X. LI
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MINIMAL IMMERSIONS OF S^2 INTO $S^{2m}(1)$ WITH DEGREE $2m+2$

XINGXIAO LI

1. Introduction

In a previous paper [1], by solving completely the totally isotropic condition, we obtained explicit repressentations for all full minimal immersions with area $2\pi[m(m+1)+2]$ of the 2- sphere S^2 into the $2m$-dimensional Euclidean sphere $S^{2m}(1)$ of radius 1, from which we get a classification theorem. It turned out that all those immersions are of degree $2m+2$ (for definition, see Section 2 below). Naturally one asks: Are there any more full minimal immersions x: $S^2 \to S^{2m}(1)$ of degree $2m+2$ other than those of area $2\pi[m(m+1)+2]$? How would they be like?

To answer these questions, it needs further analysis. Using the method developed in [1] we first give, in Section 4 of the present paper, much more concrete examples, and then obtain the corresponding explicit representations in terms of independent parameters. At the same time we complete, in Section 5, the classification of the minimal immersions of degree $2m+2$.

2. Preliminaries

Barbosa [2] established a bijection between the set of all full, generalized minimal immersions $x: S^2 \to S^{2m}(1)$ and that of all linearly full, totally isotropic curves $\Xi: S^2 \to \boldsymbol{C}P^{2m}$, where $\boldsymbol{C}P^{2m}$ is the $2m$-dimensional complex projective space of constant holomorphic curvature 4, with such immersions corresponding to their directrices.

Therefore, we can naturally define the degree of a minimal immersion x: $S^2 \to S^{2m}(1)$ to be the degree of its directrix Ξ as a holomorphic curve in $\boldsymbol{C}P^{2m}$, and denote it by $deg(x)=deg\Xi$.

Denote by $\boldsymbol{C}$ the field of complex numbers, and fix a stereo-graphic projection of $S^2(1)$ onto $\boldsymbol{C}$ to get a local complex coordinate z, $z\in\boldsymbol{C}$, such that the induced metric by x can be written as $ds^2=2F|dz|^2$. It is well known that each holomorphic curve Ξ: $S^2 \to \boldsymbol{C}P^{2m}$ has a local representation (or lift) as

$$\xi=\sum_{i=0}^{n} a_i z^i, \qquad z\in\boldsymbol{C},$$

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where a_i, $i=0, 1, 2, \cdots, n$, are vectors in C^{2m+1}, and if ξ has no zero on C, then $n=\deg\Xi$. In what follows we always agree to this.

If $(\cdot, \cdot)$ is the canonical symmetric product of C^{2m+1}, then the total isotropy of Ξ is equivalent to that [2]

$$(\xi^i, \xi^j)\equiv 0, \ 0\leqq i+j\leqq 2m-1, \tag{2.1}$$

where $\xi^i := \xi^{(i)} = d^i\xi/dz^i$.

Let $\Xi_i: S^2 \to CP^{N_i}$, $N_i=\binom{2m+1}{i+1}$, $i=0, 1, 2, \cdots, 2m-1$, be the associated curves of Ξ [3], which are determined by

$$\xi\wedge\xi'\wedge\cdots\wedge\xi^i: S^2 \longrightarrow C^{N_i+1}, \ i=0, 1, 2, \cdots, 2m-1.$$

Obviously $\Xi_0=\Xi$.

In the remains of this paper, we denote for each i, $0\leqq i\leqq 2m-1$, by σ_i the stationary index of Ξ_i, and δ_i the stationary multiplicity of Ξ at any fixed point of S^2. Then we have (one can see [2] for details):

$$\deg\Xi = 2m+\sum_{i=0}^{m-1}\sigma_i, \ \deg\Xi\geqq 2m+2\sum_{i=0}^{m-1}\delta_i. \tag{2.2}$$

LEMMA 2.1 [1]. *Let $\Xi: S^2\to CP^{2m}$ be a linearly full curve of degree $2m+2$. If $\xi=\sum_{i=0}^{2m+2}a_iz^i$ is a local lift of Ξ, then Ξ is totally isotropic if and only if the following hold*:

$$(1) \ (a_i, a_j)=0, \text{ except } i+j=2m+p, \ 0\leqq p\leqq 4, \tag{2.3}$$

$$(2) \ (a_{m-r}, a_{m+r})=(-1)^r\frac{(m!)^2}{(m-r)!(m+r)!}(a_m, a_m), \tag{2.4}$$

$$(3) \ (a_{m-r}, a_{m+r+1})=(-1)^r\frac{(2r+1)m!(m+1)!}{(m-r)!(m+1+r)!}(a_m, a_{m+1}), \tag{2.5}$$

$$(4) \ (a_{m+1-s}, a_{m+1+s})=(-1)^{s-1}\frac{1}{(m+1-s)!(m+1+s)!}\{s^2m!(m+2)!(a_m, a_{m+2}) + (s^2-1)[(m+1)!]^2(a_{m+1}, a_{m+1})\}, \ s=1, 2, \cdots, m+1, \tag{2.6}$$

$$(5) \ (a_{m+1-r}, a_{m+2+r})=(-1)^r\frac{(2r+1)m!(m+1)!}{(m-r)!(m+1+r)!}(a_{m+1}, a_{m+2}), \tag{2.7}$$

$$(6) \ (a_{m+2-r}, a_{m+2+r})=(-1)^r\frac{(m!)^2}{(m-r)!(m+r)!}(a_{m+2}, a_{m+2}), \tag{2.8}$$

where $r=1, 2, \cdots, m$.

