\nu(z_\nu')\to p$ from Theorem 2.2. □

3. Cluster set of a holomorphic map $f: \Delta^* \to M$ at 0

According to Nishino-Suzuki [16], we define and study cluser sets. We denote the punctured disk $\{z \in \boldsymbol{C}\,;\, 0<|z|<\rho \leqq 1\}$ by $\Delta^*(\rho)$. Let $f: \Delta^* \to M$ be a holomorphic map.

DEFINITION 3.1. We define the cluster set $f(0: X)$ of f at 0 by

$$f(0: X)=\bigcap_{\rho>0} \overline{f(\Delta^*(\rho))},$$

where $\overline{f(\Delta^*(\rho))}$ is the closure of $f(\Delta^*(\rho))$ in X.

It is easy to see that $f(0: X)$ is either a single point or a continuum.
From the Riemann extension theorem we have

PROPOSITION 3.2. *If $f(0: X)$ is contained in a coodinate neighborhood of X, f can be extended to a holomorphic map $f: \Delta \to X$ and then $f(0: X)$ is a single point of X.*

DEFINITION 3.3. We call a holomorphic map $f: \Delta^* \to M$ has an essential singularity at 0 if $f(0: X)$ contains at least two points.

THEOREM 3.4 (cf. Theorem 1 in [16]). *If a holomorphic map $f: \Delta^* \to M$ has an essential singularity at 0, $f(0: X)$ is a pseudoconcave set of order* 1.

Proof (The following proof is essentially the same of [16]). Assume that there is a coordinate neighborhood U in X which is biholomorphic to the polydisk $\{(z_1, z_2, \cdots, z_n) \in \boldsymbol{C}^n\,;\, |z_i|<1 (1 \leqq i \leqq n)\}$ such that $f(0: X) \cap U \neq \emptyset$ and $f(0: X) \cap U^* \neq \emptyset$, where $U^*=\{p \in U\,;\, |z_1(p)| \leqq r\} \cup \{p \in U\,;\, s \leqq \max_{2 \leq i \leq n} |z_i(p)|\} (0<r<1, 0<s<1)$. $Z_i(p)=z_i \circ f(z)$ is a holomorphic function in $D=f^{-1}(U)$ $(\neq \emptyset)$. We can choose a positive real number ε such that $\varepsilon<r<1-\varepsilon$, $s<1-2\varepsilon$ and $f(0: X) \cap U_\varepsilon \neq \emptyset$, where $U_\varepsilon=\{p \in U\,;\, |z_i(p)|<1-\varepsilon\ (1 \leqq i \leqq n)\}$. Consider in $\Delta^*(\rho)$ $(0<\rho<1)$ the inverse image $f^{-1}(U_\varepsilon) \cap \Delta^*(\rho)=D_\rho$ $(\neq \emptyset)$. We may assume that $D_\rho= \Delta^*(\rho)$ for every $\rho>0$ since if $D_\rho=\Delta^*(\rho)$ for a ρ, $f(\Delta^*(\rho)) \subset U_\varepsilon$ and $f(0: X)$ is a single point from Proposition 3.2. So the origin $z=0$ belongs to the accumulation points of the boundary γ of D_ρ in $\Delta^*(\rho)$. Since $f(0: X) \cap U^*=\emptyset$, we can find a $\rho>0$ such that

(i) for $z \in D_\rho$ $\varepsilon<|Z_1(z)|<1-\varepsilon$ and $|Z_i(z)|<1-2\varepsilon$ $(2 \leqq i \leqq n)$

and

(ii) $|Z_1(\zeta)|=1-\varepsilon$ for $\zeta \in \gamma$.

From Theorem in Tôki [21], $\lim_{z \in D_\rho, z \to 0} |Z_1(z)|=1-\varepsilon$. Then $f(0: X) \cap U_\varepsilon=\emptyset$. This contradicts to the definition of U_ε. □

From Theorem 3.4 and a property of the pseudoconcave set of order 1, we obtain

PROPOSITION 3.5 (cf. Proposition 3 in [16]). *If a holomorphic map* $f:\Delta^*\to M$ *has an essential singularity at* 0 *and* $f(0: X)$ *is contained in an analytic subset* C *of dimension* 1 *of* X, *then* $f(0: X)$ *is also an analytic subset of dimension* 1 *of* X *composed of irreducible components of* C.

From Corollary 2.3 we obtain

THEOREM 3.6 (cf. Proposition 2 in [16]). *If a holomorphic map* $f:\Delta^*\to M$ *has an essential singularity at* 0, *then* $f(0: X)\subset S_M(X)$.

4. Nonhyperbolic curve and hyperbolic curve

Let X be a compact complex manifold of dimension 2 and let A be a curve in X. In [1], we defined a nonhyperbolic curve with respect to A as the following

DEFINITION 4.1. An irreducible curve C in X is a nonhyperbolic curve with respect to A, if the following condition is satisfied:

In case $C\not\subset A$, the normalization of $C\setminus A$ is isomorphic to either a smooth elliptic curve, $\boldsymbol{P}^1$, $\boldsymbol{C}$ or $\boldsymbol{C}^*=\boldsymbol{C}\setminus\{0\}$. In case $C\subset A$, the normalization of $C\setminus A'$ is isomorphic to either a smooth elliptic curve, $\boldsymbol{P}^1$, $\boldsymbol{C}$ or $\boldsymbol{C}^*$, where A' is the union of the components of A except C. (A' may be $\emptyset$).

DEFINITION 4.2. An irreducible curve C in X is a hyperbolic curve with respect to A, if C is not a nonhyperbolic curve with respect to A.

If C is a nonhyperbolic curve with respect to A such that $C\not\subset A$, then $C\subset S_M(X)$ since there is a nonconstant holomorphic map $f:\boldsymbol{C}\to C\setminus A$ where $M=X\setminus A$. In [1] we showed the following.

