EQUIVARIANT CRITICAL POINT THEORY
AND SET-VALUED GENERA

Dedicated to Professor Seiya Sasao on his 60th birthday

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Introduction

Throughout this paper G denotes a compact Lie group. For a point x of a G-space X we set

$$Gx := \{gx \in X \mid g \in G\}, \quad \text{and} \quad G_x := \{g \in G \mid gx = x\}.$$

Gx is called the *orbit* through x, and G_x the *isotropy subgroup* at x. Its conjugacy class (G_x) is called the *orbit type* (or *isotropy type*) of x. We denote by $\mathcal{I}(X)$ the set of all orbit types on X.

Let M be a closed G-manifold, or more generally a paracompact Banach G-manifold of class at least C^1, i.e., M is a paracompact C^1 Banach manifold and G acts differentiably by diffeomorphisms. Let $f : M \to \mathbf{R}$ be a C^1 G-function, i.e., f is of class C^1 and satisfies $f(gx) = f(x)$ for all $x \in M$ and $g \in G$. If $x \in M$ is a critical point of f, i.e., $df_x = 0$, then its orbit Gx is called a *critical orbit* and its orbit type (G_x) is called a *critical orbit type*. Let $K = \{x \in M \mid df_x = 0\}$ be the critical point set of f, and set $K_c = K \cap f^{-1}(c)$ and $M_c = \{x \in M \mid f(x) \leq c\}$ for any $c \in \mathbf{R}$.

The number of critical points alone has been discussed in the nonequivariant setting. We should, however in the equivariant setting, discuss the types of critical orbits as well as the number of them. There also arises a new problem to estimate the number of critical orbit *types*. In this paper we will define the *set-valued genus* and the *numerical genus* of a G-space, and obtain some inequalities for the genus and the number of critical values and orbit types for G-functions.

Let $\mathcal{A}$ be the set of G-homeomorphism classes of compact G-ANR's (G-*equivariant Absolute Neighborhood Retract*). For a G-space X we define a subset $\alpha(X)$ of $\mathcal{A}$ to be the set of those $S \in \mathcal{A}$ for which there is a G-map $X \to S$. We call $\alpha(X)$ the *set-valued genus* of X. For two subsets α_1 and α_2 of $\mathcal{A}$, we define

$$\alpha_1 * \alpha_2 := \{S * T \mid S \in \alpha_1, \ T \in \alpha_2\}.$$

where $S*T$ denotes the join of S and T.

Suppose $f: M \to R$ is bounded below and satisfies the *Palais-Smale condition* or the conditions (R_0)–(R_2) stated below. If the critical values of f are $c_1, \cdots, c_m$, then we will obtain

(0.1) $$\alpha(K_{c_1}) * \cdots * \alpha(K_{c_m}) \subseteq \alpha(M).$$

For a G-space X and a subset S of $\mathcal{A}$, we define $\beta_S(X)$ to be the smallest number n such that $A_1 * \cdots * A_n \in \alpha(X)$ for $A_1, \cdots, A_n \in S$. We call $\beta_S(X)$ the *numerical S-genus* of X. Since a homogeneous space G/H with H a closed subgroup of G is a G-ANR, we may take as S the following subset $\mathcal{H}$ or $\mathcal{O}(X)$ of $\mathcal{A}$,

$$\mathcal{H} = \{ G/H \mid H \text{ is a proper closed subgroup of } G \},$$

$$\mathcal{O}(X) = \{ G/H \mid (H) \in \mathcal{I}(X) \}.$$

Then under some conditions on $f: M \to R$ we will obtain from (0.1),

$$\beta_{\mathcal{H}}(M) \leq \# f(K) \cdot \# \mathcal{I}(K),$$

and

$$\beta_{\mathcal{O}(M)}(M) \leq \# f(K) \cdot \# \mathcal{I}(K),$$

(see Theorem 3.1 below), where $\# f(K)$ is the number of critical values and $\# \mathcal{I}(K)$ the number of critical orbit types.