LEMMA 2.2 [1]. *Let $\xi=\sum_{i=0}^{n}a_iz^i$ be a C^{2m+1}-valued polynomial, N a non-negative integer. If we denote by p the multi-indices $(p_0, p_1, \cdots, p_N)$ such that $0\leqq p_0<p<\cdots<p_N\leqq n$, then*

$$\xi\wedge\xi'\wedge\cdots\wedge\xi^N=\sum_p\prod_{i>j}(p_i-p_j)z^{l(p)}a_{p0}\wedge a_{p1}\wedge\cdots\wedge a_{pN}$$

$$=\sum_{k\geq 0}\left[\sum_{l(p)=k}\prod_{i>j}(p_i-p_j)a_{p0}\wedge a_{p1}\wedge\cdots\wedge a_{pN}\right]z^k,$$

where $l(p)=\sum_{1\leq i\leq N}p_i-N(N+1)/2$.

LEMMA 2.3 [1] *Let* $N\geqq 2$ *be an integer, and* $a_1, a_2, \cdots, a_{N+1}$ *be vectors in a complex space* $\boldsymbol{C}^n$. *Then,* $a_1, a_2, \cdots, a_{N+1}$ *are linearly related if and only if the exterior vectors* $a_1\wedge\cdots\wedge a_N$ *and* $a_1\wedge\cdots\wedge a_{N-1}\wedge a_{N+1}$ *are parallel in* $\wedge^N\boldsymbol{C}^n$.

3. Fundamental lemmas

This section will be devoted to proving some lemmas that are essential in the present paper.

LEMMA 3.1. *Let* $\Xi: S^2\to \boldsymbol{C}P^{2m}$ *be a linearly full, totally isotropic curve with degree* $2m+2$. *If* $z=0$ *is one stationary point of an associated curve* Ξ_i *for some* i, $0\leqq i\leqq m-1$, *and* $\xi=\sum_{j=0}^{2m+2}a_jz^j$ *is a local lift of* Ξ, *then there are numbers* λ_j, $\mu_l\in\boldsymbol{C}$, $0\leqq j\leqq i$, $0\leqq l\leqq i$ *or* $i+2\leqq l\leqq 2m-i$, *such that*

$$a_{i+1}=\sum_{j=0}^{i}\lambda_j a_j, \tag{3.1}$$

$$a_{2m+1-i}=\sum_{j=0}^{i}\mu_j a_j+\sum_{j=i+2}^{2m-i}\mu_j a_j. \tag{3.2}$$

Also we have,

(1) $\{a_j,\ j\neq i,\ 2m+1-i\}$ *is a basis for* $\boldsymbol{C}^{2m+1}$,

(2) $(a_j, a_k)=0$, *except* $j+k=2m+2,\ 2m+3,\ 2m+4$, (3.3)

(3) $$(a_{m+1-s}, a_{m+1+s})=(-1)^{s-1}\frac{[(m+1)!]^2[s^2-(m-i)^2]}{(m+1-s)!(m+1+s)!(m-i)^2}(a_{m+1}, a_{m+1}),$$

$$s=1, 2, \cdots, m+1, \tag{3.4}$$

(4) (a_{m+1-r}, a_{m+1+r})

$$=(-1)^r\frac{(2r+1)[(m+1)!]^2\lambda}{(m-i)^2(2m+2-i)(m-r)!(m+1+r)!}(a_{m+1}, a_{m+1}), \tag{3.5}$$

(5) $$(a_{m+2-r}, a_{m+2+r})=(-1)^{r+1}\frac{[(m+1)!]^2}{(m-i)^2(i+2)(m-r)!(m+r)!}$$

$$\times\left[\frac{2m-3-2i}{2m+2-i}\lambda\mu_{2m-i}+\frac{4(m-1-i)}{i+3}\mu_{2m-1-i}\right](a_{m+1}, a_{m+1}), \tag{3.6}$$

where $r=0, 1, \cdots, m$, *and* $\lambda=\lambda_i$.

Proof. By Barbosa [2], for any j, $0\leqq j\leqq m-1$, the associated curves, Ξ_j and Ξ_{2m-1-j} have the same stationary points and the same multiplicity at each point. So $z=0$ is a stationary point of Ξ_{2m-1-i}. On the other hand, it is easily seen from the inequality in (2.2) that $z=0$ is not the stationary point of Ξ_j, if $j\neq i$, $2m-1-i$ and that $\delta_i=\delta_{2m-i}=1$ at $z=0$.

First, the fact that $z=0$ is not the stationary point of Ξ_{i-1} and Lemma 2.3 give

$$\xi\wedge\xi'\wedge\cdots\wedge\xi^{i}(0)\neq 0. \tag{3.7}$$

Second, the fact that $z=0$ is the stationary point of Ξ_i gives

$$\xi\wedge\xi'\wedge\cdots\wedge\xi^{i+1}(0)=0. \tag{3.8}$$

(3.7) and (3.8) are equivalent to that $a_0, a_1, \cdots, a_i$ are linearly independent and $a_0, a_1, \cdots, a_i, a_{i+1}$ are linearly related. Thus we get (3.1).

Now using (3.1) and Lemma 2.2 we can obtain

$$\xi\wedge\xi'\wedge\cdots\wedge\xi^{j}\equiv 0 \ mod(z^{j-i}), \qquad i\leqq j\leqq 2m-1-i. \tag{3.9}$$

So,

$$\xi_j:=z^{i-j}\xi\wedge\xi'\wedge\cdots\wedge\xi^{j}$$

is a nonzero local lift of Ξ_i around $z=0$, $i\leqq j\leqq 2m-1-i$.