THEOREM 4.3. *Let* A *be a curve in* $\boldsymbol{P}^2$. *Set* $X=\boldsymbol{P}^2$ *and* $M=\boldsymbol{P}^2\setminus A$. *If* $S_M(X)$ *is a curve in* X, *then* $S_M(X)$ *is composed of nonhyperbolic curves with respect to* A.

COROLLARY 4.4. *Let* A *be a curve in* $\boldsymbol{P}^2$. *Set* $X=\boldsymbol{P}^2$ *and* $M=\boldsymbol{P}^2\setminus A$. *If* $S_M(X)$ *is a curve in* X *and an irredncible curve* C *in* X *is a hyperbolic curve with respect to* A, *then* $C\not\subset S_M(X)$.

5. Fundamental lemma

Let $Y=\boldsymbol{P}^p$ and $L^p=Y\setminus(H_0\cup\cdots\cup H_{p+1})$ where $H_0, \cdots, H_{p+1}$ are $p+2$ hyperplanes in $\boldsymbol{P}^p$ in general position. Following Cartan [4], we represent Y and L^p as follows. Let $(w_0, \cdots, w_{p+1})$ be homogeneous coordinates for $\boldsymbol{P}^{p+1}$ and imbed Y in $\boldsymbol{P}^{p+1}$ as the hyperplane $Y=\{(w_0, \cdots, w_{p+1})\in\boldsymbol{P}^{p+1}; w_0+\cdots+w_{p+1}=0\}$. Without loss of generality, we may assume that $H_j=\{(w_0, \cdots, w_{p+1})\in Y; w_j=0\}$ and therefore $L^p=\{(w_0, \cdots, w_{p+1})\in\boldsymbol{P}^{p+1}; w_0+\cdots+w_{p+1}=0$ and $w_j\neq 0$ for $j=0, \cdots, p+1\}$. We now define an analytic subvariety Δd of Y. It will be the union of a particular set of hyperplanes which we shall call diagonal hyperplanes with respect to $H_0, \cdots, H_{p+1}$. Let $\mathcal{J}$ be the set of subsets of $\{0, \cdots, p+1\}$ which consist of at least two elements and not more than p elements. For $I=\{j_1, \cdots, j_k\}\in\mathcal{J}$, we set $\Delta_I=\{(w_0, \cdots, w_{p+1})\in Y; w_{j_1}+\cdots+w_{j_k}=0\}$ and define $\Delta_d=\bigcup_{I\in\mathcal{J}}\Delta_I$. Note that if I' is the subset of $\{0, \cdots, p+1\}$ complementary to $I\in\mathcal{J}$, then $\Delta_I'=\Delta_I$.

Kiernan-Kobayashi [10] showed the following

THEOREM 5.1. *L^p is hyperbolically imbedded modulo Δ_d in Y.*

Next lemma is fundamental for our work.

LEMMA 5.2 (cf. [2], pp. 454-456). *Let $A_0, \cdots, A_l$ be $l+1$ $(l\geqq n+1)$ distinct irreducible hypersurfaces in $\boldsymbol{P}^n$ and set $A=A_0\cup\cdots\cup A_l$. Then there exists a rational map $G: \boldsymbol{P}^n\to\boldsymbol{P}^p$ $(p\geqq 2)$ such that $G|_{\boldsymbol{P}^n\setminus A}: \boldsymbol{P}^n\setminus A\to L^{p-1}=Y\setminus(H_0\cup\cdots\cup H_p)$ is holomorphic, $G(\boldsymbol{P}^n\setminus A)\not\subset\Delta_d$ and the rank of G is always $\geqq 1$.*

Proof. Let $P_j(z_0, \cdots, z_n)$ $(0\leqq j\leqq l)$ be homogeneous polynomials which take zeros only on A_j respectively, where $(z_0, \cdots, z_n)$ are the homogeneous coordinates for $\boldsymbol{P}^n$. We may assume that P_j's are of the same degree d. Let F be the rational map $\boldsymbol{P}^n$ to $\boldsymbol{P}^l$ defined by $y_0=P_0, \cdots, y_l=P_l$, where $(y_0, \cdots, y_l)$ are the homogeneous coordinates for $\boldsymbol{P}^l$. Since the rank of F is $\leqq n$, the image of F is contained in a hypersurface S of $\boldsymbol{P}^l$. Let us write the defining equation of S as follows:

$$\sum_{\lambda} c_\lambda\cdot y_0^{\lambda_0}\cdot\cdots\cdot y_l^{\lambda_l}=0,$$

where $c_\lambda\neq 0$, $\lambda=(\lambda_0, \cdots, \lambda_l)$ and λ_i's are nonnegative integers such that $\lambda_0+\cdots+\lambda_l=N$ (a positive integer). Set $G_\lambda=c_\lambda\cdot P_0^{\lambda_0}\cdot\cdots\cdot P_l^{\lambda_l}$. Then $\{G_\lambda\}$ are homogeneous polynomials of $z_0, \cdots, z_n$ of degree $d\cdot N$ and satisfy $\sum_\lambda G_\lambda\equiv 0$ and $G_\lambda\neq 0$ on $\boldsymbol{P}^n\setminus A$. Let $\{G_0, \cdots, G_p\}$ be a subset of $\{G_\lambda\}$ which satisfies $G_0+\cdots+G_p\equiv 0$ and every subtotal of $G_0, \cdots, G_p$ is not identically zero. We consider the rational map G of $\boldsymbol{P}^n$ to $Y=\{(w_0, \cdots, w_p)\in\boldsymbol{P}^p; w_0+\cdots+w_p=0\}$ by $(G_0, \cdots, G_p)$.

