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KODAI MATH. J
18 (1995), 377—396

NEVANLINNA THEORY FOR MINIMAL SURFACES OF PARABOLIC TYPE

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§ 1. Introduction

Consider a minimal surface $x=(x_1, \cdots, x_m): M \rightarrow \boldsymbol{R}^m$ in $\boldsymbol{R}^m$. In the previous papers [9], [10], [11] and [12], the author gave some value-distribution-theoretic properties of the Gauss map of M in the case where M is complete (cf., [13]). On the other hand, under some assumptions, E.F. Beckenbach and his collaborators showed that the map x itself has many properties which are similar to the results in Nevanlinna theory for meromorphic functions on $\boldsymbol{C}$ in their papers [4], [3], [2] and [6]. They developed their theory for 'meromorphic minimal surfaces'. Roughly speaking, these are minimal surfaces in $\boldsymbol{R}^m$ with at worst pole-like singularities which is conformally isomorphic with the complex plane. The purpose of this paper is to extend some of their results to the case where M is conformally isomorphic with a Riemann surface of parabolic type. For brevity, we restrict ourselves to the case of regular minimal surfaces though our arguments are also available for minimal surfaces with pole-like singularities.

By definition, a Riemann surface M is of parabolic type if there is a proper map $\tau: M \rightarrow [0, +\infty)$ of class C^{∞} such that $dd^c \log\tau=0$ and $dd^c\tau \not\equiv 0$ on $M-M_s$ for some $s>0$, where $M_s := \{a \in M;\ \tau(a)<s\}$. We define the hyperspherical function by

$$m^0(r;\ c,\ M) := \frac{2}{r}\int_{\partial M_r} \log\frac{1}{|x,\ c|}\, d^c\tau - \frac{2}{s}\int_{\partial M_s} \log\frac{1}{|x,\ c|}\, d^c\tau,\ (c \in \bar{\boldsymbol{R}}^m)$$

and the order function for M by $T^0(r;\ M) := m^0(r;\ \infty,\ M)$, where $|x,\ c|$ denotes a half of the chordal distance between $\varpi^{-1}(x)$ and $\varpi^{-1}(c)$ for the stereographic projection ϖ of the unit sphere in $\boldsymbol{R}^{m+1}$ onto $\bar{\boldsymbol{R}}^m := \boldsymbol{R}^m \cup \{\infty\}$. We define also the counting function and the visibility function by

$$N(r;\ c,\ M) := \int_s^r \sum_{a \in M_t} \nu_{|x-c|}(a) \frac{dt}{t},$$

$$H(r;\ c,\ M) := \int_s^r \frac{dt}{t}\int_{M_t} dd^c \log|x-c|^2$$

Received January 10, 1994; revised September 5, 1994.

respectively for each $c \in \boldsymbol{R}^m$, where $\nu_{|x-c|}(a)$ denotes the multiplicity of zero of $|x-c|$ at a. Let $G: M \to P^{m-1}(\boldsymbol{C})$ be the Gauss map of M (cf., [13, § 1.2]). For the pull-back Ω_G of the (normalized) Fubini-Study metric on $P^{m-1}(\boldsymbol{C})$ we define the order function of G by

$$T_G(r) := \int_s^r \frac{dt}{t} \int_{M_t} \Omega_G,$$

and consider the function $E(r) := \int_s^r (\chi(\overline{M}_t)/t) dt$, where $\chi(\overline{M}_t)$ denotes the Euler characteristic of $\overline{M}_t$.

We shall give the first main theorem;

$$T^0(r; M) = m^0(r; c, M) + H(r; c, M) + N(r; c, M),$$

and the second main theorem which asserts that, for each $\varepsilon > 0$,

$$\sum_{j=1}^{q} m^0(r; c_j, M) + T_G(r) \leqq (2+\varepsilon) T^0(r; M) - E(r) + O(1)$$

for all r except in a set E with $\int_E (1/t) dt < \infty$. We give also the defect relation for minimal surfaces which is similar to that for meromorphic functions. In the last section, we study complete minimal surfaces in $\boldsymbol{R}^m$ with finite total curvature, and show that the number two of the second main theorem is sharp.

§ 2. Some integral formulas

Let M be an open Riemann surface and consider a nonzero complex-valued function u on a domain D in M possibly with isolated singularities. We call u a function with admissible singularities if u is of class $\boldsymbol{C}^\infty$ outside a discrete subset of D such that, on some neighborhood U of each $a \in D$, we can write

$$|u(z)| = |z-a|^{\sigma_a} u^*(z) \tag{2.1}$$

with a holomorphic local coordinate $z = x + iy$ on U, $\sigma_a \in \boldsymbol{R}$ and a nonnegative continuous function u^* satisfying the condition that, for $v := \log u^*$,

$$|v| = o\left(\frac{1}{|z-a|}\right), \quad \left|\frac{\partial v}{\partial x}\right| = o\left(\frac{1}{|z-a|}\right), \quad \left|\frac{\partial v}{\partial y}\right| = o\left(\frac{1}{|z-a|}\right), \tag{2.2}$$

$dd^c v$ is locally integrable.

The number σ_a in (2.1) is obviously unique. The map $\nu_u: D \to \boldsymbol{R}$ defined by $\nu_u(a) := \sigma_a (a \in D)$ for the number σ_a appearing in (2.1) gives a divisor on D which we call the divisor of u, where a divisor on D means a map $\nu: D \to \boldsymbol{R}$ which vanishes outside a discrete set in D. We mean by a pseudo-metric on M a conformal metric ds^2 possibly with isolated singularities which can be locally written as $ds^2=\lambda_z^2|dz|^2$ by using a nonnegative functions λ_z with admissible singularities, where z is a holomorphic local coordinate. The divisor of a pseudometric ds^2 is defined by $\nu_{ds}:=\nu_{\lambda_z}$ for each local expression $ds^2=\lambda_z^2|dz|^2$. Obviously, ν_{ds} is globally well-defined on M. For a nonzero meromorphic form ω on M we define the divisor of ω by setting $\nu_\omega:=\nu_{f_z}$ for local expressions $\omega=f_z dz$.

We denote by $\mathcal{D}_{p,q}$ the space of all C^∞ differential (p, q)-forms on M with compact supports. For a C^∞ (p, q)-form Ω on M we can define a (p, q)-current $[\Omega]$ by $[\Omega](\varphi)=\int_M \Omega\wedge\varphi$ $(\varphi\in\mathcal{D}_{1-p,1-q})$ and, for a locally integrable function u on M, the $(0, 0)$-current $[u]$ is defined by $[u](\varphi)=\int_M u\varphi$ $(\varphi\in\mathcal{D}_{1,1})$. They are simply denoted by Ω and u respectively if we have no confusion. Moreover, with each divisor ν we can associate the $(1, 1)$-current $[\nu]$ defined by $[\nu](\varphi)=\sum_{a\in M}\nu(a)\varphi(a)$ $(\varphi\in\mathcal{D}_{0,0})$. As usual, we define the differentials of (p, q)-current T by

$$\partial T(\varphi)=(-1)^{p+q-1}T(\partial\varphi) \qquad (\varphi\in\mathcal{D}_{-p,1-q}),$$

$$\bar\partial T(\varphi)=(-1)^{p+q-1}T(\bar\partial\varphi) \qquad (\varphi\in\mathcal{D}_{1-p,-q})$$

and

$$dT:=(\partial+\bar\partial)T,\ d^cT:=\frac{\sqrt{-1}}{4\pi}(\bar\partial-\partial)T.$$

For later use, we give the following Stokes theorem for currents.