Since $z=0$ is not the stationary point of Ξ_{2m-2-i}. We see that

$$\xi_{2m-2-i}\wedge\xi'_{2m-2-i}(0)\neq 0. \tag{3.10}$$

On the other hand, Ξ_{2m-1-i} has $z=0$ as one of its stationary points, so

$$\xi_{2m-1-i}\wedge\xi'_{2m-1-i}(0)=0. \tag{3.11}$$

(3.10) and (3.11) imply, in view of Lemma 2.3, that $a_0, a_1, \cdots, a_i, a_{i+2}, \cdots, a_{2m-i}$ are linearly independent and $a_0, a_1, \cdots, a_i, a_{i+2}, \cdots, a_{2m-i}, a_{2m+1-i}$ are linearly related, and hence we get (3.2).

Now the fullness of Ξ implies that $a_0, a_1, \cdots, a_i, a_{i+2}, \cdots, a_{2m-i}, a_{2m-i+2}, \cdots, a_{2m+2}$ must hull C^{2m+1} and so they form a basis for C^{2m+1}.

Finally, it is not hard to see, by direct calculations, that (3.3) follows from (2.3), (2.4), (2.5) and (3.1); (3.4) from (2.6), (3.1) and (3.3); (3.5) from (2.7), (3.1), (3.3) and (3.4); (3.6) from (2.8) and (3.2)-(3.5). For example, we derive (3.6) as follows:

From (3.2)-(3.5) we find

$$\begin{aligned}(a_{i+3}, a_{2m+1-i})&\\ &=\mu_{2m-i}(a_{i+3}, a_{2m-i})+\mu_{2m-1-i}(a_{i+3}, a_{2m-1-i})\\ &=\mu_{2m-i}(-1)^{m-i}\frac{[2(m-i-2)+1][(m+1)!]^2\lambda}{(m-i)^2(2m+2-i)(i+2)!(2m-i-1)!}(a_{m+1}, a_{m+1})\end{aligned}$$

$$+\mu_{2m-1-i}(-1)^{m-i-1}\frac{[(m+1)!]^2[(m-i-2)^2-(m-i)^2]}{(m-i)^2(i+3)(2m-1-i)!}(a_{m+1},\ a_{m+1})$$

$$=\frac{(-1)^{m-i}[(m+1)!]^2}{(m-i)^2(i+2)!(2m-i-1)!}\left[\frac{2m-2i-3}{2m+2-i}\mu_{2m-i}\lambda\right.$$

$$\left.+\frac{4(m-i-1)}{i+3}\mu_{2m-1-i}\right](a_{m+1},\ a_{m+1}). \tag{3.12}$$

But from (2.8) we know that

$$(a_{i+3},\ a_{2m+1-i})=(-1)^{m-1-i}\frac{(m!)^2}{(i+1)!(2m-1-i)!}(a_{m+2},\ a_{m+2}). \tag{3.13}$$

Comparing the right hand sides of (3.12) and (3.13) we get

$$(m!)^2(a_{m+2},\ a_{m+2})=-\frac{[(m+1)!]^2}{(m-i)^2(i+2)}\left[\frac{2m-2i-3}{2m+2-i}\mu_{2m-i}\lambda\right.$$

$$\left.+\frac{4(m-i-1)}{i+3}\mu_{2m-1-i}\right](a_{m+1},\ a_{m+1}). \tag{3.14}$$

Insert (3.14) into (2.8) one can obtain (3.6). Q.E.D.

LEMMA 3.2. *Let $\Xi: S^2\to CP^{2m}$ be a linearly full curve of degree $2m+2$. If Ξ is totally isotropic, then there exists some integer i, $0\leqq i\leqq m-1$, such that $\sigma_i=2$, and other $\sigma_j=0$ for $0\leqq j\leqq m-1$, $j\neq i$. That is, all associated curves Ξ_j, $0\leqq j\leqq m-1$, except some Ξ_i, are immersions.*

Proof. By the assumption and (2.2), we have

$$\sigma_0+\sigma_1+\cdots+\sigma_{m-1}=2. \tag{3.15}$$

So only two cases can occur:

CASE 1. For some i, $0\leqq i\leqq m-1$, $\sigma_i=2$, and so $\sigma_j=0$ for other j, $0\leqq j\leqq m-1$;

CASE 2. For some i, j, $0\leqq i<j\leqq m-1$, $\sigma_i=\sigma_j=1$, $\sigma_l=0$ for other l, $0\leqq l\leqq m-1$.

Thus to prove Lemma 3.2, we need only to show that Case 2 is impossible. To this end, we suppose the contrary. Let p, q be the stationary points of Ξ_i and Ξ_j respectively. Then the inequality in (2.2) implies that $p\neq q$. Change the complex coordinate z if necessary, we can make p, q be the points $z=0$ and $z=\infty$ respectively. By Lemma 3.1, Ξ has a local lift as

$$\xi=\sum_{l=0}^{2m+2}a_l z^l,$$

satisfying (3.1) and (3.2) for some complex numbers $\lambda_0, \cdots, \lambda_i, \mu_0, \cdots, \mu_i, \mu_{i+2}, \cdots, \mu_{2m-i}$.

On the other hand, if we set $w=1/z$, then w is a new complex coordinate such that $w=0$ is the stationary point of Ξ_j. So applying Lemma 3.1 again to the local lift of Ξ:

$$\eta=\sum_{l=0}^{2m+2} a_{2m+2-l}w^l,$$

we know that there are numbers $\lambda'_0, \cdots, \lambda'_j, \mu'_0, \cdots, \mu'_j, \mu'_{j+2}, \cdots, \mu'_{2m-j}$ such that

$$a_{2m+1-j}=\sum_{l=0}^{j}\lambda'_l a_{2m+2-l}, \tag{3.16}$$

$$a_{j+1}=\sum_{l=0}^{j}\mu'_l a_{2m+2-l}+\sum_{l=j+2}^{2m-j}\mu'_l a_{2m+2-l}. \tag{3.17}$$

If $\lambda'_{i+1}=0$, then (3.16) indicates that $a_{2m+1-j}, a_{2m+2-j}, \cdots, a_{2m-i}, a_{2m+2-i}, \cdots, a_{2m+2}$ are linearly related, contradicting (1) in Lemma 3.1.

