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ON POLARIZED MANIFOLDS OF SECTIONAL GENUS THREE

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§ 1. Introduction

Let L be an ample line bundle on a complex projective manifold M of dimension $n\geqq 2$. The sectional genus $g=g(M, L)$ of a polarized manifold (M, L) is defined by the formula $2g(M, L)-2=(K+(n-1)L)L^{n-1}$, where K is the canonical bundle of M. For polarized manifolds over $\boldsymbol{C}$, it is known that g takes non-negative integers ([F1; Corollary 1] or [I2; Lemma 7]).

In many papers the structure of (M, L) with low g has been studied: see [F1] or [I2] for $g\leqq 1$; [BeLP] for $g=n=2$; [F2] for $g=2$; [Ma] for $g=3$ and $n=2$. As for the case $g=3$ and $n\geqq 3$, we see from the results of [F1] or [I2] that (M, L) is one of the following types.

(1.1) There is an effective divisor E on M such that $(E, L_E)\cong(\boldsymbol{P}^{n-1}, \mathcal{O}(1))$ and $[E]_E=\mathcal{O}(-1)$.

(1.2) There is a fibration $\Phi: M\to C$ over a smooth curve C such that every fiber F of Φ is a hyperquadric in $\boldsymbol{P}^n$ and $L_F=\mathcal{O}(1)$.

(1.3) There is a fibration $\Phi: M\to C$ over a smooth curve C such that $(F, L_F)\cong(\boldsymbol{P}^2, \mathcal{O}(2))$ for every fiber F of Φ.

(1.4) (M, L) is a scroll over a smooth surface S.

(1.5) $K+(n-2)L$ is nef.

(1.6) (M, L) is a scroll over a smooth curve of genus three.

In the case (1.6), we have nothing more to say.

In the case (1.1), using the theory of minimal reduction (e.g. [I2; (0.11)], [F2; (1.9)], or [F; (11.11)]), we see (M, L) is obtained by a finite number of simple blow-ups of a polarized manifold (M', L') which is of type (1.3) or (1.5).

The cases (1.2) and (1.3) are further studied in § 2 and § 3, which is the main part of this paper. We shall see our classification results are similar to those in case $g=2$, but the computations are more complicated.

In the case (1.4), $(M, L)\cong(\boldsymbol{P}_S(\mathcal{E}), H(\mathcal{E}))$ and $g(S, \det\mathcal{E})=3$ for some vector bundle $\mathcal{E}$ on S, thus the classification of (M, L) is reduced to the classification of ample vector bundles $\mathcal{E}$ for each polarized surface with $g=3$. Under the

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additional condition that L is spanned, the classification was obtained in [BiLL]. Without this condition, however, we have only some partial results and our classification is not yet complete. The author hopes this case will be treated in a future paper.

The case (1.5) is a kind of "general type". For any fixed n, there are only finitely many deformation types of (M, L). (See [F; (13.1)].) But it seems to be difficult to enumerate all such deformation types.

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Notation and Terminology

Basically we use the customary notation in algebraic geometry as in [H2]. All varieties are defined over $\boldsymbol{C}$ and assumed to be complete. Vector bundles are often identified with locally free sheaves of their sections and these words are used interchangeably. Line bundles are identified with linear equivalence classes of Cartier divisors, and their tensor products are denoted additively, while we use multiplicative notation for intersection products in Chow rings. The linear equivalence class is denoted by [ ], and its corresponding invertible sheaf is denoted by $\mathcal{O}[\ ]$. We use { } for the homology class of an algebraic cycle.

Given a morphism $f: X \to Y$ and a line bundle A on Y, we denote f^*A by A_X, or sometimes by A for short when there is no danger of confusion. The canonical bundle of a manifold M is denoted by K^M, unlike the customary notation K_M. The $\mathcal{O}(1)$'s of projective spaces $\boldsymbol{P}_\alpha$, $\boldsymbol{P}_\beta$, $\cdots$ will be denoted by H_α, H_β, $\cdots$. Given a vector bundle $\mathcal{E}$ on X, we denote by $\boldsymbol{P}_X(\mathcal{E})$ (or $\boldsymbol{P}(\mathcal{E})$) the associated projective space bundle, and denote by $H(\mathcal{E})$ the tautological line bundle on $\boldsymbol{P}(\mathcal{E})$ in the sense of [H2]. The pair $(\boldsymbol{P}(\mathcal{E}), H(\mathcal{E}))$ is called the scroll of $\mathcal{E}$.

