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SURFACES WITH PARALLEL MEAN CURVATURE VECTOR IN $P^2(C)$

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1. Introduction

The set of surfaces with parallel mean curvature vector in Riemannian manifold, which includes all minimal surfaces in the manifold, has been studied by many geometers. Especially, Chen [1] and Yau [7] studied them in the case that the ambient space is an n-dimensional real space form $\bar{M}^n(c)$ of constant sectional curvature c. They proved that if $x: M \to \bar{M}^n(c)$ is an isometric immersion with parallel mean curvature vector of a two-dimensional Riemannian manifold M into $\bar{M}^n(c)$, then $x(M)$ is one of the following surfaces: (1) *a minimal surface in* $\bar{M}^n(c)$, (2) *a minimal surface of a small hypersphere of* $\bar{M}^n(c)$, and (3) *a surface with constant mean curvature in a* 3-*sphere of* $\bar{M}^n(c)$. This shows that the study of surfaces in $\bar{M}^n(c)$ with parallel mean curvature vector is reduced to that of minimal surfaces except the case (3).

On the other hand, concerning the surfaces with parallel mean curvature vector in a complex space form, we know several minimal surfaces in the n-dimensional complex projective space $P^n(C)$ with the Fubini-Study metric of constant holomorphic sectional curvature 4ρ. Moreover, many results characterizing them have been obtained (cf. [2], [3], [4], [5], [6]). However, when we concern with non-minimal surfaces in $P^n(C)$ with parallel mean curvature vector, not many such examples are known so far, even for $n=2$.

In Sections 1 and 2 of the previous paper [5], we developed a local theory of surfaces in $P^n(C)$ by using the Kaehler function. By applying it, in this paper we shall study non-minimal immersions $x: M \to P^2(C)$ with parallel mean curvature vector. In fact, in Section 2 we obtain basic formulas for such surfaces in a 2-dimensional Kaehler manifold of constant holomorphic sectional curvature 4ρ. Then, in Sections 3 and 4, we show a method of the local construction of such immersions. Finally, in Section 5 we determine isometric immersions with parallel mean curvature vector field of a Riemannian 2-manifold with constant Gaussian curvature into $P^2(C)$. Theorem 5.2 generalizes a theorem by Ludden, Okumura and Yano [4].

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2. The fundamental theorem of surfaces in a Kaehler manifold

Let X be a complex 2-dimensional Kaehler manifold of constant holomorphic sectional curvature 4ρ. We consider some basic properties of immersed surfaces in X. Let $\{\omega_\alpha\}$ be a local field of unitary coframes on X, so that the Kaehler metric is represented by $\sum \omega_\alpha \bar{\omega}_\alpha$. Here and in what follows, we will agree on the following range of indices: $1 \leqq \alpha, \beta, \gamma \leqq 2$. We denote by $\omega_{\alpha\beta}$ the unitary connection forms with respect to $\{\omega_\alpha\}$. The structure equations of X are given by

$$
\begin{aligned}
&d\omega_\alpha = \sum \omega_{\alpha\beta} \wedge \omega_\beta, \quad \omega_{\alpha\beta} + \bar{\omega}_{\beta\alpha} = 0, \\
&d\omega_{\alpha\beta} = \sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}, \\
&\Omega_{\alpha\beta} = -\rho(\omega_\alpha \wedge \bar{\omega}_\beta + \delta_{\alpha\beta} \sum \omega_\gamma \wedge \bar{\omega}_\gamma).
\end{aligned} \tag{2.1}
$$

Let (M, ds^2) be an oriented connected 2-dimensional Riemannian manifold. The tangent bundles of M and X are denoted by TM and TX, respectively. Let $x: M \to X$ be an isometric immersion of M into X. By means of the differential dx we may consider TM as a subbundle of the induced bundle x^*TX over M, so that we get the orthogonal decomposition $x^*TX = TM \oplus NM$, where NM denotes the normal bundle of x.