So, $\lambda'_{i+1}\neq 0$. Without loss of generality, we can assume $\lambda'_{i+1}=-1$. Then (3.16) says

$$a_{2m+1-i}=-a_{2m+1-j}+\sum_{\substack{0\leq l\leq j\\ l\neq i+1}}\lambda'_l a_{2m+2-l},$$

which with (3.2) gives

$$a_{2m+1-i}=\sum_{l=2m+2-j}^{2m-i}\mu_l a_l-a_{2m+1-j}. \tag{3.18}$$

If $\mu'_{i+1}=0$, then $a_{j+1}, a_{j+2}, \cdots, a_{2m-j}, a_{2m+2-j}, \cdots, a_{2m-i}, a_{2m+2-i}, \cdots, a_{2m+2}$ are linearly related, contradicting (1) in Lemma 3.1.

So $\mu'_{i+1}\neq 0$. Take, for example, $\mu'_{i+1}=-1$. Then (3.17) can be rewritten as

$$a_{2m+1-i}=-a_{j+1}+\sum_{\substack{0\leq l\leq j\\ l\neq i+1}}\mu'_l a_{2m+2-l}+\sum_{l=j+2}^{2m-j}\mu'_l a_{2m+2-l},$$

which compared with (3.2) implies

$$a_{2m+1-i}=-a_{j+1}+\sum_{l=2m+2-j}^{2m-i}\mu_l a_l+\sum_{l=j+2}^{2m-j}\mu_l a_l. \tag{3.19}$$

(3.18) and (3.19) give us that

$$a_{j+1}-a_{2m+1-j}=\sum_{l=j+2}^{2m-j}\mu_l a_l,$$

that is, $a_{j+1}, \cdots, a_{2m-j}, a_{2m+1-j}$ are linearly related, which also contradicts (1) in Lemma 3.1. This completes the proof. Q.E.D.

LEMMA 3.3. *Under the conditions of Lemma* 3.1, *the complex numbers* λ_r, μ_r, $0\leqq r\leqq i$, *and* μ_j, $i+2\leqq j\leqq 2m-i$ *are uniquely determined by the number* $\lambda=\lambda_i$. *In fact we have*

$$\lambda_{i-r}=(-1)^r\frac{(2m+2-i+r)!\lambda^{r+1}}{(r+1)!(2m+2-i)^r(2m+2-i)!},\quad 0\leqq r\leqq i, \tag{3.20}$$

$$\mu_r=(-1)^{i+r}\frac{(2m+2-r)!}{(i+1)!}\left(\frac{\lambda}{2m+2-i}\right)^{2m+1-i-r}\left[\frac{1}{(2m+1-i-r)!}\right.$$

$$\left.-\frac{1}{(2m-2i)!(i-r+1)!}\right], \quad 0\leqq r\leqq i, \tag{3.21}$$

$$\mu_s=(-1)^{i+s}\frac{(2m+2-s)!}{(i+1)!(2m+1-i-s)!}\left(\frac{\lambda}{2m+2-i}\right)^{2m+1-i-s},$$

$$i+2\leqq s\leqq 2m-i. \tag{3.22}$$

Proof. First we calculate μ_{2m-i}. By (3.2)-(3.4),

$$(a_{i+2}, a_{2m+1-i})=\mu_{2m-i}(a_{i+2}, a_{2m-i})$$

$$=\mu_{2m-i}(-1)^{m-i+1}\frac{[(m+1)!]^2(2m-2i-1)}{(i+2)!(2m-i)!(m-i)^2}(a_{m+1}, a_{m+1}). \tag{3.23}$$

But by (3.5),

$$(a_{i+2}, a_{2m+1-i})=(-1)^{m-i-1}\frac{[2(m-i-1)+1][(m+1)!]^2\lambda}{(m-i)^2(i+1)!(2m+2-i)(2m-i)!}(a_{m+1}, a_{m+1}). \tag{3.24}$$

If $(a_{m+1}, a_{m+1})=0$, then the $(m+1)$-dimensional subspace spanned by $a_0, a_1, \cdots, a_i, a_{i+2}, \cdots, a_{m+1}$ is totally isotropic by (3.3), contradicting the general fact that the totally isotropic subspace in $\boldsymbol{C}^{2m+1}$ is at most m-dimensional. Thus $(a_{m+1}, a_{m+1})\neq 0$. Comparing (3.23) and (3.24) we find

$$\mu_{2m-i}=\frac{(i+2)\lambda}{2m+2-i}. \tag{3.25}$$

Second, we turn to the other complex numbers. By Lemma 3.2, the stationary index $\sigma_i=2$. But it is obvious from the inequality in (2.2) that $\delta_i=1$ at $z=0$. So Ξ_i also has a stationary point $z_0\neq 0$. Since z_0 is possibly ∞, we would rather change the coordinate z to $w=1/z$, and consider

$$\eta=\sum_{l=0}^{2m+2} a_{2m+2-l}w^l,$$

the local lift of Ξ around $w=0$ ($z=\infty$). Let $w_0=1/z_0$. Since w_0 is not a stationary point of Ξ_{i-1}, but a stationary point of Ξ_i, it is not hard to see that

$$\eta\wedge\eta'\wedge\cdots\eta^i(w_0)\neq 0, \qquad \eta\wedge\eta'\wedge\cdots\eta^{i+1}(w_0)=0. \tag{3.26}$$