(2.3) *Let T be a 1-current on a Riemann surface M and D a relatively compact domain in M with smooth boundary. If T is equal to a C^∞ one form on some neighborhood of ∂D, then*

$$\int_D dT=\int_{\partial D}T.$$

To see this, we write $T=[\eta]+T'$ with a C^∞ one form η on M and a current T' which vanishes on a neighborhood of ∂D. We know that (2.3) holds for $T=[\eta]$, and we easily have

$$\int_D dT'\equiv dT'(\varphi)=T'(d\varphi)=0=\int_{\partial D}T'$$

for every $\varphi\in\mathcal{D}_{0,0}$ with $\varphi=1$ on $\mathrm{Supp}\,(T')$. These give (2.3).

For a function u with admissible singularities, we can prove the following:

PROPOSITION 2.4. $dd^c[\log|u|^2]=[\nu_u]+[dd^c\log|u|^2]$.

The proof is similar to that of [13, Proposition 4.1.4]. We omit the details.

Consider a relatively compact domain D in M with smooth real analytic boundary. We say that a meromorphic form ω on D is purely imaginary on ∂D if ω has a continuous extension to ∂D satisfying the condition that $\mathrm{Re}\,\omega=0$

and ω has no zero or pole. We note here that, by the principle of reflection, such a meromorphic form can be extended to a meromorphic form on an open neighborhood of $\bar{D}$.

PROPOSITION 2.5. *Let D be a relatively compact domain in M with smooth real analytic boundary and ω a meromorphic form on D which is purely imaginary on ∂D. Then,*

$$\sum_{a\in D}\nu_{\omega}(a)=-\chi(\bar{D}). \tag{2.6}$$

Proof. Take the double $\hat{D}$ of D, namely, the welding of the domain D and the conjugate surface of D along ∂D (cf., [1, p. 119]). Then, $\hat{D}$ is a compact Riemann surface with $\chi(\hat{D})=2\chi(\bar{D})$ and the meromorphic form ω is extended to a meromorphic form $\hat{\omega}$ on $\hat{D}$ satisfying the identity $\sum_{a\in\hat{D}}\nu_{\hat{\omega}}(a)=2\sum_{a\in D}\nu_{\omega}(a)$. By the well-known theorem for a meromorphic form on a compact Riemann surface (e.g., see [8, Theorem 17.12]), we have the desired identity (2.6).

PROPOSITION 2.7. *In the same situation as in Proposition 2.5, consider a pseudo-metric ds^2 on a neighborhood of $\bar{D}$ which has no singularities on ∂D. Then, for the nonnegative function λ with $ds^2=\lambda^2|\omega|^2$, it holds that*

$$\chi(\bar{D})-\int_{\partial D}d^c\log\lambda^2=-\int_D[dd^c\log\lambda^2]-\sum_{a\in D}\nu_{ds}(a).$$

Proof. Since $[\nu_{ds}]=[\nu_{\lambda}]+[\nu_{\omega}]$, we have

$$\int_D[dd^c\log\lambda^2]=\int_D dd^c[\log\lambda^2]-\int_D[\nu_{\lambda}]=\int_{\partial D}d^c\log\lambda^2-\sum_{a\in D}\nu_{ds}(a)-\chi(\bar{D}).$$

by using Proposition 2.4, (2.3) and (2.6). This gives Proposition 2.7.

By using Proposition 2.7, we can give another proof for the following version of the classical Gauss-Bonnet theorem.

THEOREM 2.8. *Let D be a relatively compact domain in M with real analytic smooth boundary and ds^2 a pseudo-metric on a neighborhood of $\bar{D}$ which has no singularities on ∂D. Then,*

$$\chi(\bar{D})-\frac{1}{2\pi}\int_{\partial D}\kappa=\frac{1}{2\pi}\int_D K\Omega_{ds^2}-\sum_{a\in D}\nu_{ds}(a), \tag{2.9}$$

where κ, K and Ω_{ds^2} denote the geodesic curvature form of the curve ∂D, the Gaussian curvature and the area form of ds^2 respectively.

Proof. We write $ds^2=\lambda_z^2|dz|^2$ in terms of a holomorphic local coordinate z. Then, the Gaussian curvature is locally given by $K:=-(1/\lambda_z^2)\Delta\log\lambda_z$ outside the singularities of ds^2, so that $K\Omega_{ds^2}=-2\pi dd^c\log\lambda_z^2$. On the other hand, on a sufficiently small neighborhood of each $a \in \partial D$, if we write $ds^2=\lambda_\zeta^2|d\zeta|^2$ with a holomorphic local coordinate ζ with $\mathrm{Re}(\zeta)=\mathrm{const}$ on ∂D, the geodesic curvature form is given by $\kappa=2\pi d^c \log\lambda_\zeta^2$ (cf., [18, pp. 27~28]).

Consider first the case that D is an open disk in $\boldsymbol{C}$ and $\nu_{ds}\equiv 0$. As a holomorphic local coordinate ζ with $\mathrm{Re}\,\zeta=\mathrm{const}$ on ∂D we take a branch of $\log(z-a)$ locally, where a is the center of D. Since $\lambda_\zeta=\lambda_z|dz/d\zeta|=\lambda_z|z-a|$, we have

$$\int_{\partial D} d^c \log\lambda_\zeta^2=\int_{\partial D} d^c \log\lambda_z^2+\int_{\partial D} d^c \log|z-a|^2=\int_D dd^c \log\lambda_z^2+1. \tag{2.10}$$

This gives (2.9) for this particular case, because $\chi(\bar{D})=1$. For the general domain D we may assume that ∂D is not connected, because, otherwise, we may replace D by the domain D' removed one or two sufficiently small closed disks from D and add the formulas (2.9) applied for D' and for the removed disks. Then, we can choose a meromorphic form ω on D which is purely imaginary on ∂D. In fact, as a solution of Dirichlet problem there is a non-constant continuous function h on $\bar{D}$ which is harmonic and a constant on each connected component of ∂D. It is easily seen that $\omega:=\partial h$ is holomorphic and purely imaginary on ∂D. For each $a\in\partial D$ take a holomorphic local coordinate ζ around a such that $\mathrm{Re}\,\zeta=\mathrm{const}$ on ∂D and $|d\zeta|^2=|\omega|^2$. We have $\kappa=2\pi d^c\log\lambda^2$ for a function λ with $ds^2=\lambda^2|\omega|^2$. The formula (2.9) is a direct result of Proposition 2.7.