Since $A_0, \cdots, A_l$ are all irreducible and distinct, $\lambda\neq\lambda'$ implies $G_\lambda/G_{\lambda'}\not\equiv$ constant. Therefore, we have $p\geqq 2$ and the rank of G is always $\geqq 1$. □

COROLLARY 5.3. *If rank $G=1$, there is a holomorphic rational function g on $\boldsymbol{P}^n \setminus A$ with lacunary three points.*

Proof. It is easy to see that if rank $G=1$, the normalization of $W=\overline{G(\boldsymbol{P}^n \setminus A)}$ is isomorphic to $\boldsymbol{P}^1$. Let $\pi: \boldsymbol{P}^1 \to W$ be the normalization of W. If $\pi^{-1} \circ G(\boldsymbol{P}^n \setminus A)$ is not lacunary three points, there is a nonconstant holomorphic map $h: \boldsymbol{C} \to \pi^{-1}(G(\boldsymbol{P}^n \setminus A))$. Then $\pi \circ h(\boldsymbol{C}) \subset \Delta_d$ from Theorem 5.1. By Lemma 5.2, $G(\boldsymbol{P}^n \setminus A) \not\subset \Delta_d$. So, $\Delta_d \cap G(\boldsymbol{P}^n \setminus A)$ is a set of points. This is a contradiction since h is a nonconstant map. Set $g=\pi^{-1} \circ G$. Then g is a holomorphic rational function on $\boldsymbol{P}^n \setminus A$ with lacunary three points. □

6. Rational functions of C- or C^*-type

Let A be a curve in $\boldsymbol{P}^2$, f be a nonconstant rational function on $\boldsymbol{P}^2$ and I_f be the set of inditermination points of f. According to Kashiwara [8] and Kizuka [12], we define and study rational functions of $\boldsymbol{C}$- or $\boldsymbol{C}^*$-type.

DEFINITION 6.1. We call f a rational function of $\boldsymbol{C}$-type (resp. $\boldsymbol{C}^*$-type) on $\boldsymbol{P}^2 \setminus A$ if f is a rational function on $\boldsymbol{P}^2$ and normalization of every irreducible component of all level curves of f except for a finite number of them is isomorphic to $\boldsymbol{C}$ (resp. $\boldsymbol{C}^*$) on $\boldsymbol{P}^2 \setminus (A \cup I_f)$. ($A$ may be $\emptyset$).

DEFINITION 6.2. We call a nonconstant rational function f primitive if all level curves of f are irreducible except for a finite number of them.

From the Stein factorization we have

PROPOSITION 6.3 (cf. Proposition 1 in [12]). *For every nonconstant rational function f on $\boldsymbol{P}^2$ there exists a pair of a primitive rational function f_0 on $\boldsymbol{P}^2$ and a rational function π on $\boldsymbol{P}^1$ such that $f=\pi \circ f_0$.*

It is well known that every irreducible component of all level curves of a rational function is the same type except for a finite number of them, so we have

PROPOSITION 6.4. *If there are infinite irreducible components of the level curves of a rational function f on $\boldsymbol{P}^2 \setminus A$ such that their normarizations are isomorphic to $\boldsymbol{C}$ (resp. $\boldsymbol{C}^*$) on $\boldsymbol{P}^2 \setminus (A \cup I_f)$, then f is of $\boldsymbol{C}$-type (resp $\boldsymbol{C}^*$-type).*

PROPOSITION 6.5. *Let A be a curve in $\boldsymbol{P}^2$. If there exists a rational function f of $\boldsymbol{C}$-* or *$\boldsymbol{C}^*$-type on $\boldsymbol{P}^2 \setminus A$, then A must belong to one of the following two classes of curves.*

(i) *The sum of compactifications of several irreducible components of the level curves of f of $\boldsymbol{C}$- or $\boldsymbol{C}^*$-type on $\boldsymbol{P}^2$.*

(ii) *The sum of compactifications of several irreducible components of the level curves of f of* $\boldsymbol{C}$*-type on* $\boldsymbol{P}^2$ *(which may be* $\emptyset$*) and an irreducible curve of genus* 0 *in* $\boldsymbol{P}^2$ *such that f is of* $\boldsymbol{C}^*$*-type on* $\boldsymbol{P}^2 \setminus A$.

Proof. Let us consider the case that f is of $\boldsymbol{C}$-type on $\boldsymbol{P}^2 \setminus A$ at first. In this case, f is of $\boldsymbol{C}$-type on $\boldsymbol{P}^2$. Suppose that A_1 is an irreducible component of A which is not contained in a level curve of f. Then, $f|_{A_1}$ is a nonconstant holomorphic function on A_1 to $\boldsymbol{P}^1$. Since infinite level curves of f intersect with $A_1 \setminus I_f$, this is a contradiction from Proposition 6.4.

Next, let us consider the case that f is of $\boldsymbol{C}^*$-type on $\boldsymbol{P}^2 \setminus A$. If f is of $\boldsymbol{C}^*$-type on $\boldsymbol{P}^2$, it is obvious that A is the sum of compactifications of several irreducible components of level curves of f from the same discussion above. So, we prove that f is of $\boldsymbol{C}$-type on $\boldsymbol{P}^2$ and A must belong to the class (ii) if A does not belong to the class (i). Suppose A_1 and A_2 be irreducible components of A such that they are not contained in level curves of f. Then, $f|_{A_i}$ is a nonconstant holomorphic function on A_i to $\boldsymbol{P}^1$ ($i=1, 2$). This is a contradiction since infinite level curves of f intersect with $A_i \setminus I_f$. Let A_1 be an irreducible component of A which is not contained in a level curve of f. From Proposition 6.3, we may assume that f is a primitive rational function. Then it is obvious that $f|_{A_1}: A_1 \to \boldsymbol{P}^1$ is holomorphic and one to one. So, the genus of A_1 is 0. □