Let $\{e_1, e_2\}$ be an oriented orthonormal local frame on M. Let $\langle , \rangle$ denote the Riemannian metric of X induced by the Kaehler metric $\sum \omega_\alpha \bar{\omega}_\alpha$ and J the complex structure of X. The *Kaehler function* $\cos(\alpha)$ on M is defined by

$$\cos(\alpha) = \langle Je_1, e_2 \rangle,$$

which is independent of the choice of oriented orthonormal frames on M. The immersion is said to be holomorphic if $\cos(\alpha)=1$ on M, anti-holomorphic if $\cos(\alpha)=-1$ on M, and totally real if $\cos(\alpha)=0$ on M.

Recall that in Sections 1 and 2 of [5], it was assumed that x is neither holomorphic nor anti-holomorphic at a neighbourhood of any point of M. In this paper, we also assume the same conditions on x, and use the some formulas obtained in Sections 1 and 2 of [5]. Let H be the mean curvature vector field of x, which is defined by

$$H = \frac{1}{2} \sum_{\lambda, i} h_{\lambda ii} e_\lambda,$$

where $h_{\lambda ij}$'s are the components of the second fundamental form of x (cf. Section 1 [5]), and e_i and e_λ are the adapted frames along x. The immersion x is called minimal if $H=0$ on M. Let $D^\perp$ denote the connection of the normal bundle NM. If

$$D^\perp H = 0$$

on $\boldsymbol{M}$, then $\boldsymbol{H}$ is called the *parallel* mean curvature vector field.

We assume that $\boldsymbol{H}\neq 0$, $\boldsymbol{D}^{\perp}\boldsymbol{H}=0$ on $\boldsymbol{M}$, and the Kaehler function is $\cos(\alpha)$. We can construct a unique system of global orthonormal vector fields $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ along $\boldsymbol{M}$ such that $\tilde{e}_1$ and $\tilde{e}_2$ are tangent to $\boldsymbol{M}$ by the following: First we put $\tilde{e}_3=-\boldsymbol{H}/\|\boldsymbol{H}\|$, then the normal vector field $\tilde{e}_4$ of NM is uniquely determined by choosing it to be compatible with the fixed orientations of $\boldsymbol{M}$ and $\boldsymbol{X}$. The system of vectors $\{\tilde{e}_3, \tilde{e}_4, J\tilde{e}_3, J\tilde{e}_4\}$ is linearly independent, because x is neither holomorphic nor anti-holomorphic. We have the identity

$$\cos(\alpha)=\langle J\tilde{e}_4, \tilde{e}_3\rangle$$

which is easily proved by using the fact that $\cos(\alpha)$ is independent of the choice of the oriented orthonormal frame on $\boldsymbol{M}$. By using the Schmidt orthonormalization, we get a new frame $\{\tilde{e}_1, \tilde{e}_2\}$ on $\boldsymbol{M}$, which is explicitly represented as follows

$$\tilde{e}_1=\cot(\alpha)\tilde{e}_3-\operatorname{cosec}(\alpha)J\tilde{e}_4,$$

$$\tilde{e}_2=\operatorname{cosec}(\alpha)J\tilde{e}_3+\cot(\alpha)\tilde{e}_4.$$

It is easy to see that $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ is an adapted frame on $\boldsymbol{M}$ in $\boldsymbol{X}$, that is, $\tilde{e}_1$ and $\tilde{e}_2$ are sections on $T\boldsymbol{M}$ and $\tilde{e}_3$ and $\tilde{e}_4$ are sections on NM. Moreover, we define vector fields e_1 and e_3 as follows:

$$e_1=\frac{1}{2}\sec\left(\frac{\alpha}{2}\right)(\tilde{e}_1-J\tilde{e}_2),$$

$$e_3=\frac{1}{2}\operatorname{cosec}\left(\frac{\alpha}{2}\right)(\tilde{e}_1+J\tilde{e}_2),$$

and put

$$e_2=Je_1 \quad \text{and} \quad e_4=Je_3.$$

Then $\{e_1, e_2, e_3, e_4\}$ is a J-canonical frame along x (cf. Section 1 [5]). We extend $\{\tilde{e}_A\}$ and $\{e_A\}$ to a neighbourhood of $\boldsymbol{M}$ in $\boldsymbol{X}$, where A, B and C run from 1 through 4.