§ 3. Sum to product estimates

Let $x=(x_1, \cdots, x_m): M\to\boldsymbol{R}^m$ be a regular minimal surface in $\boldsymbol{R}^m$. With each positively oriented isothermal coordinates (u, v) associating a holomorphic local coordinate $z=u+\sqrt{-1}v$, we can regard M as a Riemann surface with conformal metric, and the functions $x_i (1\leqq i\leqq m)$ are harmonic on M. We first note the following:

PROPOSITION 3.1. *For each $c\in\boldsymbol{R}^m$, $h_c(z):=|x(z)-c|$ is a function with admissbile singularities on M.*

Proof. The function $h_c(z)$ is obviously of class C^∞ on $\{a\in M;\ x(a)\neq c\}$. For a point a with $x(a)=c$, take a holomorphic local coordinate $z=u+\sqrt{-1}v=a+re^{i\theta}$ on a neighborhood of a. Since x_j's are harmonic functions in z, we can expand $x(u, v)$ as

$$x(u, v)=c+\sum_{j=n}^{\infty} r^j(d_j\cos j\theta+e_j\sin j\theta),$$

where $d_j,\ e_j\in\boldsymbol{R}^m$, $d_n\neq 0$ or $e_n\neq 0$. Here, by the assumption of the regularity of M, we have $n=1$. Since u, v are isothermal coordinates, we have

$$|x_u|=|x_v|, \qquad (x_u, x_v)=0 \tag{3.2}$$

and so

$$r|x_r|=|x_\theta|, \qquad (x_r, x_\theta)=0,$$

where x_u, x_v, x_r and x_θ denote the partial derivatives of x with respect to u, v, r and θ respectively. This gives $|d_1|=|e_1|$ and $(d_1, e_1)=0$. Then, we can write

$$|x(z)-c|^2=|z-a|^2 u^*(z),$$

where u^* is a function written as

$$u^*(z)=|d_1|^2+\sum_{j=1}^{\infty} P_j(\cos\theta, \sin\theta)r^j$$

with some polynomials $P_j(X, Y)$. We can easily check that $v:=\log|u^*|$ satisfies the condition (2.2), and so h_c has an admissible singularity around a.

Take a holomorphic local coordinate $\zeta=u+\sqrt{-1}v$ and set $ds^2=\lambda^2|d\zeta|^2$. Then, by (3.2) $\lambda=|x_u|=|x_v|$ and the vector-valued functions

$$e_1:=\frac{x_u}{\lambda}, \qquad e_2:=\frac{x_v}{\lambda} \tag{3.3}$$

give an orthonormal basis of the tangent plane of M at each point of M.

PROPOSITION 3.4 ([4]). *For each $c\in \boldsymbol{R}^m$ it holds that*

$$\Delta\log|x-c|^2=\frac{4\lambda^2(|x-c|^2-(x-c, e_1)^2-(x-c, e_2)^2)}{|x-c|^4}\geqq 0.$$

Proof. Since $\Delta x\equiv x_{uu}+x_{vv}=0$, we have

$$\begin{aligned}\Delta\log|x-c|^2&=\frac{2((|x_u|^2+|x_v|^2)|x-c|^2-2(x-c, x_u)^2-2(x-c, x_v)^2)}{|x-c|^4}\\ &=\frac{4|x_u|^2(|x-c|^2-(x-c, e_1)^2-(x-c, e_2)^2)}{|x-c|^4}.\end{aligned}$$

On the other hand, if we take vectors e_i $(3\leqq i\leqq m)$ such that $e_1, e_2, \cdots, e_m$ give an orthonormal basis of $\boldsymbol{R}^m$, then

$$|x-c|^2-(x-c, e_1)^2-(x-c, e_2)^2=\sum_{i=3}^{m}(x-c, e_i)^2\geqq 0,$$

which completes the proof of Proposition 3.4.

Remark 3.5. In the case $m=3$, we consider the angle θ between the vector $x(a)-c$ and the normal vector of M at each point $a\in M$. By Proposition 3.4, we have

$$dd^c\log|x-c|^2=\frac{1}{\pi}\frac{\lambda^2\cos^2\theta}{|x-c|^2}du\wedge dv.$$

By the same calculations as the above, we obtain also

$$\Delta\log(1+|x|^2)=\frac{4\lambda^2(1+|x|^2-(x,e_1)^2-(x,e_2)^2)}{(1+|x|^2)^2}\geqq\frac{4\lambda^2}{(1+|x|^2)^2}. \tag{3.6}$$

Now, we regard the space $\boldsymbol{R}^m$ as a subspace of $\boldsymbol{R}^{m+1}$ by identifying a point $(c_1, \cdots, c_m)\in\boldsymbol{R}^m$ with the point $(c_1, \cdots, c_m, 0)\in\boldsymbol{R}^{m+1}$. Consider the one point compactification $\bar{\boldsymbol{R}}^m:=\boldsymbol{R}^m\cup\{\infty\}$ and the streographic projection $\varpi: S^m\rightarrow\bar{\boldsymbol{R}}^m$, namely, the map which maps $N:=(0, \cdots, 0, 1)$ to ∞ and each $C(\neq N)\in S^m$ to the point $c\in\boldsymbol{R}^m$ such that C, c and N are collinear, where S^m is the unit sphere in $\boldsymbol{R}^{m+1}$ with center at the origin. For $c, d\in\boldsymbol{R}^m$ we denote a half of the distance between $\varpi^{-1}(c)$ and $\varpi^{-1}(d)$ by $|c, d|$. By elementary calculations, we have

$$|c, d|:=\frac{|c-d|}{\sqrt{1+|c|^2}\sqrt{1+|d|^2}}(\leqq 1)$$

and $|c, \infty|:=1/\sqrt{1+|c|^2}$.

We give the following analogue to [13, Proposition 2.5.1].

PROPOSITION 3.7. *For each $\varepsilon>0$ there exists a constant δ_0 depending only on ε such that, for every $c\in\boldsymbol{R}^m$ and $\delta\geqq\delta_0$,*

$$\Delta\log\frac{1}{\log(\delta/|x, c|^2)}\geqq\left(\frac{1}{1+|c|^2}\frac{1}{|x, c|^2\log^2(\delta/|x, c|^2)}-\varepsilon\right)\Delta\log(1+|x|^2).$$

Proof. For $\delta>e$ set $\varphi:=-\log\log(\delta/|x, c|^2)$. By direct calculations, we have

$$\Delta\varphi=\frac{\Delta\log|x-c|^2-\Delta\log(1+|x|^2)}{\log(\delta/|x, c|^2)} +4\frac{\left(\frac{(x, x_u)}{1+|x|^2}-\frac{(x-c, x_u)}{|x-c|^2}\right)^2+\left(\frac{(x, x_v)}{1+|x|^2}-\frac{(x-c, x_v)}{|x-c|^2}\right)^2}{\log^2(\delta/|x, c|^2)}.$$

We can rewrite this as

$$\Delta\varphi=\left(\frac{1}{\log(\delta/|x, c|^2)}-\frac{1}{\log^2(\delta/|x, c|^2)}\right)\Delta\log|x-c|^2 -\left(\frac{1}{\log(\delta/|x, c|^2)}+\frac{1}{\log^2(\delta/|x, c|^2)}\right)\Delta\log(1+|x|^2)+\frac{\phi}{\log^2(\delta/|x, c|^2)},$$

where

$$\phi:=\Delta\log|x-c|^2+\Delta\log(1+|x|^2) +4\left(\frac{(x, x_u)}{1+|x|^2}-\frac{(x-c, x_u)}{|x-c|^2}\right)^2+4\left(\frac{(x, x_v)}{1+|x|^2}-\frac{(x-c, x_v)}{|x-c|^2}\right)^2.$$

Take $\delta_0>e$ such that $1/\log\delta_0+1/\log^2\delta_0<\varepsilon$. Then, for every $\delta\geqq\delta_0$ the first term is nonnegative and the second term is not less than $-\varepsilon\Delta\log(1+|x|^2)$. Choose

$e_1, \cdots, e_m$ as in the proof of Proposition 3.4, and set $x_i := (x, e_i)$ and $c_i := (c, e_i)$. By using (3.3), Proposition 3.4 and (3.6), we obtain

$$\begin{aligned}\phi &= \frac{4\lambda^2(1+|x|^2+\sum_{j=1}^m (x_i-c_i)^2-2(x_1^2+x_2^2-c_1x_1-c_2x_2))}{(1+|x|^2)|x-c|^2}\\ &= \frac{4\lambda^2(1+|x|^2-x_1^2-x_2^2+c_1^2+c_2^2+\sum_{i=3}^m (x_i-c_i)^2)}{(1+|x|^2)|x-c|^2}\\ &\geqq \frac{(1+|x|^2)\,\Delta\log(1+|x|^2)}{|x-c|^2}\\ &= \frac{1}{1+|c|^2}\frac{1}{|x,\ c|^2}\Delta\log(1+|x|^2).\end{aligned}$$

These give Proposition 3.7.