§ 2. The case of a hyperquadric fibration over a curve

In this section, we study the case (1.2), following the idea in [F2; § 3].

(2.1) Since $h^0(F, L_F)=n+1$, $\mathcal{E}:=\Phi_*\mathcal{O}_M[L]$ is a locally free sheaf of rank $n+1$ on C and a natural map $\Phi^*\mathcal{E}\to L$ is surjective. This yields a C-morphism $\rho: M\to \boldsymbol{P}_C(\mathcal{E})$ and for every point x on C the restriction of ρ to $F_x:=\Phi^{-1}(x)$ is an embedding of F_x into $\boldsymbol{P}^n$. Hence ρ itself is an embedding and M is a member of $|2H(\mathcal{E})+B_{\boldsymbol{P}(\mathcal{E})}|$ for some line bundle B on C. We put $d=L^n$, $e=c_1(\mathcal{E})$, $b=\deg B$ and denote by $g(C)$ the genus of C. After simple computation, we get $d=2e+b$, $2g(C)+e+b=4$, and $s:=2e+(n+1)b\geqq 0$. Furthermore in the last inequality, equality holds if and only if every fiber of Φ is smooth by [F2; (3.3)]. From these results, we have $(n+1)d+s+4ng(C)=8n$, hence $g(C)$

$=0$ or 1.

(2.2) We first study the case $g(C)=1$. In this case, C is an elliptic curve and we have $e=d-2$ and $b=4-d$ from the equality above. Hence we obtain $d\leqq 6$, since $s\geqq 0$ and $n\geqq 3$.

(2.3) We consider the ampleness of $\mathcal{E}$. If $\mathcal{E}$ is ample, then $\det\mathcal{E}$ is ample and $e=c_1(\mathcal{E})>0$. It follows that $d>2$, hence $\mathcal{E}$ is not ample when $d\leqq 2$. On the other hand, $\mathcal{E}$ is ample when $d\geqq 5$ by the argument in [F2; (3.13)]. In general, for any indecomposable vector bundle $\mathcal{F}$ on an elliptic curve, $\mathcal{F}$ is ample if and only if $c_1(\mathcal{F})>0$ (for a proof, see e.g. [H1]). Thus when $d=3$ or 4, $\mathcal{E}$ is ample if it is indecomposable.

(2.4) When $d=3$ or 4, we can find an example of (M, L) similarly as in [F2; (3.12)]. We can also find an example of (M, L) with $d=6$ as follows. Let C be a smooth elliptic curve and take a line bundle $\mathcal{L}$ on C with $\deg\mathcal{L}=1$. We put $\mathcal{E}=\mathcal{L}^{\oplus 4}$, then $\mathcal{E}$ is ample, $c_1(\mathcal{E})=4$, $\boldsymbol{P}_C(\mathcal{E})\cong C\times\boldsymbol{P}^3_\sigma$, and $H(\mathcal{E})=H_\sigma+\mathcal{L}_{\boldsymbol{P}(\mathcal{E})}$, where H_σ is the pullback of $\mathcal{O}(1)$ on $\boldsymbol{P}^3_\sigma$. Putting $B=-2\mathcal{L}$, we have $\deg B=-2$ and $2H(\mathcal{E})+B_{\boldsymbol{P}(\mathcal{E})}=2H_\sigma$. Then a general member M of $|2H(\mathcal{E})+B_{\boldsymbol{P}(\mathcal{E})}|$ is smooth and, putting $L=[H(\mathcal{E})]_M$, we obtain an expected example of (M, L) with $d=6$.