7. Kizuka's theorem

THEOREM 7.1 (Theorem 1 in Kizuka [11] and Theorem 0 in [12]). *Let A be a curve in* $\boldsymbol{P}^2$. *Suppose that there exists a holomorphic map* $\varphi: \Delta^* \to \boldsymbol{P}^2 \setminus A$ *such that* φ *has an essential singularity at* 0 *and* $\varphi(0: \boldsymbol{P}^2) \subset A$. *Then A must be a nonsingular cubic curve or there exists a rational function f of* $\boldsymbol{C}$*- or* $\boldsymbol{C}^*$*-type on* $\boldsymbol{P}^2 \setminus A$. *In the latter case, A contains at least one irreducible component of a level curve of f.*

Since each tangent line to a nonsingular cubic curve A through any points of $\boldsymbol{P}^2 \setminus A$ intersects with A at most two points, it is easy to see the following

COROLLARY 7.2. *Let A be a curve in* $\boldsymbol{P}^2$. *Set* $X=\boldsymbol{P}^2$ *and* $M=\boldsymbol{P}^2 \setminus A$. *Suppose that there exists a holomorphic map* $\varphi: \Delta^* \to \boldsymbol{P}^2 \setminus A$ *such that* φ *has an essential singularity at* 0 *and* $\varphi(0: \boldsymbol{P}^2) \subset A$. *Then* $S_M(X)=X$.

8. Hyperbolicity of $\boldsymbol{P}^2 \setminus A$

In this section, let A be a curve with l ($l \geqq 4$) irreducible components in $\boldsymbol{P}^2$. Set $X=\boldsymbol{P}^2$ and $M=\boldsymbol{P}^2 \setminus A$. From Corollary of Theorem in [1] we have

THEOREM 8.1. *There are following three cases.*

(i) $S_M(X)=\emptyset$.
(ii) $S_M(X)$ *is a curve in* $\boldsymbol{P}^2$.
(iii) $S_M(X)=X$.

PROPOSITION 8.2. *If* $S_M(X)=X$, *then there exists a holomorphic rational function* g *of* $\boldsymbol{C}$- *or* $\boldsymbol{C}^*$-*type on* $\boldsymbol{P}^2\setminus A$ *with lacunary three points.*

Proof. From Lemma 5.2, there exists a rational map $G: \boldsymbol{P}^2\to\boldsymbol{P}^p$ ($p\geqq 2$) such that $G|_{\boldsymbol{P}^2\setminus A}: \boldsymbol{P}^2\setminus A\to L^{p-1}=Y\setminus(H_0\cup\cdots\cup H_p)$ is holomorphic and $G(\boldsymbol{P}^2\setminus A)\not\subset\Delta_d$. From Theorem 1 in [2], rank $G=1$ if $S_M(X)=X$. From Corollary 5.3, there exists a holomorphic rational function g on $\boldsymbol{P}^2\setminus A$ with lacunary three points. From the little Picard theorem, if h is a nonconstant holomorphic map of $\boldsymbol{C}$ to $\boldsymbol{P}^2\setminus A$, $g\circ h\equiv$constant. From Corollary of Theorem in [1] there exist infinite nonhyperbolic curves with respect to A, so they are contained respectively in level curves of g. From Proposition 6.4, g is a holomorphic rational function of $\boldsymbol{C}$- or $\boldsymbol{C}^*$-type on $\boldsymbol{P}^2\setminus A$. □

COROLLARY 8.3. *If* $S_M(X)=X$, *A must belong to one of two classes* (i), (ii) *of Proposition* 6.5.

Consequently, from Corollary 8.3 and Proposition 6.5 there are criterions that the case (i) or (ii) of Theorem 8.1 occurs as the following

PROPOSITION 8.4.

(1) *If at least one irreducible component of* A *is of genus* $\geqq 1$, $S_M(X)$ *is a curve or an empty set.*

(2) *If at least two irreducible components of* A *are hyperbolic curves with respect to* A, $S_M(X)$ *is a curve or an empty set.*

(3) *If the singularities of* A *are at most normal crossings,* $S_M(X)$ *is a curve or an empty set.*

Chapter II. A generalization of the big Picard theorem (1)

9. Cluster set of a holomorphic map $f: \Delta^*\to\boldsymbol{P}^2\setminus A$ at 0

THEOREM 9.1. *Let* A *be a curve with* l ($l\geqq 4$) *irreducible components in* $\boldsymbol{P}^2$. *If a holomorphic map* $f: \Delta^*\to\boldsymbol{P}^2\setminus A$ *has an essential singularity at* 0, *then* $f(0:\boldsymbol{P}^2)$ *is a curve in* $\boldsymbol{P}^2$ *which consists of nonhyperbolic curves with respect to* A.

Proof. Let us consider cases (ii) and (iii) in Theorem 8.1, since in case (i) 0 is a removable singularity of f from Corollary 2.3. In case (ii), it is easy to prove statements of Theorem 9.1 from Theorem 3.6, Proposition 3.5 and Theorem 4.3. In case (iii), there exists a holomorphic rational function g of $\boldsymbol{C}$- or $\boldsymbol{C}^*$-type on $\boldsymbol{P}^2\setminus A$ which omits $\{0, 1, \infty\}$ from Proposition 8.2. Then $g\circ f: \Delta^*\to$

$\boldsymbol{P}^1 \setminus \{0, 1, \infty\}$ can be extended to a holomorphic map $g \circ f : \Delta \to \boldsymbol{P}^1$ from the big Picard theorem. It is clear that $f(0 : \boldsymbol{P}^2) \subset \overline{g^{-1}(d)}$, where $d = g \circ f(0)$. Since $\overline{g^{-1}(d)}$ consists of finite nonhyperbolic curves with respect to A, $f(0 : \boldsymbol{P}^2)$ consists of finite nonhyperbolic curves with respect to A from Proposition 3.5. □

Remark. In Theorem 9.1, $l \geqq 4$ is a necessary condition for which $f(0 : \boldsymbol{P}^2)$ is a curve in $\boldsymbol{P}^2$. For example, $f(e^{1/z}, e^{e^{1/z}}) : \Delta^* \to \boldsymbol{C}^2(x, y) \setminus \{x=0\} \cup \{y=0\}$ has an essential singularity at 0 and $f(0 : \boldsymbol{P}^2) \supset \{y = e^x\}$.