Take mutually distinct points $c_1, \cdots, c_q \in \boldsymbol{R}^m$ and set $L := \min_{i<j} |c_i,\ c_j|$.

(3.8) *For every $w \in \boldsymbol{R}^m$ it holds that $|w,\ c_i| \geqq L/2$ for all i but at most one.*

In fact, if there are two distinct i and j such that $|w,\ c_i| < L/2$, then we have an absurd conclusion $L \leqq |c_i,\ c_j| \leqq |c_i,\ w| + |w,\ c_j| < L$.

We can give the following:

PROPOSITION 3.9. *Let $c_1, \cdots, c_q$ be mutually distinct points in $\boldsymbol{R}^m$. For an arbitrarily given $\varepsilon > 0$ take some $\delta_0 (> e^2)$ with $1/\log^2\delta_0 + 1/\log\delta_0 < \varepsilon/q$. Then there is a positive constant $C > 0$ depending only on $c_1, \cdots, c_q$ such that*

$$\Delta\log\frac{(1+|x|^2)^\varepsilon}{\prod_{j=1}^q \log(\delta_0/|x,\ c_j|^2)} \geqq C\Delta\log(1+|x|^2)\prod_{j=1}^q \frac{1}{|x,\ c_j|^2\log^2(\delta_0/|x,\ c_j|^2)}.$$

Proof. By Proposition 3.7, we can find a positive constant C such that

$$\begin{aligned}\Delta\log\frac{(1+|x|^2)^\varepsilon}{\prod_{j=1}^q \log(\delta_0/|x,\ c_j|^2)} &= \varepsilon\Delta\log(1+|x|^2) + \sum_{j=1}^q \Delta\log\frac{1}{\log(\delta_0/|x,\ c_j|^2)}\\ &\geqq \left(\varepsilon + \sum_{j=1}^q \left(\frac{1}{(1+|c_j|^2)|x,\ c_j|^2\log^2(\delta/|x,\ c_j|^2)} - \frac{\varepsilon}{q}\right)\right)\Delta\log(1+|x|^2)\\ &\geqq C\left(\sum_{j=1}^q \frac{1}{|x,\ c_j|^2\log^2(\delta_0/|x,\ c_j|^2)}\right)\Delta\log(1+|x|^2).\end{aligned}$$

For an arbitrarily fixed point $a_0 \in M$ we change the indices of c_j's so that

$$|x(a_0),\ c_1| \leqq |x(a_0),\ c_2| \leqq \cdots \leqq |x(a_0),\ c_q|.$$

Then, for $j \geqq 2$, we have $|x,\ c_j| \geqq L/2$ by (3.8). Since the function $h(u) := u\log^2(\delta_0/u)$ in u is increasing on $(0, 1]$, $|x,\ c_j|^2 \log^2(\delta_0/|x,\ c_j|^2)$ $(j \geqq 2)$ are bounded from below by a positive constant depending only on L. Therefore, we can easily find a positive constant C' such that

$$\sum_{j=1}^{q} \frac{1}{|x, c_j|^2 \log^2(\delta_0/|x, c_j|^2)}$$

$$\geq \frac{1}{|x, c_1|^2 \log^2(\delta_0/|x, c_1|^2)} \geq C' \prod_{j=1}^{q} \frac{1}{|x, c_j|^2 \log^2(\delta_0/|x, c_j|^2)}.$$

This gives Proposition 3.9.

COROLLARY 3.10. *In the same situation as in Proposition* 3.9, *for each* $\varepsilon>0$ *there exists positive constants* δ *and* C *such that*

$$\Delta \log \frac{(1+|x|^2)^{\varepsilon}}{\prod_{j=0}^{q} \log(\delta/|x, c_j|^2)} \geq C^2 \frac{|x_u|^2}{(1+|x|^2)^2} \prod_{j=1}^{q} \frac{1}{|x, c_j|^2 \log^2(\delta/|x, c_j|^2)}.$$

This is an immediate consequence of Proposition 3.9 and (3.6).

§4. The first main theorem for minimal surfaces

Let M be an open Riemann surface. By an exhaustion τ of M we mean that τ is a continuous map of M into $[0, +\infty)$ which is proper, namely, whose inverse image of every compact set is compact. For an exhaustion τ of M we set $M_r := \{x ; \tau(x)<r\}$ and $M_{s,r} := M_r - M_s$. We call τ a parabolic exhaustion of M if it is an exhaustion of M satisfying the condition that τ^2 is of class C^∞ on M and that $dd^c \log\tau \equiv 0$ on an open neighborhood of $M_{s,+\infty}$ for some fixed $s \geq 0$ and $dd^c\tau \not\equiv 0$. For example, the function $\tau(z) := |z|$ $(z \in \boldsymbol{C})$ is a parabolic exhaustion of $\boldsymbol{C}$, where we may take $s=0$. If an open Riemann surface M is of finite type, namely, biholomorphic to a compact Riemann surface $\overline{M}$ with finitely many points $a_1, \cdots, a_k$ removed, then M has a parabolic exhaustion. In fact, as a parabolic exhaustion we can take a C^∞ function τ on M which equals $1/|z_l|$ on some neighborhood of each a_l, where z_l is a holomorphic local coordinate with $z_l(a_l)=0$. It is known that an open Riemann surface M has a parabolic exhaustion if and only if M is of type O_G, or there is no nonconstant negative subharmonic function on M (cf., [17, Theorem 10.12] and [16]).

Let M be an open Riemann surface on which a fixed parabolic exhaustion τ is defined. For brevity, we assume that $s>0$ and ∂M_s is smooth in the following. We note that, on $M_{s,+\infty}$,

$$\tau dd^c\tau = \tau^2 dd^c \log\tau + d\tau \wedge d^c\tau = d\tau \wedge d^c\tau = |\partial\tau/\partial z|^2 (\sqrt{-1}/2\pi) dz \wedge d\bar{z} \geq 0.$$

Moreover, since $\log\tau$ is harmonic, $d\tau$ vanishes only on a discrete subset of $M_{s,+\infty}$. For each $r>s$ set $C = \int_{\partial M_r} d^c \log\tau^2$. The constant C is independent of r, because

$$\int_{\partial M_{r_2}} d^c \log\tau^2 - \int_{\partial M_{r_1}} d^c \log\tau^2 = \int_{M_{r_1,r_2}} dd^c \log\tau^2 = 0 \quad (s<r_1<r_2).$$

On the other hand,

$$C=(2/r)\int_{\partial M_r} d^c\tau=(2/r)\int_{M_r} dd^c\tau>0\,.$$

After replacing τ by τ^σ for a suitable $\sigma>0$, we assume that $C=1$ in the following.

Let ν be a divisor on M. We define the counting function of ν by

$$N(r,\nu):=\int_s^r \sum_{z\in M_t}\nu(z)\frac{dt}{t}\,.$$

We give here the following version of famous Jensen's formula (cf., [13, Proposition 3.1.3], or [15, p. 128]).