(3.26) implies that there are numbers $c_0, c_1, \cdots, c_i$, such that

$$\eta^{i+1}(w_0)=\sum_{i=0}^{i} c_l\eta^l(w_0). \tag{3.27}$$

Now using (3.1) and (3.2) we can express $\eta^l(w_0)$, $0\leqq l\leqq i+1$, in terms of $a_0, a_1, \cdots, a_i, a_{i+2}, \cdots, a_{2m-i}, a_{2m+2-i}, \cdots, a_{2m+2}$ as:

$$\eta^l(w_0)=a_0\left[\frac{(2m+2)!}{(2m+2-l)!}w_0^{2m+2-l}+\lambda_0\frac{(2m+1-i)!}{(2m+1-i-l)!}w_0^{2m+1-i-l}\right.$$
$$\left.+\mu_0\frac{(i+1)!}{(i+1-l)!}w_0^{i+1-l}\right]+\cdots+a_i\left[\frac{(2m+2-i)!}{(2m+2-i-l)!}w_0^{2m+2-i-l}\right.$$
$$\left.+\lambda_i\frac{(2m+1-i)!}{(2m+1-i-l)!}w_0^{2m+1-i-l}+\mu_i\frac{(i+1)!}{(i+1-l)!}w_0^{i+1-l}\right]$$
$$+a_{i+2}\left[\frac{(2m-i)!}{(2m-i-l)!}w_0^{2m-i-l}+\mu_{i+2}\frac{(i+1)!}{(i+1-l)!}w_0^{i+1-l}\right]+\cdots$$
$$+a_{2m-i}\left[\frac{(i+2)!}{(i+2-l)!}w_0^{i+2-l}+\mu_{2m-i}\frac{(i+1)!}{(i+1-l)!}w_0^{i+1-l}\right]$$
$$+a_{2m+2-i}\frac{i!}{(i-l)!}w_0^{i-l}$$
$$+\cdots+a_{2m+2}\frac{0!}{(0-l)!}w_0^{0-l},\qquad 0\leqq l\leqq i+1,\tag{3.28}$$

where we put $p!/q!=0$ in case $q<0$.

From (3.27) and (3.28) we see that $c_l=0$ for $0\leqq l\leqq i$, so

$$\eta^{i+1}(w_0)=0,$$

which, by (3.28) with $l=i+1$, is equivalent to the following identities:

$$\frac{(2m+2-l)!}{(2m+1-i-l)!}w_0^{2m+1-i-l}+\lambda_l\frac{(2m+1-i)!}{(2m-2i)!}w_0^{2m-2i}$$
$$+\mu_l(i+1)!=0,\qquad 0\leqq l\leqq i,\tag{3.29}$$

$$\frac{(2m+2-l)!}{(2m+1-i-l)!}w_0^{2m+1-i-l}+\mu_l(i+1)!=0,\qquad i+2\leqq l\leqq 2m-i.\tag{3.30}$$

Set $l=2m-i$ in (3.30) and use (3.25) we get

$$w_0=-\frac{\mu_{2m-i}}{i+2}=-\frac{\lambda}{2m+2-i}.\tag{3.31}$$

Thus (3.30) gives (3.22).

To prove (3.20), we first note from (3.3) that for each r, $0\leqq r\leqq i-2$,

$$(a_{i+1},\ a_{2m+4+r-i})=0,$$

which, with (3.1), implies

$$\lambda(a_i,\ a_{2m+4+r-i})+\lambda_{i-1}(a_{i-1},\ a_{2m+4+r-i})+\cdots+\lambda_{i-r-2}(a_{i-r-2},\ a_{2m+4+r-i})=0.$$

This identity and Lemma 3.1 give us

$$\frac{\lambda_{i-r}\lambda^2}{(2m+2-i)^2(2m+2-i+r)!}+\frac{(2m+5-2i+2r)\lambda_{i-r-1}\lambda}{(2m+2-i)(2m+3-i+r)!}$$
$$+\frac{(r+3)(r+3+2m-2i)\lambda_{i-r-2}}{(2m+4-i+r)!}=0. \tag{3.32}$$

On the other hand, we know from Lemma 3.1 that

$$(a_{i+1},\ a_{2m+3-i})=\lambda_i(a_i,\ a_{2m+3-i})+\lambda_{i-1}(a_{i-1},\ a_{2m+3-i})$$
$$=(-1)^{m+1-i}\frac{[(m+1)!]^2}{(m-i)^2(2m+2-i)!(i-1)!}\Big[\frac{2m+3-2i}{2m+2-i}\lambda^2$$
$$+\frac{4(m+1-i)}{2m+3-i}\lambda_{i-1}\Big](a_{m+1},\ a_{m+1}),$$

$$(a_{i+1},\ a_{2m+3-i})=(-1)^{m-i}\frac{[(m+1)!]^2}{(m-i)^2(i+2)(i-1)!(2m+1-i)!}$$
$$\times\Big[\frac{(2m-3-2i)(i+2)\lambda^2}{(2m+2-i)^2}+\frac{4(m-1-i)}{i+3}\mu_{2m-i-1}\Big](a_{m+1},\ a_{m+1}).$$

Comparing these two identities and using (3.22) we get

$$\lambda_{i-1}=-\frac{(2m+3-i)\lambda^2}{2(2m+2-i)}. \tag{3.33}$$

(3.33) indicates that (3.20) holds for $r=1$. An induction using (3.32) then proves (3.20) for all r, $0\leqq r\leqq i$.