10. The big Picard theorem for a holomorphic map $f : \Delta^* \to \boldsymbol{P}^2 \setminus A$

Let A be a curve with l ($l \geqq 4$) irreducible components in $\boldsymbol{P}^2$ and let $f : \Delta^* \to \boldsymbol{P}^2 \setminus A$ be a holomorphic map. Set $X = \boldsymbol{P}^2$ and $M = \boldsymbol{P}^2 \setminus A$.

From Theorem 8.1 there are three cases (i) $S_M(X) = \emptyset$, (ii) $S_M(X)$ is a curve and (iii) $S_M(X) = X$. In case (i), f is always extended holomorphically to $f : \Delta \to \boldsymbol{P}^2$ from Corollary 2.3. In case (iii), let us consider $f = (z, e^{1/z}) : \Delta^* \to \boldsymbol{C}^2(x, y) \setminus \{x=2\} \cup \{x=3\} \cup \{y=0\}$ for example. Then $f(\Delta^*)$ is contained in a transendental curve $\{y = e^{1/x}\}$. In case (ii), we show that if f has an essential singularity at 0, $f(\Delta^*)$ is contained in a nonhyperbolic curve with respect to A in $\boldsymbol{P}^2$ and then f is regarded as a function of one variable. Namely, we have the following

THEOREM 10.1. *Suppose that $S_M(X)$ is a curve and $f : \Delta^* \to \boldsymbol{P}^2 \setminus A$ is a holomorphic map. Then f can be extended to a holomorphic map $f : \Delta \to \boldsymbol{P}^2$ or $f(\Delta^*) \subset C$, where C is a nonhyperbolic curve with resprct to A such that $C \not\subset A$.*

Proof. From the Lemma 5.2, there is a rational map $G : \boldsymbol{P}^2 \dashrightarrow \boldsymbol{P}^p$ ($p \geqq 2$) such that $G|_{\boldsymbol{P}^2 \setminus A} : \boldsymbol{P}^2 \setminus A \to L^{p-1} = Y \setminus (H_0 \cup \cdots \cup H_p)$ is holomorphic where $H_0, \cdots, H_p$ are $p+1$ hyperplanes in general position in $Y \cong \boldsymbol{P}^{p-1}$ and $G(\boldsymbol{P}^2 \setminus A) \not\subset \Delta_d$. Set $V = G(\boldsymbol{P}^2 \setminus A)$ and $W = \overline{G(\boldsymbol{P}^2 \setminus A)}$. There are two cases such that (1) rank $G = 2$ or (2) rank $G = 1$.

At first, let us consider the case (1). According to applying $G \circ f : \Delta^* \to L^{p-1}$ for Fujimoto's theorem, there are following two cases.

(a) $G \circ f$ can be extended to a holomorphic map $G \circ f : \Delta \to W$.

(b) $G \circ f(\Delta^*) \subset \Delta_d \cap V$.

In case (a), set $G \circ f(0) = q \in W$. Then $f(0 : \boldsymbol{P}^2) \subset G^{-1}(q)$. If 0 is an essential singularity of f, then $f(0 : \boldsymbol{P}^2) = C_1 \cup \cdots \cup C_k$ where C_j ($1 \leqq j \leqq k$) is a nonhyperbolic curve with respect to A such that $G(C_j) = q$ from Theorem 9.1. In this case if $G \circ f(\Delta^*) \equiv q$, then $f(\Delta^*) \subset C_j$ for some j. If $G \circ f(\Delta^*) \not\equiv q$, there exists a positive real number $\rho \leqq 1$ such that $G \circ f(\Delta^*(\rho)) \not\ni q$. Therefore, $f(\Delta^*(\rho)) \cap C_j = \emptyset$ for every j. Now set $A' = A \cup C_1 \cup \cdots \cup C_k$ and $M' = \boldsymbol{P}^2 \setminus A'$. If $f : \Delta^*(\rho) \to \boldsymbol{P}^2 \setminus A'$ has an essential singularity at 0, then $S_{M'}(X) = X$ from Corollary 7.2 since $f(0 : \boldsymbol{P}^2) \subset A'$. Then it is clear that $S_M(X) = X$, so this is absurd. In case (b), there are two cases such that i) $\dim(\Delta_d \cap V) = 0$ or ii) $\dim(\Delta_d \cap V) = 1$. In case i), $G\circ f(\Delta^*)\equiv q\in\Delta_d\cap V$. Then $f(\Delta^*)\subset G^{-1}(q)$. If 0 is an essential singularity of f, $f(\Delta^*)\subset C$ where C is an irreducible component of $G^{-1}(q)$ which is a nonhyperbolic curve with respect to A from Theorem 3.6, Proposition 3.5 and Theorem 4.3. In case ii), $B=\overline{G^{-1}(\Delta_d\cap V)}$ is a curve in $\boldsymbol{P}^2$. Then, if 0 is an essential singularity of f, $f(\Delta^*)\subset C$ where C is an irreducible component of B and a nonhyperbolic curve with respect to A from Theorem 3.6, Proposition 3.5 and Theorem 4.3.