PROPOSITION 4.1. *Let u be a function with admissible singularities on M. Then, for $r>s>0$,*

$$\begin{aligned}\int_s^r\frac{dt}{t}\int_{M_t}dd^c[\log|u|^2]&=\int_s^r\frac{dt}{t}\int_{M_t}[dd^c\log|u|^2]+N(r,\nu_u)\\&=\frac{2}{r}\int_{\partial M_r}\log|u|\,d^c\tau-\frac{2}{s}\int_{\partial M_s}\log|u|\,d^c\tau\,.\end{aligned}\tag{4.2}$$

The first identity is due to Proposition 2.4. To see the second, we first use (2.3) to see $\int_{M_t}dd^c[\log|u|^2]=\int_{\partial M_t}d^c\log|u|^2$ for the case where ∂M_t are smooth and u has no singularity on ∂M_t. Since $d\log\tau\wedge d^c\log|u|^2=d\log|u|^2\wedge d^c\log\tau=d(\log|u|^2\wedge d^c\log\tau)$, we have

$$\begin{aligned}\int_{r_1}^{r_2}\frac{dt}{t}\int_{M_t}dd^c[\log|u|^2]&=\int_{r_1}^{r_2}\frac{dt}{t}\Big(\int_{\partial M_t}d^c\log|u|^2\Big)\\&=\int_{M_{r_1,r_2}}d\log\tau\wedge d^c\log|u|^2=\int_{\partial M_{r_1,r_2}}\log|u|^2d^c\log\tau\\&=\frac{2}{r_2}\int_{\partial M_{r_2}}\log|u|\,d^c\tau-\frac{2}{r_1}\int_{\partial M_{r_1}}\log|u|\,d^c\tau\end{aligned}$$

in the case where u has no singularities on $\overline{M}_{r_1,r_2}(s\leqq r_1<r_2)$. For general cases, take numbers r_i with $r_0(:=s)<r_1<\cdots<r_k:=r$ such that u has no singularities on the interior of $M_{r_i,r_{i+1}}$. For all s', r' with $r_i<s'<r'<r_{i+1}$ we have

$$\int_{s'}^{r'}\frac{dt}{t}\int_{M_t}dd^c[\log|u|^2]=\frac{2}{r'}\int_{\partial M_{r'}}\log|u|\,d^c\tau-\frac{2}{s'}\int_{\partial M_{s'}}\log|u|\,d^c\tau\,.$$

This remains valid for $s'=r_i$ and $r'=r_{i+1}$ because both sides of (4.2) are continuous as functions in s and r. These conclude that

$$\begin{aligned}\int_s^r \frac{dt}{t}\int_{M_t} dd^c[\log|u|^2] &= \sum_{i=0}^{k-1}\int_{r_i}^{r_{i+1}}\frac{dt}{t}\int_{M_t} dd^c[\log|u|^2]\\ &= \sum_{i=0}^{k-1}\frac{2}{r_{i+1}}\int_{\partial M_{r_{i+1}}}\log|u|\,d^c\tau-\frac{2}{r_i}\int_{\partial M_{r_i}}\log|u|\,d^c\tau\\ &= \frac{2}{r}\int_{\partial M_r}\log|u|\,d^c\tau-\frac{2}{s}\int_{\partial M_s}\log|u|\,d^c\tau.\end{aligned}$$

Now, we consider a regular minimal surface $x=(x_1, \cdots, x_m): M\to \boldsymbol{R}^m$ and assume that the surface M, considered as a Riemann surface with conformal metric, has a parabolic exhaustion τ. We regard M as a surface immersed in $\bar{\boldsymbol{R}}^m$. We define the hyperspherical proximity function for M by

$$m^0(r\,;\, c,\, M):=\frac{2}{r}\int_{\partial M_r}\log\frac{1}{|x,\, c|}d^c\tau-\frac{2}{s}\int_{\partial M_s}\log\frac{1}{|x,\, c|}d^c\tau$$

for each $c\in\bar{\boldsymbol{R}}^m$. For the particular case $c=\infty$, we see

$$\begin{aligned}m^0(r\,;\,\infty,\, M)&=\frac{2}{r}\int_{\partial M_r}\log(1+|x|^2)^{1/2}d^c\tau-\frac{2}{s}\int_{\partial M_s}\log(1+|x|^2)^{1/2}d^c\tau\\ &=\int_s^r\frac{dt}{t}\int_{M_t}dd^c\log(1+|x|^2)\geqq 0.\end{aligned}\tag{4.3}$$

We also define the counting function and the visibility function by

$$N(r\,;\, c,\, M):=N(r,\, \nu_{|x-c|}),$$

$$H(r\,;\, c,\, M):=\int_s^r\frac{dt}{t}\int_{M_t}[dd^c\log|x-c|^2]$$

respectively for each $c\in\boldsymbol{R}^m$, and set $N(r\,;\,\infty,\, M)\equiv H(r\,;\,\infty,\, M)\equiv 0$.

For geometric meanings of $H(r\,;\, c,\, M)$, see Remark 3.5.

Moreover, we define the hyperspherical affinity of $c(\in\bar{\boldsymbol{R}}^m)$ by

$$A^0(r\,;\, c,\, M):=m^0(r\,;\, c,\, M)+N(r\,;\, c,\, M)+H(r\,;\, c,\, M)$$

and the order function of M by

$$T^0(r\,;\, M):=A(r\,;\,\infty,\, M)(=m^0(r\,;\,\infty,\, M)).$$

We can prove the following:

PROPOSITION 4.4. *The function $T^0(r\,; M)$ is increasing and convex with respect to* $\log r$ *and tends to* ∞ *as* $r\to\infty$.

Proof. To see the first assertion, consider the fcnction h with $dd^c\log(1+|x|^2)=h\,d\log\tau\wedge d^c\log\tau$ and observe the identities

$$\frac{dT^0(r;\, M)}{d\log r}=r\frac{T^0(r;\, M)}{dr}=\int_{M_r}dd^c\log(1+|x|^2)$$

$$\frac{d^2T^0(r;\, M)}{(d\log r)^2}=r\frac{d}{dr}\int_0^r\frac{dt}{t^2}\int_{\partial M_t}hd^c\tau=\frac{1}{r}\int_{\partial M_r}hd^c\tau\geqq 0\,.$$

To see the latter, take some $t_0>s$ with $K:=\int_{M_{t_0}}dd^c\log(1+|x|^2)>0$. We then have

$$T^0(r;\, M)\geqq\int_{t_0}^r\frac{dt}{t}\int_{M_{t_0}}dd^c\log(1+|x|^2)=K\log\frac{r}{t_0}\longrightarrow\infty\quad(\text{as } r\to\infty)\,.$$

Now, we give the first main theorem for minimal surfaces, which was given by E. F. Beckenbach and T. A. Cootz in [3] for the case where M is conformally isomorphic with $\boldsymbol{C}$.