Let $\{\tilde{\theta}_A\}$ and $\{\theta_A\}$ be the dual coframes of $\{\tilde{e}_A\}$ and $\{e_A\}$ respectively. Let $\tilde{\theta}_{AB}$ and θ_{AB} be the Riemannian connection forms with respect to the canonical 1-forms $\{\tilde{\theta}_A\}$ and $\{\theta_A\}$, respectively and put

$$\omega_\alpha=\theta_{2\alpha-1}+i\theta_{2\alpha},$$

$$\omega_{\alpha\beta}=\theta_{2\alpha-1,2\beta-1}+i\theta_{2\alpha,2\beta-1}, \quad \text{where } i=\sqrt{-1}.$$

Then we have the following relations (cf. [5]):

$$\tilde{\theta}_{12}=i\left(\cos^2\left(\frac{\alpha}{2}\right)\omega_{11}-\sin^2\left(\frac{\alpha}{2}\right)\omega_{22}\right), \tag{2.2}$$

$$\tilde{\theta}_{34}=-i\left(\sin^2\left(\frac{\alpha}{2}\right)\omega_{11}-\cos^2\left(\frac{\alpha}{2}\right)\omega_{22}\right),$$

$$\tilde{\theta}_{13}+i\tilde{\theta}_{23}=-\omega_{12}-\frac{1}{2}(d\alpha-\sin(\alpha)(\omega_{11}+\omega_{22})),$$

$$\tilde{\theta}_{14}+i\tilde{\theta}_{24}=i\left\{\omega_{12}-\frac{1}{2}(d\alpha-\sin(\alpha)(\omega_{11}+\omega_{22}))\right\}.$$

We denote the restriction of $\{\tilde{\theta}_A\}$ to M by the same letters and put

$$\phi=\tilde{\theta}_1+i\tilde{\theta}_2.$$

By the assumptions, $\tilde{e}_3$ is a parallel vector field along M, hence so is $\tilde{e}_4$. This implies

$$\tilde{\theta}_{34}=0. \tag{2.3}$$

Then it is proved that there exist a positive number b, complex-valued smooth functions a and c defined locally on M, which satisfy the followings (cf. (2.1) and (2.2) in [5]):

$$\begin{aligned}
&\tilde{\theta}_{12}=i\cot(\alpha)\{(a-b)\phi-(\bar{a}-b)\bar{\phi}\},\\
&d\alpha=(a+b)\phi+(\bar{a}+b)\bar{\phi},\\
&(da-ia\tilde{\theta}_{12})\wedge\phi=-\left\{\cot(\alpha)(\bar{a}-b)a+\frac{3}{4}\rho\sin(2\alpha)\right\}\phi\wedge\bar{\phi},\\
&(dc+3ic\tilde{\theta}_{12})\wedge\bar{\phi}=\cot(\alpha)(b-a)c\phi\wedge\bar{\phi},\\
&\boldsymbol{H}=-2b\tilde{e}_3.
\end{aligned} \tag{2.4}$$

The third and fourth formulas of (2.4) are the Codazzi equations of x.

Denoting by K the Gaussian curvature of M, the Gauss equation is written as

$$K=6\rho\cos^2(\alpha)-4(|a|^2-b^2). \tag{2.5}$$

Let K_N be the normal curvature of x defined by

$$d\tilde{\theta}_{34}=-K_N\tilde{\theta}_1\wedge\tilde{\theta}_2.$$

By taking the exterior derivative of the second formula of (2.2) and using the formula (2.1) in [5], we have

$$K_N=(3\cos^2(\alpha)-1)\rho+2(|c|^2-|a|^2).$$

Since now the normal curvature vanishes, we get

$$|c|^2=|a|^2-\frac{\rho}{2}(3\cos^2(\alpha)-1). \tag{2.6}$$

Combining formulas (2.5) and (2.6), we get

$$K=(1+3\cos^2(\alpha))\rho-2(|a|^2-2b^2+|c|^2). \tag{2.7}$$

For a neighbourhood U of a point of $\boldsymbol{M}$, there exists an isothermal coordinate