Next, let us consider case (2). From Corollary 5.3, there exists a holomorphic rational function g on $\boldsymbol{P}^2\setminus A$ with lacunary $\{0, 1, \infty\}$. Then $g\circ f:\Delta^*\to\boldsymbol{P}^1\setminus\{0, 1, \infty\}$ can be extended to a holomorphic map $g\circ f:\Delta\to\boldsymbol{P}^1$. Set $a=g\circ f(0)$. If $a\in\{0, 1, \infty\}$, then $f(0:\boldsymbol{P}^2)\subset A$. From Corollary 7.2, 0 is a removable singularity of f. When $a\notin\{0, 1, \infty\}$, $g\circ f\equiv a$ or $g\circ f\not\equiv a$. In the former case, $f(\Delta^*)\subset g^{-1}(a)$. If 0 is an essential singularity of f, $f(\Delta^*)\subset C$ where C is an irreducible component of $g^{-1}(a)$ and a nonhyperbolic curve with respect to A from Theorem 3.6, Proposition 3.5 and Theorem 4.3. In the latter case, there exists a positive real number $\rho\leqq 1$ such that $g\circ f(\Delta^*(\rho))\cap\{0, 1, \infty, a\}=\emptyset$. Set $A'=A\cup g^{-1}(a)$ and $M'=\boldsymbol{P}^2\setminus A'$. Suppose that $f:\Delta^*(\rho)\to\boldsymbol{P}^2\setminus A'$ has an essential singularity at 0. Then from Corollary 7.2, $S_{M'}(X)=X$ since $f(0:\boldsymbol{P}^2)\subset A'$. This is absurd since it is clear that $S_M(X)=X$. □

Chapter III. A generalization of the big Picard theorem (2)

11. Meromorphic maps

It is well known the following

PROPOSITION 11.1. *If the Cousin II problem is solvable in a domain of D in $\boldsymbol{C}^k$ and f is a holomorphic map of D to $\boldsymbol{P}^n$, then there are holomorphic functions f_i $(0\leqq i\leqq n)$ in D such that there is no common zero of f_i and $f=[f_0:\cdots:f_n]$.*

DEFINITION 11.2. Let X be a complex manifold, let Y be a compact complex manifold and $f:X\to Y$ be a meromorphic map. We denote by I_f the set of the indetermination points of f (i.e. the set of all points $\{x\}$ of X such that $f(x)$ is not a single point).

The following proposition is well known. (cf. Noguchi-Ochiai [17], Chapt. IV).

PROPOSITION 11.3. *Let X, Y and f be the same in Definition* 11.2. *Then*

i) *I_f is an analytic subset of X such that* codim $I_f\geqq 2$,

ii) *$f(x)$ is a compact connected analytic subset of Y such that* $\dim f(x)\geqq 1$ *if $x\in I_f$ and*

iii) *$f|_{X\setminus I_f}:X\setminus I_f\to Y$ is a holomorphic map.*

PROPOSITION 11.4. *If the Cousin II problem is solvable in a domain D in $\boldsymbol{C}^k$ and f is a meromorphic map of D to $\boldsymbol{P}^n$, then there are holomorphic functions f_i $(0 \leqq i \leqq n)$ in D such that $f=[f_0:\cdots:f_n]$ and codimension $\{z \in D; f_0(z)= \cdots f_n(z)=0\}$ $(=I_f)$ is greater than 1.*

Proof. From Proposition 11.3, I_f is an analytic subset of D such that codim $I_f \geqq 2$ and $f: D \setminus I_f \to \boldsymbol{P}^n$ is a holomorphic map. Since Cousin II problem is solvable in $D \setminus I_f$, there are holomorphic functions f_i $(0 \leqq i \leqq n)$ in $D \setminus I_f$ such that there is no common zero of f_i in $D \setminus I_f$ and $f|_{D \setminus I_f}=[f_0:\cdots:f_n]$. By Hartogs theorem, f_i's are holomorphic in D, so $f=[f_0:\cdots:f_n]$ in D and $I_f= \{z \in D; f_0(z)= \cdots =f_n(z)=0\}$. □

DEFINITION 11.5. Let D be a domain of $\boldsymbol{C}^k$ and f be a meromorphic map of D to $\boldsymbol{P}^n$. $[f_0:\cdots:f_n]$ is a reduced representation of f on D if f_i's $(0 \leqq i \leqq n)$ are holomorphic functions in D, $f=[f_0:\cdots:f_n]$ and $I_f=\{z \in D; f_0(z)= \cdots = f_n(z)=0\}$.

It easy to see the following

PROPOSITION 11.6. *Let N be a complex manifold* (dim $N=k \geqq 2$), *let A be an analytic subset of N such that* codim $A \geqq 2$ *and f be a meromorphic map of $N \setminus A$ to $\boldsymbol{P}^n$. Then f can be uniquely extended to a meromorphic map of N to $\boldsymbol{P}^n$.*

It is also easy to see the following

PROPOSITION 11.7. *Let f be a meromorphic map of $\Delta^* \times \Delta^{k-1}$ to $\boldsymbol{P}^n$ $(k \geqq 1)$ and let $f=[f_0:\cdots:f_n]$ be a reduced representation of f on $\Delta^* \times \Delta^{k-1}$. Suppose $f_0 \not\equiv 0$. Then f can be extended to a meromorphic map of Δ^k to $\boldsymbol{P}^n$ if and only if, $f_1/f_0, \cdots, f_n/f_0$ can be extended to meromorphic functions in Δ^k.*

It is well known the following

PROPOSITION 11.8 (cf. Theorem (Levi) in Green [7] and Corollary of Theorem 4 in Terada [20]). *Let f be a meromorphic function in $\Delta^* \times \Delta^k$ $(k \geqq 1)$ and not meromorphic in Δ^{k+1}. We denote by E the set of $(y^0) \in \Delta^k$ such that $f(x, (y^0))$ is a meromorphic function of Δ. Then* mes $E=0$.