THEOREM 4.5 (cf., [3], [6]). *$T^0(r;\, M)=A(r;\, c,\, M)$ for all $c\in\bar{\boldsymbol{R}}^m$.*

Proof. Apply Proposition 4.1 to the function $u:=1/|x,\, c|$ to see

$$\begin{aligned}T^0(r;\, M)&=\int_s^r\frac{dt}{t}\int_{M_t}dd^c\log(1+|x|^2)\\&=\int_s^r\frac{dt}{t}\int_{M_t}dd^c[\log u^2]+\int_s^r\frac{dt}{t}\int_{M_t}dd^c[\log|x-c|^2]\\&=\frac{2}{r}\int_{\partial M_r}\log|u|d^c\tau-\frac{2}{s}\int_{\partial M_s}\log|u|d^c\tau\\&\quad+\int_s^r\frac{dt}{t}\int_{M_t}[dd^c\log|x-c|^2]+\int_s^r\frac{dt}{t}\int_{M_t}[\nu_{|x-c|}]\\&=m^0(r;\, c,\, M)+H(r;\, c,\, M)+N(r;\, c,\, M)\,.\end{aligned}$$

Thus we obtain Theorem 4.5.

As is stated in the previous section, we identify the extended euclidean space $\bar{\boldsymbol{R}}^m$ with the unit sphere S^m in $\boldsymbol{R}^{m+1}$ by the stereographic projection and denote by dV the volume form on $\bar{\boldsymbol{R}}^m$ induced from the standard volume form on S^m which is normalized so as the total volume is 1. We can prove the following:

THEOREM 4.6. *It holds that*

$$T^0(r;\, M)=\int_{c\in\overline{\boldsymbol{R}^m}}N(r;\, c,\, M)dV+\int_{c\in\overline{\boldsymbol{R}^m}}H(r;\, c,\, M)dV\,.$$

Proof. To see this, consider the function $u(x,\, c):=1/|x,\, c|$ of $x\in M$ and $c\in\boldsymbol{R}^m$. Then, by Theorem 4.5 we have

$$T^0(r;\, M)-N(r;\, c,\, M)-H(r;\, c,\, M)$$
$$=\frac{2}{r}\int_{a\in\partial M_r}\log|u(x(a),\, c)|\, d^c\tau-\frac{2}{s}\int_{a\in\partial M_s}\log|u(x(a),\, c)|\, d^c\tau.$$

Consider each term of these identities as a function in c and integrate it with respect to dV. The function $G(a)=\int_{c\in\bar{R}^m}u(x(a),\, c)dV$ in a is a constant because $\int_{c\in\bar{R}^m}\log|x,\, c|dV$ does not depend on the choice of a vector x. Since the volume of the total space with respect to dV is 1, we can easily obtain the desired conclusion by using Fubini's theorem.

§ 5. The second main theorem for minimal surfaces

The purpose of this section is to give the second main theorem for minimal surfaces which is a generalization of the result given by E. F. Beckenbach and collaborators to the case where minimal surfaces are of parabolic type.

Let $x=(x_1,\, \cdots,\, x_m)\colon M\rightarrow\boldsymbol{R}^m$ be a regular minimal surface in $\boldsymbol{R}^m$ and assume that the surface M considered as a Riemann surface with a conformal metric has a parabolic exhaustion τ. We consider the Gauss map $G\colon M\rightarrow P^{m-1}(\boldsymbol{C})$ of M, which is locally written as

$$G=\left(\frac{\partial x_1}{\partial z}:\cdots:\frac{\partial x_m}{\partial z}\right),$$

where we denote by $(w_1:\cdots:w_m)$ homogeneous coordinates on $P^{m-1}(\boldsymbol{C})$. The (normalized) Fubini-Study metric form on $P^{m-1}(\boldsymbol{C})$ is given by $dd^c\log(|w_1|^2+|w_2|^2+\cdots+|w_m|^2)$. We denote by Ω_G the pull-back of the Fubini-Study metric form on $P^{m-1}(\boldsymbol{C})$ via G, which is locally given by

$$\Omega_G:=dd^c\log\left(\left|\frac{\partial x_1}{\partial z}\right|^2+\cdots+\left|\frac{\partial x_m}{\partial z}\right|^2\right).$$

We define the order function of the Gauss map of G by

$$T_G(r):=\int_s^r\frac{dt}{t}\int_{M_t}\Omega_G,$$

and the function

$$E(r):=\int_s^r\chi(\bar{M}_t)\frac{dt}{t}.$$

We now state the second main theorem for minimal surfaces.

THEOREM 5.1 (cf., [3], [6]). *Let M be a regular minimal surface in $\boldsymbol{R}^m$ which has a parabolic exhaustion τ. For mutually distinct points $c_1,\, \cdots,\, c_q\in\bar{\boldsymbol{R}}^m$ and a positive number ε there is a set E_ε with $\int_{E_\varepsilon}d\rho/\rho<+\infty$ such that*

(5.2) $$\sum_{j=1}^{q} m^0(r\,;\, c_j,\, M)+T_G(r)\leqq 2(1+\varepsilon)T^0(r\,;\, M)-E(r)+O(1)$$

for all $r\notin E_\varepsilon$.

For the proof of Theorem 5.1, we use the following:

PROPOSITION 5.3. *Let h be a locally summable nonnegative function on M and set*

$$T^h(r):=\int_s^r \frac{dt}{t}\int_{M_t} h\sqrt{-1}\omega\wedge\bar{\omega},$$

where $\omega:=\partial\log\tau$. Then, there is a positive constant C such that

$$\frac{2}{r}\int_{\partial M_r} \log h d^c\tau \leqq C\log T^h(r)+C$$

for each r except in a set with finite logarithmic measure.

This is proved by minor changes of the proof of [13, Proposition 3.2.4]. By the same argument as in the proof of Proposition 4.4, we can show that $T^h(r)$ is an increasing function in r. Moreover, we have

$$\frac{d^2T^h}{(d\log r)^2}=\frac{1}{r}\int_{\partial M_r} hd^c\tau.$$

Using the concavity of the logarithm, we get

$$\frac{2}{r}\int_{\partial M_r}\log hd^c\tau\leqq \log\left(\frac{2}{r}\int_{\partial M_r} hd^c\tau\right)=\log\left(2\frac{d^2T^h}{(d\log r)^2}\right).$$

On the other hand, as in the proof of [13, Proposition 3.2.4], by the use of [13, Lemma 3.2.5] we can show that, for each $\varepsilon>0$

$$\frac{d^2T^h(r)}{(d\log r)^2}\leqq T^h(r)^{(1+\varepsilon)^2}$$

for all r except in a set E with $\int_E (1/t)dt<+\infty$. From these facts, we easily conclude Proposition 5.3.

Proof of Theorem 5.1. The form $\omega:=\partial\log\tau$ is holomorphic on $M-M_s$ and purely imaginary on each smooth ∂M_r. Consider the functions g_i with $\partial x_i= g_i\omega$ $(1\leqq i\leqq m)$, which are meromorphic on $M_{s,+\infty}$. Set $|G|:=(|g_1|^2+\cdots+|g_m|^2)^{1/2}$. Then $ds^2=2|G|^2|\omega|^2$ ([13, § 1.1]). Therefore, $\nu_{|G|}+\nu_\omega=\nu_{ds}=0$ and so $\nu_\omega=-\nu_{|G|}$. For a given $\varepsilon>0$ take some $\delta>0$ such that the inequality in Corollary 3.10 holds, and set $h:=(1+|x|^2)^\varepsilon/\prod_{j=1}^q \log(\delta/|c_j,\, x|^2)$. Consider the function h^* such that $dd^c\log h=h^*\sqrt{-1}\omega\wedge\bar{\omega}$. As a consequence of Corollary 3.10 we get

$$\frac{|G|^2h^2}{(1+|x|^2)^{2+2\varepsilon}\prod_{j=1}^q|x,\, c_j|^2}\leqq Ch^*$$