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1. Set-valued genera

A G-ANR is a metrizable G-space S such that every G-map $Z \to S$ from a G-invariant closed subspace Z of a metrizable G-space Y into S has a G-extension $U \to S$ to a G-neighborhood U of Z in Y. Any point of the join $S*T$ of two G-ANS's S and T is represented by the form $[(\lambda, x), (\mu, y)]$ with the usual identification, where $x \in S$, $y \in T$, $0 \leq \lambda$, $\mu \leq 1$ and $\lambda + \mu = 1$. Then $S*T$ becomes a G-space via the action

$$g[(\lambda, x), (\mu, y)] = [(\lambda, gx), (\mu, gy)]$$

for any $g \in G$. Moreover $S*T$ is a G-ANR (see Murayama [4; Corollary 12.3]). This implies that the join $\alpha_1 * \alpha_2$ of subsets α_1 and α_2 of $\mathcal{A}$ is also a subset of $\mathcal{A}$.

Convention 1.1. We make the following conventions concerning the empty set $\emptyset$ and the set-valued genus $\alpha(X)$ of a G-space X.
  (i) The empty set $\emptyset$ is assumed to belong to $\mathcal{A}$.
  (ii) If $X = \emptyset$ then we assume $\alpha(X) = \mathcal{A}$, and if $X \neq \emptyset$ then $\emptyset \notin \alpha(X)$.
  (iii) $X * \emptyset = X = \emptyset * X$ for any space X. Then we see $\alpha \subseteq \alpha * \mathcal{A}$ for any subset

α of $\mathcal{A}$, since $\emptyset \in \mathcal{A}$.

(iv) Considering $\emptyset$ as the empty subset of $\mathcal{A}$, $\alpha * \emptyset = \emptyset = \emptyset * \alpha$ for any subset α of $\mathcal{A}$.

The set-valued genus possesses the following simple properties.

PROPOSITION 1.2. (i) *Monotonicity*: *If there is a G-map $X \to Y$, then $\alpha(Y) \subseteq \alpha(X)$.*

(ii) *Subadditivity*: *If $X = A \cup B$ is a metrizable G-space and A, B are G-invariant closed subsets of X, then $\alpha(A) * \alpha(B) \subseteq \alpha(X)$.*

(iii) *Subcontinuity*: *Let X be a metrizable G-space and A a G-invariant closed subset of X. For any finite subset $\mathcal{F}$ of $\alpha(A)$, there is a G-neighborhood $U_{\mathcal{F}}$ of A in X such that $\mathcal{F} \subseteq \alpha(U) \subseteq \alpha(A)$ for any G-neighborhood U of A with $U \subseteq U_{\mathcal{F}}$.*

Proof. (i) Easy.

(ii) Assume $\alpha(A) \neq \emptyset$ and $\alpha(B) \neq \emptyset$, and take $S \in \alpha(A)$ and $T \in \alpha(B)$. There are G-maps $\varphi: A \to S$ and $\psi: B \to T$. Since S and T are G-ANR's, we have their G-extensions $\tilde{\varphi}: N_A \to S$ and $\tilde{\psi}: N_B \to T$, where N_A and N_B are open G-neighborhood of A and B in X, respectively. There is a function $\lambda: X \to [0, 1]$ such that $\lambda = 0$ on $X - N_A$ and $\lambda = 1$ on $X - N_B$. We may assume that λ is G-invariant since we may integrate it over G if necessary. We may define a G-map $\eta: X \to S * T$ by

$$\eta(x) = [(\lambda(x), \tilde{\varphi}(x)), (1 - \lambda(x), \tilde{\psi}(x))]$$

for $x \in X$. Thus $S * T \in \alpha(X)$.

(iii) If $\mathcal{F}$ is empty, then $\mathcal{F} \subseteq \alpha(U) \subseteq \alpha(A)$ for all G-neighborhood U of A. So assume $\mathcal{F} \neq \emptyset$, and take $S \in \mathcal{F}$ and a G-map $\varphi: A \to S$. Then φ has a G-extension $U_S \to S$ to a G-neighborhood U_S of A, and thus $S \in \alpha(U_S)$. Set $U_{\mathcal{F}} := \bigcap_{S \in \mathcal{F}} U_S$. Then $U_{\mathcal{F}}$ is a G-neighborhood of A since $\mathcal{F}$ is finite. For any $S \in \mathcal{F}$ and G-neighborhood U of A with $U \subseteq U_{\mathcal{F}}$, we see $\alpha(U_S) \subseteq \alpha(U_{\mathcal{F}}) \subseteq \alpha(U)$ by the monotonicity property. Thus $\mathcal{F} \subseteq \bigcup_{S \in \mathcal{F}} \alpha(U_S) \subseteq \alpha(U)$. $\square$