It is easy to see the following

PROPOSITION 11.9. *Let N be an arbitrary complex manifold of* dim$=k$ $(k \geqq 2)$ *and let f be a meromorphic map of N to $\boldsymbol{P}^n$. If C is an irreducible and locally irreducible analytic subset of* dim$=1$ *in N and $C \not\subset I_f$, then $f|_C: C \to \boldsymbol{P}^n$ is holomorphic.*

LEMMA 11.10. *Let f be a meromorphic map of $\Delta^* \times \Delta^k$ $(k \geqq 1)$ to $\boldsymbol{P}^n$ and it* *can not be extended meromorphically to* Δ^{k+1}. *Then, there is a subset* E *of* Δ^k *such that* mes $E=0$ *and for every* $(y^0)\notin E$, $f|_{(y)=(y^0)}:\Delta^*\to \boldsymbol{P}^n$ *is holomorphic and* 0 *is an essential singular point.*

Proof. It is easy to see the demonstration from Propositions 11.7, 11.8 and 11.9, since the set of (y^0) such that $\{(y)=(y^0)\}\subset I_f$ is contained in an analytic subset of Δ^k.

12. The big Picard theorem for a meromorphic map $f: N\setminus B\to \boldsymbol{P}^2\setminus A$

THEOREM 12.1. *Let* N *be an arbitrary complex manifold of* $\dim=k$ $(k\geqq 1)$ *and let* B *be a proper analytic subset of* N. *Let* A *be a curve in* $\boldsymbol{P}^2$ *with* l $(l\geqq 4)$ *irreducible components. And set* $X=\boldsymbol{P}^2$ *and* $M=\boldsymbol{P}^2\setminus A$. *Assume that* $S_M(X)$ *is a curve. If* f *be a meromorphic map of* $N\setminus B$ *to* M, *then* f *can be extended to a meromorphic map of* N *to* X *or* $f(N\setminus B)\subset C$, *where* C *is a nonhyperbolic curve with respect to* A *such that* $C\not\subset A$.

Proof. If $k=1$, f is a holomorphic map of $N\setminus B$ to M. So, f can be extended to a holomorphic map of N to X or $f(N\setminus B)\subset C$, where C is a nonhyperbolic curve with respect to A from Theorem 10.1. Therefore we assume that $k\geqq 2$. Suppose that f can not be extended meromorphically to a neighborhood U of a regular point of B. Since we can consider $U\cong\Delta^*\times\Delta^{k-1}$, we may assume that $f|_U$ is a meromorphic map of $\Delta^*\times\Delta^{k-1}$ to M and f can not be extended to a meromorphic map of Δ^k to X. From Lemma 11.10 there is a subset of E of Δ^{k-1} such that mes $E=0$ and for every $(y^0)\notin E$, $f|_{(y)=(y^0)}:\Delta^*\to M$ is holomorphic and 0 is an essential singular point. From Theorem 10.1, $f(\Delta^*, (y^0))\subset C$ for fixed $(y^0)\notin E$ where C is a nonhyperbolic curve with respect to A. Since f is holomorphic in $\Delta^*\times\Delta^{k-1}\setminus I_f$, mes $E=0$ and the number of nonhyperbolic curve with respect to A in M is finite, $f(\Delta^*\times\Delta^{k-1}\setminus I_f)\subset C$. Therefore, $f(\Delta^*\times\Delta^{k-1})\subset C$. From the theorem of invariance of analytic relations and $N\setminus B$ is connected, $f(N\setminus(B\cup I_f))\subset C$. Therefore, $f(N\setminus B)\subset C$. If f can be extended meromorphically to every regular point of B, f can be extended to a meromorphic map of N to X from Proposition 11.6. □

Remark. If $S_M(X)=\emptyset$ in the same situation above, f can be always extended to a meromorphic map of N to X because there is no nonhyperbolic curve with respect to A.

And if $S_M(X)=X$, Theorem 12.1 does not hold for example, $f=(z, e^{1/z})$: $\Delta^*\to \boldsymbol{C}^2(x, y)\setminus\{x=2\}\cup\{x=3\}\cup\{y=0\}$.

COROLLARY 12.2. *Let* N, B, A, X, M *and* f *be the same in Theorem* 12.1. *Suppose that* $S_X(X)$ *is a curve or an empty set and rank* $f=2$, *then* f *can be extended to a meromorphic map of* N *to* X.

COROLLARY 12.3. *Let* A, X *and* M *be the same in Theorem* 12.1. *Suppose* *that $S_M(X)$ is a curve or an empty set, then any analytic automorphism of M is the restriction to M of a birational map of X.*

13. Application

THEOREM 13.1. *Let N be an arbitrary complex manifold of* $\dim = k$ $(k \geqq 1)$ *and let B be a proper analytic subset of N. Let X be an arbitrary compact complex manifold and let M be a relatively compact domain of X. Suppose that f is a holomorphic map of $N \setminus B$ to M and f can be extended to a meromorphic map of N to X. If a point $o \in B \cap I_f$ is at most a normal crossing singularity of B, then $f(o) \subset S_M(X)$.*

Proof. We may consider locally, so we assume that $N \setminus B = \Delta^* \times ((\Delta^*)^{k-l-1} \times \Delta^l)$ $(0 \leqq l \leqq k-1)$ and $o = (0, \cdots, 0)$. Suppose that $\Delta^* \times ((\Delta^*)^{k-l-1} \times \Delta^l) \ni (x^m, (y^m)) \to o$ and $f(x^m, (y^m)) \to p \notin S_M(X)$ $(m \to \infty)$. Set $f_m(z) = f(x^m, z \times (y_1{}^m / |y^m|), \cdots, z \times (y_{k-1}{}^m / |y^m|))$, where $(y^m) = (y_1{}^m, \cdots, y_{k-1}{}^m)$. Then f_m is a holomorphic map of Δ^* to M, $f_m(|y^m|) \to p \notin S_M(X)$ and $|y^m| \to 0$ $(m \to \infty)$, where $|y^m| = \sqrt{|y_1{}^m|^2 + \cdots + |y_{k-1}{}^m|^2}$. Since $f_m(z)$ can be extended to a holomorphic map of Δ to X from Proposition 11.9, $f_m(0) = f_m(x^m, (0)) \to p$ $(m \to \infty)$ from Corollary 2.4. Since $f(x, (0))$ can be extended to a holomorphic map of Δ to X from Proposition 11.9, $f(x, (0)) \to p$ $(x \to 0)$. Suppose that $\Delta^* \times ((\Delta^*)^{k-l-1} \times \Delta^l) \ni (\tilde{x}^m, (\tilde{y}^m)) \to o$ and $f(\tilde{x}^m, (\tilde{y}^m)) \to q \notin S_M(X)$ $(m \to \infty)$, then we conclude that $f(x, (0)) \to q$ $(x \to 0)$ by the same discussion above. Therefore $p = q$. Since $f(o)$ is a connected analytic subset of X and $\dim f(o) \geqq 1$ from Proposition 11.3, $f(o) \subset S_M(X)$. □