for some $C>0$. By the monotonicity of the integral, we have

$$\frac{2}{r}\int_{\partial M_r}\log|G|d^c\tau+\sum_{j=1}^{q}\frac{2}{r}\int_{\partial M_r}\log\frac{1}{|x,\ c_j|}d^c\tau+\frac{2}{r}\int_{\partial M_r}\log h d^c\tau$$
$$\leqq\frac{1}{r}\int_{\partial M_r}\log h^* d^c\tau+(2+2\varepsilon)\frac{2}{r}\int_{\partial M_r}\log(1+|x|^2)^{1/2}d^c\tau+O(1)$$

for every $r>s$. On the other hand, by Proposition 4.1 we obtain

$$T_G(r)=\int_s^r\frac{dt}{t}\int_{M_t}[dd^c\log|G|^2]$$
$$=\int_s^r\frac{dt}{t}\int_{M_t}dd^c[\log|G|^2]-N(r,\ \nu_{|G|})$$
$$=\frac{2}{r}\int_{\partial M_r}\log|G|d^c\tau-\frac{2}{s}\int_{\partial M_r}\log|G|d^c\tau+N(r,\ \nu_\omega).$$

Here, we may replace $N(r,\ \nu_\omega)$ by $-E(r)$ owing to Proposition 2.5. Since

$$\sum_{j=1}^{q}m^0(r\,;\ c_j,\ M)\leqq\sum_{j=1}^{q}\frac{2}{r}\int_{\partial M_r}\log\frac{1}{|x,\ c_j|}d^c\tau+O(1),$$

we have

$$T_G(r)+\sum_{j=1}^{q}m^0(r\,;\ c_j,\ M)$$
$$\leqq 2(1+\varepsilon)T^0(r\,;\ M)+\frac{1}{r}\int_{\partial M_r}\log h^* d^c\tau-\frac{2}{r}\int_{\partial M_r}\log h d^c\tau-E(r)+O(1).$$

We apply Proposition 5.3 to find some constant $C>0$ such that

$$\frac{1}{r}\int_{\partial M_r}\log h^* d^c\tau\leqq C\log T^{h^*}(r)+O(1)$$

for all r except in a set E_ε with $\int_{E_\varepsilon}(1/\rho)d\rho<\infty$. On the other hand, by using Proposition 4.1,

$$\log T^{h^*}(r)=\log\int_s^r\frac{dt}{t}\int_{M_t}dd^c\log h\leqq\log\left(\frac{2}{r}\int_{\partial M_r}\log h d^c\tau+\text{const}\right).$$

Since $O(\log((2/r)\int_{\partial M_r}\log h d^c\tau))\leqq(2/r)\int_{\partial M_r}\log h d^c\tau+O(1)$,

$$T_G(r)+\sum_{j=1}^{q}m^0(r\,;\ c_j,\ M)$$
$$\leqq\frac{1}{r}\int_{\partial M_r}\log h^* d^c\tau-\frac{2}{r}\int_{\partial M_r}\log h d^c\tau+2(1+\varepsilon)T^0(r,\ M)-E(r)+O(1)$$
$$\leqq 2(1+\varepsilon)T^0(r,\ M)-E(r)+O(1)$$

for all r except in a set with finite logarithmic measure. This gives Theorem 5.1.

Now, we define the defect of c for M by

$$\delta(c, M):=\liminf_{r\to\infty}\frac{m^0(r; c, M)}{T^0(r; M)}(\leqq 1).$$

We note here that the defect for minimal surfaces is defined by an analogy of that of meromorphic functions in the classical Nevanlinna theory. However, the geometric meanings are something different. Indeed, it does not always hold that $\delta(c, M)=1$ if $c\notin M$.

Moreover, we consider the quantites

$$\Psi(M):=\liminf_{r\to\infty}\frac{T_G(r)}{T^0(r; M)},\quad \mathcal{E}(M):=\liminf_{r\to\infty}\frac{-E(r)}{T^0(r; M)}.$$

We have the following defect relation for minimal surfaces:

THEOREM 5.4. *Let M be a regular minimal surface which have a parabolic exhaustion. Then, for arbitrarily given distinct points $c_1, \cdots, c_q$ in $\bar{\boldsymbol{R}}^m$, it holds that*

$$\sum_{j=1}^{q}\delta(c_j, M)+\Psi(M)\leqq 2+\mathcal{E}(M).$$

Proof. Divide each term of (5.2) by $T^0(r; M)$ and observe the limit as r and ε tend to $+\infty$ and 0 respectively. We then have the desired conclusion.

§6. Complete minimal surfaces with finite total curvature

The purpose of this section is to observe some geometric meanings of (5.2) for a particular case of complete minimal surfaces with finite total curvature and to show that the number two of the right hand side of (5.2) is best-possible. We first note the following:

(6.1) *Let M be an open Riemann surface with a parabolic exhaustion. Consider a nonnegative* (1, 1)*-current Ω on M and set*

$$T^{\Omega}(r):=\int_s^r\frac{dt}{t}\int_{M_t}\Omega.$$

Then, it holds that

$$\lim_{r\to\infty}\frac{T^{\Omega}(r)}{\log r}=\int_M\Omega.$$

We consider a complete minimal surface M in $\boldsymbol{R}^m$ which is of finite type as an open Riemann surface, namely which is biholomorphic to a compact Riemann surface $\bar{M}$ with finitely many points $a_1, \cdots, a_k$ removed. As is mentioned in §4, if we take a C^∞ function τ on M which equals $1/|z_l|^{1/k}$ on some neighborhood of each a_l for a holomorphic local coordinate z_l with $z_l(a_l)=0$, then τ is a parabolic exhaustion of M. Here, the exponent $1/k$ is added so that $C\equiv\int_{\partial M_r} d^c \log\tau^2=1$. In what follows, τ means such a parabolic exhaustion of M.

PROPOSITION 6.2. *Let M be a complete minimal surface in $\boldsymbol{R}^m$ which is of finite type as an open Riemann surface. If* $\lim_{r\to\infty}T^0(r;M)/\log r<+\infty$, *then the (normalized) total curvature $C(M)$ of M is finite.*

Proof. Taking arbitray mutually distinct points $c_1, \cdots, c_q\in\boldsymbol{R}^m$ $(q>1)$ and $\varepsilon>0$, we apply Theorem 5.1 to see

$$T_G(r)\leqq 2(1+\varepsilon)T^0(r;M)-E(r)+O(1)$$

for all r except in a set with finite logarithmic measure. By the assumption, we have

$$\lim_{r\to\infty}\frac{T_G(r)}{\log r}=\liminf_{r\to\infty}\frac{T_G(r)}{\log r}\leqq\lim_{r\to\infty}\frac{2(1+\varepsilon)T^0(r;M)-E(r)}{\log r}<+\infty,$$

which gives the desired conclusion

$$C(M)\equiv\frac{1}{2\pi}\int_M K\Omega_{ds^2}=-\int_M \Omega_G>-\infty$$

as a result of (6.1).

Now, we restrict ourselves to the study of a complete regular minimal surface $x=(x_1, \cdots, x_m)\colon M\to\boldsymbol{R}^m$ with finite total curvature. As is shown in [7], M is biholomorphic to a compact Riemann surface $\overline{M}$ with finitely many points $a_1, \cdots, a_k$ removed, and each form $\omega_i:=\partial x_i$ $(1\leqq i\leqq m)$ is extended to a meromorphic form $\tilde{\omega}_i$ (cf., [13, Theorem 5.1.3]). Then, we can define the divisor ν_{ds} of ds^2 on $\overline{M}$ by setting $\nu_{ds}:=\min(\nu_{\tilde{\omega}_1}, \cdots, \nu_{\tilde{\omega}_m})$.