2. The genera of critical point sets

Throughout this paper M denotes a paracompact C^1 Banach G-manifold and $f: M \to \mathbf{R}$ a C^1 G-function with the critical point set K. Consider the following conditions (R_0)-(R_2) for $f: M \to \mathbf{R}$ at $c \in \mathbf{R}$:

(R_0) *There are an $\varepsilon > 0$ and a G-map $M_{c+\varepsilon} \to M_c$.*

(R_1) *K_c is compact.*

(R_2) *For every G-neighborhood U of K_c there are an $\varepsilon > 0$ with $\varepsilon < 1$ and a G-map $\varphi: M_{c+\varepsilon} - U \to M_{c-\varepsilon}$ (with $\varphi | M_{c-1} = \mathrm{id}$ if f is unbounded above and $c > \sup f(K)$).*

The conditions (R_0)–(R_2) are weaker than the conditions (D_0)–(D_2) in Clapp-Puppe [2; 3.1]. As is shown in Clapp-Puppe [2; Appendix A] and Krawcewicz-Marzantowicz [3; Lemma 1.9], the conditions (R_1) and (R_2) at any $c \in \mathbf{R}$ is a consequence of the Palais-Smale condition under suitable assumptions on differentiability and completeness. [2; Appendix A] shows that if c is a regular value of f then (R_0) is also a consequence of the Palais-Smale condition. Even if c is not a regular value we can see that (R_0) follows from the Palais-Smale condition under the assumption c is an isolated critical value.

LEMMA 2.1. *Suppose that a C^1 G-function $f : M \to \mathbf{R}$ satisfies (R_1) and (R_2) at c. For a given finite subset $\mathcal{F}$ of $\alpha(K_c)$ there is an $\varepsilon = \varepsilon(\mathcal{F}) > 0$ such that*

$$\alpha(M_{c-\varepsilon}) * \mathcal{F} \subseteq \alpha(M_{c+\varepsilon}).$$

In particular, if K_c is empty then $\alpha(M_{c-\varepsilon}) = \alpha(M_{c+\varepsilon})$.

Proof. By the assumption, K_c is compact and thus closed. M is metrizable by Palais [5; Corollary 3.4]. By the subcontinuity property of α there is a G-neighborhood $U_{\mathcal{F}}$ of K_c in M such that $\mathcal{F} \subseteq \alpha(U) \subseteq \alpha(K_c)$ for any G-neighborhood U of K_c with $U \subseteq U_{\mathcal{F}}$. Take an open G-neighborhood V of K_c such that $\overline{V} \subseteq U_{\mathcal{F}}$, and take an $\varepsilon > 0$ satisfying (R_2) for this V. Then we see

$$\alpha(M_{c+\varepsilon}) = \alpha((M_{c+\varepsilon} - V) \cup \overline{V})$$

$$\supseteq \alpha(M_{c+\varepsilon} - V) * \alpha(\overline{V})$$

$$\supseteq \alpha(M_{c-\varepsilon}) * \mathcal{F}.$$

If K_c is empty, then there are an inclusion map $M_{c-\varepsilon} \to M_{c+\varepsilon}$ and a G-map $M_{c+\varepsilon} \to M_{c-\varepsilon}$ by (R_2). Using the monotonicity property twice, we see $\alpha(M_{c-\varepsilon}) = \alpha(M_{c+\varepsilon})$. □

Consider an interval $(a, b] = \{t \in \mathbf{R} \mid a < t \leq b\}$ for $-\infty < a < b \leq \infty$. A C^1 G-function $f : M \to \mathbf{R}$ is said to satisfy $R(a, b)$ if f satisfies (R_0) at a and (R_1), (R_2) at every $c \in (a, b]$.

THEOREM 2.2. *Suppose that a C^1 G-function $f : M \to \mathbf{R}$ satisfies $R(a, b)$ for $-\infty < a < b \leq \infty$, and that $f(K)$ is bounded above if $b = \infty$. Given a family $\{\mathcal{F}_c \mid c \in (a, b]\}$ of finite subsets $\mathcal{F}_c$ of $\alpha(K_c)$ such that $\mathcal{F}_c = \{\emptyset\}$ if $K_c = \emptyset$, there are a finite number of critical values $c_1, \cdots, c_k \in (a, b]$ such that*

$$\alpha(M_a) * \mathcal{F}_{c_1} * \cdots * \mathcal{F}_{c_k} \subseteq \alpha(M_b).$$