$$z=u+iv \quad \text{such that} \quad ds^2=\lambda^2|dz|^2,$$

w ere λ is a positive function defined on U, and we have

$$\phi=\lambda dz.$$

Then the set of the first three formulas of (2.4) is rewritten as the following system of differential equations:

$$\begin{aligned}
&\frac{\partial\lambda}{\partial z}=-\lambda^2\cot(\alpha)(a-b), \\
(2.8) \qquad &\frac{\partial\alpha}{\partial z}=\lambda(a+b), \\
&\frac{\partial a}{\partial\bar{z}}=\lambda\left\{2\cot(\alpha)(\bar{a}-b)a+\frac{3}{4}\rho\sin(2\alpha)\right\}.
\end{aligned}$$

By using (2.8), we have that

$$(2.9) \qquad \frac{\partial^2\lambda}{\partial z\partial\bar{z}}=\frac{\partial^2\lambda}{\partial\bar{z}\partial z} \quad \text{if and only if } \bar{a}=a.$$

Therefore a is a real-valued function defined locally on $\boldsymbol{M}$. This implies that λ, α and a are functions of single variable, and (2.8) is seen to be a system of ordinary differential equations. Consequently, if $\boldsymbol{M}$ is a non-minimal surface with parallel mean curvature in X, then there exists a positive number b and real-valued smooth functions of single variable λ, α and a which are defined locally on $\boldsymbol{M}$ and satisfy the system of ordinary differential equations (cf. 3.1).

Remark. The fourth formula of (2.4) is equivalent to the equation

$$(2.10) \qquad \frac{\partial(\lambda^2 c)}{\partial z}=0.$$

In the next section, we shall consider a converse problem to the result obtained above, that is, a local existence problem for non-minimal surface in X with parallel mean curvature vector. To this end, we need the fundamental theorem of surfaces theory in X. When X is a real space form, the fundamental theorem of submanifolds is well-known. On the other hand, for a surface in a complex 2-dimensional Kaehler manifold of constant holomorphic sectional curvature 4ρ, the following fundamental theorem is proved by Eschenburg et al. [2], which is in essential use in this paper:

THEOREM 2.1 ([2]). *Let* $(\boldsymbol{M}, ds^2)$ *be a connected, simply connected 2-dimensional Riemannian manifold. Given complex-valued 1-forms* ω_1, ω_2, ω_{11}, ω_{22} *and* ω_{12} *defined on* $\boldsymbol{M}$ *satisfying the structure equations* (2.1) *and*

$$ds^2=\omega_1\bar{\omega}_1+\omega_2\bar{\omega}_2,$$

there exist an isometric immersion $x: \boldsymbol{M}\to\boldsymbol{X}$ *and a unitary frame* $\{E_1, E_2\}$ *along* x *such that* $\{\omega_1, \omega_2\}$ *is the unitary coframe of* $\{E_1, E_2\}$ *and* ω_{11}, ω_{22} *and* ω_{12} *are the unitary connection forms with respect to* $\{\omega_1, \omega_2\}$.

3. Local construction of surfaces in $\boldsymbol{P}^2(\boldsymbol{C})$

It was B. Y. Chen who constructed surfaces with constant mean curvature in a 3-dimensional real space form (cf. [1], p. 121). In Theorem 3.11 of [2], Eschenburg et al. proved a local existence theorem for minimal surface in $\boldsymbol{P}^2(\boldsymbol{C})$. In this section, we consider a method of the local construction of a non-minimal surface with parallel mean curvature vector in a complex 2-dimensional Kaehler manifold.

THEOREM 3.1. *Let* b *and* ρ *be real numbers* ($b>0$), *and* λ, α *and* a *be real-valued smooth functions of single variable* u *defined on an interval* I, *which satisfy the following system of ordinary differential equations*:

$$\begin{aligned}
&\frac{d\lambda}{du}=-2\lambda^2\cot(\alpha)(a-b),\\
(3.1)\qquad &\frac{d\alpha}{du}=2\lambda(a+b),\\
&\frac{da}{du}=2\lambda\left\{2\cot(\alpha)(a-b)a+\frac{3}{4}\rho\sin(2\alpha)\right\}.
\end{aligned}$$