DEFINITION 6.3. For each end a_l we define the multiplicity of M at a_l by $I_l:=-(\nu_{ds}(a_l)+1)$.

PROPOSITION 6.4 ([7, Lemma 2]). *Each multiplicity I_l is a positive integer.*

For the proof, refer to the original paper [7] or [13, Proposition 5.1.8].
We can show also the following:

PROPOSITION 6.5. *Let $\{D_\nu;\ \nu=1, 2, \cdots\}$ be a sequence of simply connected open neighborhoods of an end a_l of M such that they have real analytic smooth boundary and $\bigcap_{\nu=1}^{\infty}\bar{D}_\nu=\{a_l\}$. Then,*

$$I_l=\lim_{\nu\to\infty}-\frac{1}{2\pi}\int_{\partial D_\nu}\kappa_{ds},$$

where κ_{ds} denotes the geodesic curvature of ∂D_ν.

Proof. Taking a holomorphic local coordinate z on a neighborhood of a_l with $z(a_l)=0$, we consider discs $\Delta_\nu:=\{z;\ |z|<\delta_\nu\}$ such that $D_\nu\supset\bar{\Delta}_\nu$ and $\lim_{\nu\to\infty}\delta_\nu=0$. Then, by (2.9) we have

$$\chi(\bar{D}_\nu-\Delta_\nu)-\frac{1}{2\pi}\left(\int_{\partial D_\nu}\kappa-\int_{\partial\Delta_\nu}\kappa\right)=-\frac{1}{2\pi}\int_{D_\nu-\Delta_\nu}K\Omega_{ds^2},$$

the right hand side of which tends to zero as ν tends to ∞ because M has finite total curvature. Therofore, we get

$$\lim_{\nu\to\infty}\frac{1}{2\pi}\int_{\partial D_\nu}\kappa=\lim_{\nu\to\infty}\frac{1}{2\pi}\int_{\partial\Delta_\nu}\kappa.$$

It suffices to show that $I_l=\lim_{\nu\to\infty}(-1/2\pi)\int_{\partial\Delta_\nu}\kappa$. By definition, we can write $ds^2=|z|^{-2(I_l+1)}v(z)|dz|^2$ for some positive function $v(z)$. In view of (2.10), we have

$$\frac{1}{2\pi}\int_{\partial\Delta_\nu}\kappa_{ds}=-(I_l+1)\int_{\partial\Delta_\nu}d^c\log|z|^2+\int_{\partial\Delta_\nu}d^c\log v(z)+1,$$

which tends to $-I_l$ as ν tends to ∞.

We now prove the following:

THEOREM 6.6. *If a complete regular minimal surface M in $\boldsymbol{R}^m$ with finite total curvature $C(M)$ has k ends with multiplicites $I_l(1\leqq l\leqq k)$, then*

$$C(M)=\chi(M)-\sum_{l=1}^{k}I_l. \tag{6.7}$$

Proof. For each end a_l choose a holomorphic local coordinate z_l with $z_l(a_l)=0$ and set $D_\delta:=\bar{M}-\bigcup_{l=1}^{k}\bar{\Delta}_\delta^l$ for a sufficiently small $\delta>0$, where $\Delta_\delta^l:=\{z_l;\ |z_l|<\delta\}$. By applying (2.9) to the domain D_δ, we obtain

$$\frac{1}{2\pi}\int_{D_\delta}K\Omega_{ds^2}=\chi(\bar{D}_\delta)-\frac{1}{2\pi}\int_{\partial D_\delta}\kappa=\chi(\bar{D}_\delta)+\sum_{l=1}^{k}\frac{1}{2\pi}\int_{\partial\Delta_\delta^l}\kappa.$$

As $\delta\to 0$, we have the desired results by the use of Proposition 6.5.

Remark 6.8. For a complete regular minimal surface M in $\boldsymbol{R}^3$ with ends $a_1,\cdots,a_k$, it is shown that the multiplicities I_l as in Definition 6.3 coincide with the multiplicities m_l appearing in [14, Theorem 1]. The formula (6.7) is nothing but the result given in [14, §4].

Now, taking $q-1$ mutually distinct points $c_1,\cdots,c_{q-1}$ in $\boldsymbol{R}^m$ and setting, $c_q:=\infty$, we apply (5.2) to show that, for a given $\varepsilon>0$, there exists a set E_ε with finite logarithmic measure such that

$$\sum_{j=1}^{q-1}m^0(r;\ c_j,\ M)+T_G(r)\leqq(1+\varepsilon)T^0(r;\ M)-E(r)+O(1)\quad(r\notin E_\varepsilon). \tag{6.9}$$

First, we shall study the term $T^0(r; M)$. For each end $a_l(1\leqq l\leqq k)$ taking a holomorphic local coordinate $z_l=te^{i\theta}$ with $z_l(a_l)=0$, we have an expression

$$\frac{\partial x}{\partial z}=\sum_{j\geqq n_l}\alpha_j z^j \quad (\alpha_j\in \boldsymbol{C}^m)$$

around a_l, where $n_l=\nu_{ds}(a_l)(\leqq -2)$. Therefore, we have

$$x=\mathrm{Re}(\alpha_{-1}\log z)+\sum_{j\geqq n_l+1}t^j(d_j\cos j\theta+e_j\sin j\theta),$$

where d_j, $e_j\in \boldsymbol{R}^m$, $d_{n_l+1}\neq 0$ or $e_{n_l+1}\neq 0$. Then, by the argument in the proof of [7, Lemma 2] we can show that α_{-1} is a real vector. Since $n_l=-I_l-1$ by Definition 6.3, we can write

$$(1+|x(z)|^2)^{1/2}=|z|^{-I_l}e^{v(z)}$$

with a function v satisfying the condition (2.2). Using (4.3), (6.1), (2.3) and Proposition 2.4, we have

$$\begin{aligned}\lim_{r\to\infty}\frac{T^0(r; M)}{\log r}&=\lim_{\varepsilon\to 0}\int_{M-\cup_l\{z_l;|z_l|\leqq\varepsilon\}}dd^c\log(1+|x|^2)\\&=-\lim_{\varepsilon\to 0}\sum_{l=1}^{k}\int_{|z_l|=\varepsilon}d^c\log(1+|x|^2)\\&=-\lim_{\varepsilon\to 0}\sum_{l=1}^{k}\int_{|z_l|\leqq\varepsilon}dd^c[\log(1+|x|^2)]\\&=\sum_{l=1}^{k}I_l.\end{aligned}$$

On the other hand, by (6.1) we have

$$\lim_{r\to\infty}\frac{T_G(r)}{\log r}=\int_M\Omega_G=-C(M),\quad \lim_{r\to\infty}\frac{E(r)}{\log r}=\chi(M).$$

Since $\lim_{r\to\infty}m(c; r_j, M)/\log r\geqq 0(1\leqq j\leqq q-1)$, we can conclude that

$$-C(M)\leqq(1+\varepsilon)\sum_{l=1}^{k}I_l-\chi(M)$$

and hence, as $\varepsilon\to 0$,

$$-C(M)\leqq\sum_{l=1}^{k}I_l-\chi(M),$$

which is nothing but one half of (6.7). This shows that the number two of (5.2) is sharp.

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