Proof. First assume $b < \infty$. Let $\varepsilon(a)$ be such an $\varepsilon > 0$ as in (R_0) at a. For any $c \in (a, b]$ let $\varepsilon(c)$ be such an $\varepsilon > 0$ as in Lemma 2.1, i.e.,

$$\alpha(M_{c-\varepsilon(c)}) * \mathcal{F}_c \subseteq \alpha(M_{c+\varepsilon(c)}).$$

Let V_c denote the open interval $(c - \varepsilon(c), c + \varepsilon(c))$ for any $c \in [a, b]$. Then $\{V_c \mid c \in [a, b]\}$ is an open covering of $[a, b]$. Since $[a, b]$ is compact, there

are a finite number of $d_1, \cdots, d_m \in [a, b]$ such that

$$[a, b] \subseteq V_{d_1} \cup \cdots \cup V_{d_m} .$$

By the monotonicity property and the assumption for $\varepsilon(b)$ we have

$$\alpha(M_b) \supseteqq \alpha(M_{b+\varepsilon(b)}) \supseteqq \mathcal{F}_b * \alpha(M_{b-\varepsilon(b)}) .$$

$b - \varepsilon(b)$ is contained in V_d for some $d \in \{d_1, \cdots, d_m\}$. Since $b - \varepsilon(b) < d + \varepsilon(d)$ we have

$$\alpha(M_{b-\varepsilon(b)}) \supseteqq \alpha(M_{d+\varepsilon(d)}) \supseteqq \mathcal{F}_d * \alpha(M_{d-\varepsilon(d)}) .$$

Thus we have

$$\alpha(M_b) \supseteqq \mathcal{F}_b * \mathcal{F}_d * \alpha(M_{d-\varepsilon(d)}) .$$

Repeating this argument, we have

$$(2.3) \qquad\qquad \alpha(M_b) \supseteqq \mathcal{F}_{c_1} * \cdots * \mathcal{F}_{c_k} * \alpha(M_a)$$

for some $c_1, \cdots, c_k \in (a, b]$. If c is not a critical value then $\mathcal{F}_c = \{0\}$ and so we may assume that $c_1, \cdots, c_k$ in (2.3) are all critical values. Thus the theorem is proved for the case $b < \infty$.

Now assume $b = \infty$. Take an $r > 0$ such that $\sup f(K) < r < \infty$. By the above argument we see that there are a finite number of critical values $c_1, \cdots, c_k \in (a, r]$ such that

$$\alpha(M_a) * \mathcal{F}_{c_1} * \cdots * \mathcal{F}_{c_k} \subseteq \alpha(M_r) .$$

Since there are no critical values in $[r, \infty)$ we can see by (R_2) that there is a G-map $M_b = M \to M_r$. Thus $\alpha(M_r) = \alpha(M_b) = \alpha(M)$. Thus the theorem is also proved for the case $b = \infty$. $\qquad\qquad \square$

COROLLARY 2.4. *If, in Theorem 2.2, the critical values of f are $c_1, \cdots, c_m$ in $(a, b]$, then*

$$\alpha(M_a) * \alpha(K_{c_1}) * \cdots * \alpha(K_{c_m}) \subseteq \alpha(M_b) .$$

Proof. Let $\mathcal{F}_i$ be an arbitrary finite subset of $\alpha(K_{c_i})$ for $1 \leqq i \leqq m$. By Theorem 2.2 we have

$$\alpha(M_a) * \mathcal{F}_{i_1} * \cdots * \mathcal{F}_{i_k} \subseteq \alpha(M_b)$$

for some $1 \leqq i_1, \cdots, i_k \leqq m$. This implies

$$\alpha(M_a) * \mathcal{F}_1 * \cdots * \mathcal{F}_m \subseteq \alpha(M_b) .$$

Since $\mathcal{F}_i$ is an arbitrary finite subset of $\alpha(K_{c_i})$, we have

$$\alpha(M_a) * \alpha(K_{c_1}) * \cdots * \alpha(K_{c_m}) \subseteq \alpha(M_b) . \qquad\qquad \square$$

If, in Corollary 2.4, we take $b=\infty$ and if $f: M \to \mathbf{R}$ is bounded below, then we have

$$\alpha(K_{c_1}) * \cdots * \alpha(K_{c_m}) \subseteqq \alpha(M).$$

This means that if there are G-maps $K_{c_i} \to S_i (S_i \in \mathcal{A}, 1 \leq i \leq m)$, then there is a G-map $M \to S_1 * \cdots * S_m$. The special case in which G is finite and its action on M is free is discussed in R.-Nowakowska [6].