Let $\boldsymbol{M}$ *be an open domain of* (u, v)*-plane contained in* $I\times(-1, 1)$. *Define*

$$ds^2=\lambda^2(du^2+dv^2)$$

on $\boldsymbol{M}$. *Suppose that for any constants* k_1 *and* k_2, λ, α *and* a *satisfy*

$$\log\left(\lambda^4\left(a^2-\frac{\rho}{2}(3\cos^2(\alpha)-1)\right)\right)=k_1u+k_2. \tag{3.2}$$

Then we can construct an isometric immersion $x: \boldsymbol{M}\to\boldsymbol{X}$ *of* $\boldsymbol{M}$ *into a complex 2-dimensional Kaehler manifold* $\boldsymbol{X}$ *which satisfies the following*:

(1) *x has a non-zero parallel mean curvature vector field whose length is* $2b$,

(2) *the Kaehler function of* x *is* $\cos(\alpha)$,

(3) *the second fundamental form of* x *is explicitly written in terms of* a, b, λ *and* α.

Proof. Let (r, s, t) be the standard coordinate of $\boldsymbol{R}^3$ and $\boldsymbol{D}$ a domain in $\boldsymbol{R}^3$ such that $r>0$ and $0<s<\pi$. We define a $\boldsymbol{R}^3$-valued function $f(r, s, t)$ on $\boldsymbol{D}$ by

$$f(r,\ s,\ t)=\begin{pmatrix} -r^2\cot(s)(t-b) \\ r(t+b) \\ r\{2\cot(s)(t-b)t+3\rho\sin(2s)/4\} \end{pmatrix}.$$

$f(r, s, t)$ has continuous partial derivatives on $\boldsymbol{D}$, so that it satisfies Lipschitz condition on $\boldsymbol{D}$. Hence, a solution of the system (3.1) exists and is unique under preassigned initial conditions.

Let (λ, α, a) be a solution of (3.1) which satisfy (3.2) and we put

$$z=u+iv \quad \text{and} \quad \phi=\lambda dz.$$

We define a complex-valued function c on $\boldsymbol{M}$ by

$$c=\frac{\beta}{\lambda^2}\exp\left(\frac{k_1}{2}(u-iv)\right) \tag{3.3}$$

where β is a complex constant. Then it is proved that $\lambda^2 c$ is anti-holomorphic, which is equivalent to (2.10) and $|c|^2$ satisfies (2.6). We define ω_1, ω_2, ω_{11}, ω_{22} and ω_{12} on $\boldsymbol{M}$ as follows:

$$\begin{aligned} &\omega_1=\cos\left(\frac{\alpha}{2}\right)\phi, \\ &\omega_2=\sin\left(\frac{\alpha}{2}\right)\bar{\phi}, \\ &\omega_{11}=\frac{1}{2}\cot\left(\frac{\alpha}{2}\right)\{(a-b)\phi-(a-b)\bar{\phi}\}, \\ &\omega_{22}=\frac{1}{2}\tan\left(\frac{\alpha}{2}\right)\{(a-b)\phi-(a-b)\bar{\phi}\}, \\ &\omega_{12}=-\bar{\omega}_{21}=b\phi+c\bar{\phi}. \end{aligned} \tag{3.4}$$

Note that these satisfy (2.1) because of (3.1). Therefore, by Theorem 2.1, we have an isometric immersion $x: \boldsymbol{M}\rightarrow \boldsymbol{X}$ which has a non-zero, parallel mean curvature vector field and $\cos(\alpha)$ the Kaehler function. The second fundamental form of x is explicitly written in terms of a, b, λ and α by (2.2) of [5].

q. e. d.

4. Associated family of isometric immersion

It is well known that there exists a one-parameter family of isometric surfaces in $\boldsymbol{R}^3$ with the same constant mean curvature. The following theorem shows that an analogous property holds in the case that the ambiant space is a 2-dimensional complex space form $\boldsymbol{X}$ and that the mean curvature vector field $\boldsymbol{H}$ of an immersed surface is parallel. Note that Eschenburg et al. [2] have proved that there exists a one-parameter family of isometric minimal immersions of a simply connected surface into $P^2(C)$ with the same normal curvature and

Kaehler function. Theorem 4.1 is an extension of Theorem B in [2] stated above.