COROLLORY 13.2. *Let A be a curve in $\boldsymbol{P}^2$ with l $(l \geqq 4)$ irreducible components and its singularities are normal crossings. Then the number of analytic automorphisms of $\boldsymbol{P}^2 \setminus A$ is finite.*

Proof. Set $\boldsymbol{P}^2 = X$ and $M = \boldsymbol{P}^2 \setminus A$. Since each irreducible component of A is a hyperbolic curve with respect to A, $S_M(X)$ is a curve or an empty set from Proposition 8.4. Let φ be an automorphism of $\boldsymbol{P}^2 \setminus A$. From Corollary 12.3, φ can be extended to a birational map of $\boldsymbol{P}^2$. Since the image of an indetermination point consists of nonhyperbolic curves with respect to A from Theorem 13.1 and it must be contained in A, φ is an automorphism of $\boldsymbol{P}^2$. It is easy to see that the number of automorphism φ of $\boldsymbol{P}^2$ such that $\varphi(A) = A$ where A consists of 4 or more hyperbolic curves is finite. □

REFERENCES

[1] Y. ADACHI, On the hyperbolicity of projective plane with lacunary curves, J. Math. Soc. Japan, **46** (1994), 185-193.

[2] Y. ADACHI AND M. SUZUKI, On the family of holomorphic mappings into projective space with lacunary hypersurfaces, J. Math. Kyoto Univ., **30** (1990), 451-458.

[3] Y. ADACHI AND M. SUZUKI, Degeneracy points of the Kobayashi pseudodistances on complex manifolds, Several Complex Variables and Complex Geometry Part 2, Proceedings of Symposia in Pure Mathmatics, **52**, 1991, 41-51.

[4] H. CARTAN, Sur les systèmes de fonctions holomorphes à varietés linéaires, Ann. de l'École Normale, **45** (1928), 255-346.

[5] H. FUJIMOTO, Extensions of the big Picard's theorem, Tôhoku Math. J., **24** (1972), 415-422.

[6] O. FUJITA, Sur les familles d'ensembles analytiques, J. Math. Soc. Japan, **16** (1964), 379-405.

[7] M. GREEN, Some Picard theorems for holomorphic maps to algebraic varieties, Amer. J. Math., **97** (1975), 43-75.

[8] H. KASHIWARA, Fonctions rationnelles de type (0,1) sur le plan projectif complexe, Osaka J. Math., **24** (1987), 521-577.

[9] P. KIERNAN, Extensions of holomorphic maps, Trans. Amer. Math. Soc., **172** (1972), 345-355.

[10] P. KIERNAN AND S. KOBAYASHI, Holomorphic mappings into projective space with lacunary hyperplanes, Nagoya Math. J., **50** (1973), 199-216.

[11] T. KIZUKA, Analytic autumorphisms of the complement of an algebraic curve in the complex projective plane, Tôhoku Math. J., **33** (1981), 77-89.

[12] T. KIZUKA, Rational functions of C^*-type on the two-dimensional complex projective space, Tôhoku Math. J., **38** (1986), 123-178.

[13] S. KOBAYASHI, Invariant distances on complex manifolds and holomorphic mappings, J. Math. Soc. Japan, **19** (1967), 460-480.

[14] S. KOBAYASHI, Hyperbolic Manifolds and Holomorphic Mappings, Marcel Dekker, New York, 1970.

[15] S. LANG, Introduction to Complex Hyperbolic Spaces, Springer-Verlag, 1987.

[16] T. NISHINO AND M. SUZUKI, Sur les singularités essentielles et isolées des applications holomorphes à valeurs dans une surface complexe, Publ. Res. Inst. Math. Sci. Kyoto Univ., **16** (1980), 461-497.

[17] J. NOGUCHI AND T. OCHIAI, Geometric Function Theory in Several Complex Variables, Transl. Math. Monographes, **80**, Amer. Math. Soc., Providence, 1990.

[18] H. ROYDEN, Remarks on the Kobayashi metric, Several Complex Variables II, Lecture Notes in Math., 185, Springer, 1971.

[19] M. TADOKORO, Sur les ensembles pseudoconcaves généraux, J. Math. Soc. Japan, **17** (1965), 281-290.

[20] T. TERADA, Analyticitès relatives à chaque variable, J. Math. Kyoto Univ., **12** (1972), 263-296.

[21] Y. TÔKI, On the behavior of a meromorphic function in the neighborhood of a transendental singularity, Proc. Japan Acad., **17** (1941), 296-300.

[22] T. UEDA, Pseudoconcave sets of general order and essential singularities of analytic sets, preprint.

HIMEJI GAKUIN WOMEN'S JUNIOR COLLEGE
FUKUSAKI-CHO, KANZAKI-GUN, HYOGO
679-22 JAPAN

Current address:
12-29 KURAKUEN 2BAN-CHO
NISHINOMIYASHI, HYOGO 662
JAPAN