3. Numerical genera

Let $\mathcal{S}$ be a subset of $\mathcal{A}$. As in Introduction, the numerical $\mathcal{S}$-genus $\beta_{\mathcal{S}}(X)$ of a G-space X is defined to be the smallest number n such that there exists a G-map $X \to A_1 * \cdots * A_n$ for $A_i \in \mathcal{S}$ ($1 \leq i \leq n$). If there does not exist such a G-map we define $\beta_{\mathcal{S}}(X) = \infty$, and if $X = \emptyset$ then define $\beta_{\mathcal{S}}(X) = 0$.

Clapp-Puppe [2; §2] defined the $\mathcal{S}$-category denoted $\mathcal{S}$-cat(f) for a G-map $f: (X, X') \to (Y, Y')$. If $c_X: X \to \mathrm{pt}$ is the constant map, then $\mathcal{S}$-cat(c_X) coincides with our $\beta_{\mathcal{S}}(X)$ (see [2; Proposition 2.4]).

If we take as $\mathcal{S}$ the set $\mathcal{H}$ or $\mathcal{O}(X)$ given in Introduction, we have two genera $\beta_{\mathcal{H}}(X)$ and $\beta_{\mathcal{O}(X)}(X)$. Let $\gamma(X)$ denote $\beta_{\mathcal{O}(X)}(X)$. We easily see that if $X^G = \emptyset$ then $\beta_{\mathcal{H}}(X) \leq \gamma(X)$, and if $X^G \neq \emptyset$ then $\beta_{\mathcal{H}}(X) = \infty$ $\gamma(X) = 1$.

THEOREM 3.1. *Suppose that a C^1 G-function $f: M \to \mathbf{R}$ satisfies R(a, b) for* $-\infty < a < b \leq \infty$. *If $K_{(a,b]} = K \cap f^{-1}(a, b]$ consists of a finite number of orbits, then*

$$\gamma(M_b) - \gamma(M_a) \leq \# f(K_{(a,b]}) \cdot \# \mathcal{J}(K_{(a,b]}).$$

If, moreover, $K^G_{(a,b]} = \emptyset$, then

$$\beta_{\mathcal{H}}(M_b) - \beta_{\mathcal{H}}(M_a) \leq \# f(K_{(a,b]}) \cdot \# \mathcal{J}(K_{(a,b]}).$$

Here $K^G_{(a,b]}$ denotes the G-fixed point set of $K_{(a,b]}$.

Proof. By the assumption, $m = \# f(K_{(a,b]}) < \infty$ and $n = \# \mathcal{J}(K_{(a,b]}) < \infty$. Let $c_1, \cdots, c_m$ be the critical values in $(a, b]$. We obtain from Corollary 2.4,

$$\gamma(M_b) - \gamma(M_a) \leq \gamma(K_{c_1}) + \cdots + \gamma(K_{c_m}),$$

and

$$\beta_{\mathcal{H}}(M_b) - \beta_{\mathcal{H}}(M_a) \leq \beta_{\mathcal{H}}(K_{c_1}) + \cdots + \beta_{\mathcal{H}}(K_{c_m}).$$

We see for each $1 \leq i \leq m$, $\gamma(K_{c_i}) \leq n$, and $\beta_{\mathcal{H}}(K_{c_i}) \leq n$ if $K^G_{c_i} = \emptyset$, and obtain

$$\gamma(M_b) - \gamma(M_a) \leq mn,$$

and

$$\beta_{\mathcal{H}}(M_b) - \beta_{\mathcal{H}}(M_a) \leq mn \qquad \text{if} \quad K^G_{(a,b]} = \emptyset. \qquad \square$$

We conclude with some remarks. Our genera $\beta_{\mathcal{H}}(X)$ and $\gamma(X)$ are respecti-

vely the same as $\tilde{\gamma}_G(X)$ and $\gamma_G(X)$ in his notation of Bartsch [1]. There he gives lower bounds for these genera of some G-spaces where G is a cyclic group Z_n. Clapp-Puppe [2] also gives lower bounds for some G-spaces and the exact values for representation spheres of G where G is a p-torus $Z_p \times \cdots \times Z_p$ with p a prime, or a real torus $S^1 \times \cdots \times S^1$.

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