Finally, we can derive (3.21) readily from (3.29), (3.31) and (3.20), thus complete the proof. Q.E.D.

By (3.22), (3.6) in Lemma 3.3 can be rewritten as

$$(a_{m+2-r},\ a_{m+2+r})=(-1)^r\frac{[(m+1)!]^2\lambda^2}{(2m+2-i)^2(m-i)^2(m-r)!(m+r)!}(a_{m+1},\ a_{m+1}). \tag{3.34}$$

4. Examples

This section devotes itself to constructing concrete examples. For this, we need the following result:

PROPOSITION 4.1. *For each integer i satisfying $0\leqq i\leqq m-1$ and each complex number λ, let numbers $\lambda_0, \cdots, \lambda_i$ and $\mu_0, \cdots, \mu_i, \mu_{i+2}, \cdots, \mu_{2m+2}$ be determined by* (3.20)-(3.22). *Then the holomorphic polynomial*

$$\xi=\sum_{j=0}^{2m+2}a_jz^j$$

satisfying (3.1)-(3.5) *and* (3.34) *is totally isotropic, and thus defines a totally iso-*

tropic curve in CP^{2m}.

Proof. It is readily verified that, under the restrictions of the proposition, all the conditions (2.3)-(2.8) are satisfied, and so Lemma 2.1 proves our conclusion.

Examples 4.2. Let $\boldsymbol{e}=\{e_1, e_2, \cdots, e_m, e_0, e_m, e_{m+1}, \cdots, e_{2m}\}^t$ be an orthonormal basis for $\boldsymbol{R}^{2m+1}$, and set

$$E_j=\frac{1}{\sqrt{2}}(e_j+\sqrt{-1}e_{m+j}), \qquad 1\leqq j\leqq m, \qquad E_0=e_0. \tag{4.1}$$

Then $\boldsymbol{E}=\{E_1, \cdots, E_m, E_0, \bar{E}_1, \cdots, \bar{E}_m\}^t$ is a unitrary basis for $\boldsymbol{C}^{2m+1}$, the complexication of $\boldsymbol{R}^{2m+1}$. Fix one integer i, $0\leqq i\leqq m-1$, and a complex number λ. Let the numbers $\lambda_0, \cdots, \lambda_i, \mu_0, \cdots, \mu_i, \mu_{i+2}, \cdots, \mu_{2m-i}$ be as in Section 3. To construct an example of totally isotropic curves in CP^{2m}, we need to find a suitable set of $\boldsymbol{C}^{2m+1}$-valued vectors $a_0, a_1, \cdots, a_{2m+2}$ satisfying the conditions of Proposition 4.1. In terms of the basis $\boldsymbol{E}$, we can write $\boldsymbol{Q}=\{a_0, a_1, \cdots, a_i, a_{i+2}, \cdots, a_{2m-i}, a_{2m+2-i}, \cdots, a_{2m+2}\}^t$ as

$$\boldsymbol{Q}=T\cdot\boldsymbol{E}, \tag{4.2}$$

where T is a nonsingular matrix of order $2m+1$. Thus we need only to find a special value of T.

Write T in a block form as

$$T=\begin{pmatrix} T_1 & *_1 & *_2 \\ V & *_3 & *_4 \\ T_2 & *_5 & *_6 \end{pmatrix} \tag{4.3}$$

with T_1, T_2, $*_2$ and $*_6$ being $m\times m$-matrices, $*_3$ a number. Set

$$\alpha_{m+1-s}=(-1)^{s-1}\frac{[(m+1)!]^2[s^2-(m-i)^2]}{(m+1-s)!(m+1+s)!(m-i)^2}, \quad s=1, 2, \cdots, m+1,$$

$$\beta_{m+1-r}=(-1)^r\frac{(2r+1)[(m+1)!]^2\lambda}{(m-i)^2(2m+2-i)(m-r)!(m+1+r)!}, \quad 0\leqq r\leqq m,$$

$$\gamma_{m+2-r}=(-1)^r\frac{[(m+1)!]^2\lambda^2}{(2m+2-i)^2(m-i)^2(m-r)!(m+r)!}, \quad 0\leqq r\leqq m.$$

In order that the conditions in Proposition 4.1 be satisfied, we put

$$(\boldsymbol{Q}, \boldsymbol{Q}^t)=\begin{pmatrix} 0 & 0 & D_1 \\ 0 & 1 & D_2 \\ D_3 & D_4 & D_5 \end{pmatrix},$$

where

$$D_1=\begin{pmatrix} 0 & \cdots & \cdots & & & & & & 0 & \alpha_0 \\ 0 & \cdots & & & & & & 0 & \alpha_1 & \beta_1 \\ 0 & \cdots & & & & & 0 & \alpha_2 & \beta_2 & \gamma_2 \\ \vdots & & & & & & & & & \vdots \\ 0 & \cdots & & & 0 & \alpha_i & \beta_i & \gamma_i & \cdots & \\ 0 & \cdots & & 0 & \alpha_{i+2} & \gamma_{i+2} & 0 & \cdots & & \\ 0 & \cdots & 0 & \alpha_{i+3} & \beta_{i+3} & 0 & \cdots & & & \\ 0 & \cdots\ 0 & \alpha_{i+4} & \beta_{i+4} & \gamma_{i+4} & 0 & & & & \vdots \\ \vdots & & & & & & & & & \\ 0 & \alpha_{m-1} & \beta_{m-1} & \gamma_{m-1} & 0 & \cdots & & & & 0 \\ \alpha_m & \beta_m & \gamma_m & 0 & \cdots & & & & & 0 \end{pmatrix},$$