THEOREM 4.1. *Let* $(\boldsymbol{M}, ds^2)$ *be a simply connected oriented 2-dimensional Riemannian manifold,* $x:\boldsymbol{M}\to\boldsymbol{X}$ *an isometric immersion with non-zero parallel mean curvature vector field* $\boldsymbol{H}$ *and* $\cos(\alpha)$ *the Kaehler function. Assume that the immersion* x *is neither holomorphic nor anti-holomorphic. Then there exists a one-parameter family of isometric immersions* $x_t:\boldsymbol{M}\to\boldsymbol{X}$, $t\in(-\pi, \pi)$, *which satisfies the following properties*:

(1) $x_0=x$,

(2) x_t *is isometric to* x *for each* t,

(3) $\|\boldsymbol{H}_t\|=\|\boldsymbol{H}\|\neq 0$, *where* $\boldsymbol{H}_t$ *denotes the mean curvature vector field of* x_t,

(4) $\boldsymbol{D}_t^{\perp}\boldsymbol{H}_t=0$, *where* $\boldsymbol{D}_t^{\perp}$ *is the normal connection of* x_t,

(5) $\cos(\alpha_t)=\cos(\alpha)$,

(6) x_t *is not congruent to each other.*

Proof. By the assumptions, we can use results in Section 2. The first formula in (2.4) implies

$$a\phi\wedge\bar{\phi}=i\tan(\alpha)\tilde{\theta}_{12}\wedge\bar{\phi}+b\phi\wedge\bar{\phi}\,.$$

This shows that the real valued function a is uniquely determined by the Riemannian metric ds^2, the mean curvature vector $\boldsymbol{H}$ and $\cos(\alpha)$. By (2.6), $|c|^2$ is also uniquely determined by ds^2, $\boldsymbol{H}$ and $\cos(\alpha)$. We put

$$c=|c|e^{i\tau},\quad 0\leqq\tau<2\pi$$

where τ is a real-valued function on $\boldsymbol{M}$. Then (3.3) shows that τ is uniquely determined by ds^2, $\boldsymbol{H}$ and $\cos(\alpha)$, up to additive constants. Hence, if we put

$$c_t=ce^{it}\quad\text{for some } t\in(-\pi, \pi),$$

then c_t also satisfies the fourth formula in (2.4). We put

$$\omega_{12}=-\bar{\omega}_{21}=b\phi+c_t\bar{\phi}\,,$$

and the other connection forms are defined similarly as in (3.4). Then ω_1, ω_2, ω_{11}, ω_{22} and ω_{12} satisfy (2.1) for each t. Hence, by Theorem 2.1, for each t we have an isometric immersion $x_t:\boldsymbol{M}\to\boldsymbol{X}$ for which the adapted frame

$$\{\tilde{e}_1(t),\ \tilde{e}_2(t),\ \tilde{e}_3(t),\ \tilde{e}_4(t)\}$$

along x_t satisfies

$$\boldsymbol{H}_t=-2b\tilde{e}_3(t),\quad \boldsymbol{D}_t^{\perp}\boldsymbol{H}_t=0\,,$$

and $\cos(\alpha)$ is the Kaehler function of x_t for each t. q.e.d.

COROLLARY 4.2. *Let* $x_i:\boldsymbol{M}\to\boldsymbol{X}$ $(i=1, 2)$ *be an isometric immersion with non-zero, parallel mean curvature vector field* $\boldsymbol{H}_i$ *and the Kaehler function* $\cos(\alpha_i)$.

Assume that x_i are neither holomorphic nor anti-holomorphic, and that x_1 is isometric to x_2. Then x_1 is congruent to x_2 if and only if

$$\cos(\alpha_1)=\cos(\alpha_2), \quad \|H_1\|=\|H_2\| \quad \text{and} \quad c_1=c_2 .$$

5. Complete flat surface with parallel mean curvature vector

In this section we apply the results obtained in this paper for the case that (M, ds^2) is a Riemannian manifold of constant Gaussian curvature. As a result, we determine all isometric immersions of the (M, ds^2) into $P^2(C)$ with parallel mean curvature vector field. We put $\rho=1$ for simplicity.

Let $M^2[K]$ denote an oriented connected 2-dimensional Riemannian manifold of constant Gaussian curvature K and $x: M^2[K]\to P^2(C)$ be an isometric immersion whose mean curvature vector field H is parallel but non-vanishing. Differentiating (2.5) and using $\bar{a}=a$, we have

$$2a\frac{da}{du}+3\cos(\alpha)\sin(\alpha)\frac{d\alpha}{du}=0 . \tag{5.1}$$

Since the system (3.1) is valid for the immersion x, the formulas (5.1) and (3.1) give

$$\cos(\alpha)\equiv 0$$

or

$$3\sin^2(\alpha)=\frac{-4a^2(a-b)}{2a+b}.$$

It follows from these formulas and the Gauss equation (2.5) that a is constant, $\alpha=\pi/2$ and hence $K=0$. Note that we have $k_1=0$ in (3.2). In consequence, we obtain the following.

PROPOSITION 5.1. *Let $M^2[K]$ be an oriented 2-dimensional Riemannian manifold of constant Gaussian curvature K and $x: M^2[K]\to P^2(C)$ an isometric immersion such that the mean curvature vector field is parallel and not zero. Then x is totally real and $K=0$.*

Now we are going to determine isometric immersions with parallel mean curvature vector field of a complete flat surface into $P^2(C)$. Let R^2 be the Euclidean 2-plane with the standard flat metric $ds^2=du^2+dv^2$. We put

$$\phi=dz \quad \text{and} \quad z=u+iv .$$

Let $x: R^2\to P^2(C)$ be an isometric immersion with non-zero parallel mean curvature vector field. It follows from Proposition 5.1 that the x must be totally real and $\alpha=\pi/2$. By (2.4), we have $a=-b$. By (2.6) and (3.2), c is a complex constant with

$$|c|^2=b^2+\frac{1}{2}.$$

On account of Theorem 4.1, we may assume that c is real. Therefore we have

$$\begin{aligned}
&\omega_1=\frac{1}{\sqrt{2}}\phi,\\
&\omega_2=\frac{1}{\sqrt{2}}\bar{\phi},\\
(5.2)\qquad &\omega_{11}=-b(\phi-\bar{\phi}),\\
&\omega_{22}=-b(\phi-\bar{\phi}),\\
&\omega_{12}=-\bar{\omega}_{21}=b\phi+c\bar{\phi},
\end{aligned}$$

where b and c are real constants such that $b>0$ and $c=\sqrt{b^2+1/2}$.

We can solve the system (5.2) in the same way as in Kenmotsu [3, p.p. 679-681]: Let λ_i, $i=0, 1, 2$, be the eigenvalues of the matrix A defined by

$$(5.3)\qquad A=\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & b & c \\ -\frac{1}{\sqrt{2}} & -b & b \end{pmatrix}.$$

It is easy to see that, if necessary renumbering λ_i, λ_0 is a non-zero real number which is not rational, λ_1 is a complex number which is not real and λ_2 is the complex conjugate of λ_1. Put

$$G=\{(\exp(\lambda_i z-\bar{\lambda}_i\bar{z})\delta_{ij}) \mid z=u+iv,\ (u, v)\in \boldsymbol{R}^2\}.$$

Then $x(\boldsymbol{R}^2)$ is an orbit of the abelian Lie subgroup G of the unitary group $U(3)$. We remark that G is homeomorphic to the cylinder $S^1\times\boldsymbol{R}^1$.

Summarizing our results of this section, we obtain the following.

THEOREM 5.2. *Let* $x:\boldsymbol{R}^2\rightarrow\boldsymbol{P}^2(\boldsymbol{C})$ *be an isometric immersion with non-zero parallel mean curvature vector field* $\boldsymbol{H}$. *Then* $x(\boldsymbol{R}^2)$ *is an orbit of the abelian Lie subgroup* G *of* $U(3)$ *and* G *is algebraically determined by the constant* b, *where* $2b$ *is the length of* $\boldsymbol{H}$.

It should be remarked that when x is minimal and totally real, this theorem was proved in [4].

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