$$D_2=(\beta_{m+1}, \gamma_{m+1}, 0, \cdots, 0), \qquad D_3=D_1^t, \qquad D_4=D_2^t,$$

and

$$D_5=\begin{pmatrix} \gamma_{m+2} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since

$$(\boldsymbol{E}, \boldsymbol{E}^t)=\begin{pmatrix} 0 & 0 & I_m \\ 0 & 1 & 0 \\ I_m & 0 & 0 \end{pmatrix},$$

where I_m is the identity matrix of order m, we see from (4.2) and (4.3) that

$$\begin{cases} *_2\cdot T_1^t+*_1\cdot *_1^t+T_1\cdot *_2^t=0, & *_2\cdot V^t+*_1\cdot *_3+T_1\cdot T_4^t=0, \\ *_2\cdot T_2^t+*_1\cdot *_5^t+T_1\cdot *_6^t=D_1, & *_4\cdot V^t+*_3^2+V\cdot *_4^t=1, \\ *_4\cdot T_2^t+*_3\cdot *_5^t+V\cdot *_6^t=D_2, & *_6\cdot T_2^t+*_5\cdot *_5^t+T_2\cdot *_6^t=D_5. \end{cases} \tag{4.5}$$

To simplify $\boldsymbol{Q}$, we put $V=*_1^t=*_4=0$, $*_2=0$, and $*_3=1$. Then (4.5) becomes

$$T_1\cdot *_6^t=D_1, \quad *_6\cdot T_2^t+*_5\cdot *_5^t+T_2\cdot *_6^t=D_5, \quad *_5=D_2^t. \tag{4.6}$$

Suppose also that $*_6=I_m$ and that T_2 is symmetric, then

$$T_1=D_1, \qquad T_2=\frac{1}{2}(D_5-D_2^t\cdot D_2). \tag{4.7}$$

Thus we obtain a value $T(i, \lambda)$ of the matrix T, or, a special solution $\boldsymbol{Q}(i, \lambda)$ of $\boldsymbol{Q}$ as

$$a_0=\alpha_0E_m, \quad a_1=\alpha_1E_{m-1}+\beta_1E_m, \quad a_2=\alpha_2E_{m-2}+\beta_2E_{m-1}+\gamma_2E_m, \cdots,$$

$$a_i=\alpha_iE_{m-i}+\beta_iE_{m-i+1}+\gamma_iE_{m-i+2}, \quad a_{i+2}=\alpha_{i+2}E_{m-i-1}+\gamma_{i+2}E_{m-i},$$

$$a_{i+3}=\alpha_{i+3}E_{m-i-2}+\beta_{i+3}E_{m-i-1}, \quad a_{i+4}=\alpha_{i+4}E_{m-i-3}+\beta_{i+4}E_{m-i-2}+\gamma_{i+4}E_{m-i-1},$$

$$\cdots, \quad a_m = \alpha_m E_1 + \beta_m E_2 + \gamma_m E_3, \quad a_{m+1} = E_0,$$

$$a_{m+2} = \bar{E}_1 + \frac{1}{2}(\gamma_{m+2} - \beta_{m+1}^2)E_1 - \frac{1}{2}\beta_{m+1}\gamma_{m+1}E_2 + \beta_{m+1}E_0,$$

$$a_{m+3} = \bar{E}_2 - \frac{1}{2}\beta_{m+1}\gamma_{m+1}E_1 - \frac{1}{2}\gamma_{m+1}^2 E_2 + \gamma_{m+1}E_0,$$

$$a_{m+4} = \bar{E}_3, \cdots, \quad a_{2m-i} = \overline{E_{m-i-1}}, \quad a_{2m+2-i} = \overline{E_{m-i}}, \cdots, \quad a_{2m+2} = \overline{E_m}.$$

Now we define

$$a_{i+1} = \lambda_0 a_0 + \cdots + \lambda_i a_i,$$

$$a_{2m+1-i} = \mu_0 a_0 + \cdots + \mu_i a_i + \mu_{i+2} a_{i+2} + \cdots + \mu_{2m-i} a_{2m-i}.$$

Then it is directly verified that the polynomial

$$\xi = \sum_{j=0}^{2m+2} a_j z^j$$

satisfies all conditions of Proposition 4.1, therefore defines a totally isotropic curve $\Xi(i, \lambda)$. Also, when $0 \leqq i \leqq m-1$, $\Xi(i, \lambda)$ corresponds to a minimal immersion $x(i, \lambda): S^2 \to S^{2m}(1)$, the degree of which is, by definition, $2m+2$.

Remark 4.3. By a change of order of $E_0, \cdots, E_m$ as $E_m, \cdots, E_1$, it is easily seen that the special curve $\Xi(0, \lambda)$ is nothing but the curve Ξ_λ given in [1].

5. Explicit representations and the classification theorem

By the methods used in [1], we can obtain explicit representations for all full minimal immersions $x: S^2 \to S^{2m}(1)$ of degree $2m+2$, which we state briefly as follows:

For any of such immersion x, let Ξ be the directrix of it. Then Ξ is a totally isotropic curve in CP^{2m} of degree $2m+2$. Recall the discussions in Section 3, there exist an integer i with $0 \leqq i \leqq m-1$, a complex coordinate z on S^2, and a complex number λ, such that Ξ has a local lift

$$\xi = \sum_{j=0}^{2m+2} a_j z^j,$$

in which, $(a_{m+1}, a_{m+1}) = 1$, $\boldsymbol{Q} = \{a_0, \cdots, a_i, a_{i+2}, \cdots, a_{2m-i}, a_{2m+2-i}, \cdots, a_{2m+2}\}^t$ is a basis for $\boldsymbol{C}^{2m+1}$, and (3.1)-(3.5), (3.34) hold for numbers $\lambda_0, \cdots, \lambda_i, \mu_0, \cdots, \mu_i, \mu_{i+2}, \cdots, \mu_{2m-i}$ given by (3.20)-(3.22). Let $\boldsymbol{A}$ be in $GL(2m+1, \boldsymbol{C})$, such that

$$\boldsymbol{Q} = \boldsymbol{A}(\boldsymbol{Q}(i, \lambda)),$$

and A the matrix of $\boldsymbol{A}$ with respect to $\boldsymbol{E}$. Then

$$\boldsymbol{Q}=\boldsymbol{A}(\boldsymbol{Q}(i, \lambda))=T(i, \lambda)\boldsymbol{A}(\boldsymbol{E})=[T(i, \lambda)\cdot A]\boldsymbol{E}. \tag{5.1}$$

On the other hand, by Lemmas 3.1 and 3.3, we know that

$$(\boldsymbol{Q}(i, \lambda), \boldsymbol{Q}^t(i, \lambda))=(\boldsymbol{Q}, \boldsymbol{Q}^t)=(\boldsymbol{A}(\boldsymbol{Q}(i, \lambda)), \boldsymbol{A}(\boldsymbol{Q}^t(i, \lambda))),$$

which is equivalent to

$$(\boldsymbol{A}^t\cdot\boldsymbol{A}(\boldsymbol{Q}(i, \lambda), \boldsymbol{Q}^t(i, \lambda))=(\boldsymbol{Q}(i, \lambda), \boldsymbol{Q}^t(i, \lambda))$$

or,

$$((\boldsymbol{A}^t\cdot\boldsymbol{A}-\boldsymbol{I})(\boldsymbol{Q}(i, \lambda)), \boldsymbol{Q}(i, \lambda))=0,$$

with $\boldsymbol{I}$ the identity of $GL(2m+1, \boldsymbol{C})$. Since $\boldsymbol{Q}(i, \lambda)$ is a basis for $\boldsymbol{C}^{2m+1}$, we can get easily that

$$\boldsymbol{A}^t\cdot\boldsymbol{A}=\boldsymbol{I}. \tag{5.2}$$

Using (5.2) and suitably choosing $\boldsymbol{E}$, one finds (for details, see [1]):

$$A=\pm\begin{pmatrix} A_1 & 0 & 0 \\ V & 1 & 0 \\ A_2 & (A_1^{-1})^t\cdot V^t & (A_1^{-1})^t \end{pmatrix} \tag{5.3}$$

with A_1 a lower triangular matrix of order m, V a row vector of dimension m, and A_2 determined by

$$A_2=-\frac{1}{2}(A_1^{-1})^t\cdot V^t\cdot V+C\cdot A_1, \tag{5.4}$$

where C is an antisymmetric $(m\times m)$-matrix.

Conversely, given arbitarily an integer i with $0\leqq i\leqq m-1$, a complex number λ, a nonsingular, lower triangular matrix A_1 of order m, an m-dimensional vector V and an antisymmetric $(m\times m)$-matrix C, we obtain by (5.3) and (5.4) a matrix A of order $2m+1$, and then a basis for $\boldsymbol{C}^{2m+1}$ by (5.1). A direct verification using Proposition 4.1 shows that the polynomial

$$\xi=\sum_{i=0}^{2m+2} a_j z^j$$

defined by $\boldsymbol{Q}$, (3.1), (3.2) and (3.20)-(3.22) is totally isotropic, therefore gives a minimal immersion $x: S^2\to S^{2m}(1)$ of degree $2m+2$.

Thus (5.1), (5.3) and (5.4) define an explicit representation of full minimal immersions $x: S^2\to S^{2m}(1)$ of degree $2m+2$, in terms of independent parameters.

Finally, for each $i=0, 1, \cdots, m-1$, we can use the same arguments as in Remark 5.8 of [1] to obtain the following result:

PROPOSITION 5.1. *Up to isometries, the set of all totally isotropic, linearly full curves* $\Xi: S^2\to \boldsymbol{C}P^2$ *of degree* $2m+2$, *of which only the i-th associated curve fails to be an immersion, has a natural structure diffeomorphic to the trivial bundle*

$$GB = \&_2 \times \frac{SO(2m+1, \boldsymbol{C})}{SO(2m+1, \boldsymbol{R})},$$

with $\&_2$ *the quotient space of* $\& = S^2 \times \boldsymbol{C}$ *by an action of the modulo group* $Z_2 = \{-1, 1\}$.

As for the directrices of the full minimal immersions x of S^2 into $S^{2m}(1)$, the case $i=m-1$ can not occur. Hence, combining Lemma 3.2 and Proposition 5.1, we complete our classification. The conclusion is,

THEOREM 5.2. *The set of full minimal immersions* $x: S^2 \to S^{2m}(1)$ *of degree* $2m+2$ *is, modulo isometries, diffeomorphic to a disjoint union of* $m-1$ *copies of the trivial bundle* GB.

Remark 5.3. The area formula (See [1, 2]) gives the area of immersions x in Theosem 5.3 as

$$A(x) = 2\pi[m(m+1) + 2(i+1)], \qquad 0 \leqq i \leqq m-2. \tag{5.5}$$

So the case $i=0$ is just that discussed in [1].

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DEPARTMENT OF MATHEMATICS
SICHUAN UNIVERSITY
CHENGDU 610064, SICHUAN
P. R. CHINA

Current Address
DEPARTMENT OF MATHEMATICS
HENAN NORMAL UNIVERSITY, XINXIANG 453002
HENAN, P. R. CHINA