(2.5) From now on, we study the case $g(C)=0$. In this case, $C\cong\boldsymbol{P}^1_\xi$ and we have $e=d-4$ and $b=8-d$ from the equality in (2.1). Hence we obtain $d\leqq 12$, since $s\geqq 0$ and $n\geqq 3$. Furthermore when $d=11$ or 12, we have $n=3$; when $d=12$, we have $s=0$ and Φ is a $\boldsymbol{P}^1\times\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1_\xi$.

(2.6) We put $P=\boldsymbol{P}_C(\mathcal{E})$, $H=H(\mathcal{E})$, and denote by H_ξ the pullback of $\mathcal{O}(1)$ on $\boldsymbol{P}^1_\xi$. Since $C\cong\boldsymbol{P}^1_\xi$, we can describe $\mathcal{E}\cong\mathcal{O}(e_0)\oplus\cdots\oplus\mathcal{O}(e_n)$, where $e_0, \cdots, e_n\in\boldsymbol{Z}$, $e_0\leqq\cdots\leqq e_n$, and $\sum_{i=0}^n e_i=e$. We denote $\mathcal{O}(e_0)\oplus\cdots\oplus\mathcal{O}(e_n)$ by $\mathcal{O}(e_0, \cdots, e_n)$ for simplicity. We shall classify $\mathcal{E}\cong\mathcal{O}(e_0, \cdots, e_n)$ for each $d=1, 2, \cdots, 12$.

(2.7)
LEMMA. *$2(e_{n-1}+e_n)<d$ when $e_0\leqq 0$.*

Proof. (cf. [F2; (3.24)]). A natural surjection $\mathcal{E}\to\mathcal{O}(e_0, \cdots, e_{n-1})$ gives a prime divisor $D_1:=\boldsymbol{P}(\mathcal{O}(e_0, \cdots, e_{n-1}))$ on P. Similarly $\mathcal{E}\to\mathcal{O}(e_0, \cdots, e_{n-2}, e_n)$ gives a prime divisor $D_2:=\boldsymbol{P}(\mathcal{O}(e_0, \cdots, e_{n-2}, e_n))$ on P and $\mathcal{E}\to\mathcal{O}(e_0, \cdots, e_{n-2})$ gives a subvariety $W:=\boldsymbol{P}(\mathcal{O}(e_0, \cdots, e_{n-2}))$ of P. We have $D_1\in|H-e_nH_\xi|$, $D_2\in|H-e_{n-1}H_\xi|$, and $W=D_1\cap D_2$ as schemes. When $e_0\leqq 0$, we have $W\not\subset M$ since H_W is not ample. Hence $\dim(M\cap W)=n-2$ and $0<L^{n-2}\{M\cap W\}=H^{n-2}(2H+bH_\xi)(H-e_nH_\xi)(H-e_{n-1}H_\xi)=d-2(e_{n-1}+e_n)$. □

(2.8) Suppose that $d=1$. We have $e=-3$, $b=7$, and $M\in|2H+7H_\xi|$. By (2.7), $\mathcal{E}\cong\mathcal{O}(-3, 0, \cdots, 0)$, $\mathcal{O}(-2, -1, 0, \cdots, 0)$, or $\mathcal{O}(-1, -1, -1, 0, \cdots, 0)$.

(2.8.1) When $\mathcal{E}\cong\mathcal{O}(-1, -1, -1, 0, \cdots, 0)$, we have $n\leqq 4$ by the argument in [F2; (3.21)]. Indeed, we have

$$P\cong\left\{\begin{array}{c}(\xi_0:\xi_1)\times(\sigma_0:\sigma_1:\sigma_2:\sigma_{30}:\sigma_{31}:\cdots:\sigma_{n0}:\sigma_{n1})\in\boldsymbol{P}^1_\xi\times\boldsymbol{P}^{2n-2}_\sigma\\ |\xi_0:\xi_1=\sigma_{30}:\sigma_{31}=\cdots=\sigma_{n0}:\sigma_{n1}\end{array}\right\},$$

$H=H_\sigma-H_\xi$, and $M\in|2H_\sigma+5H_\xi|$. Thus we can describe

$$M=\{q_0(\sigma)\xi_0^5+q_1(\sigma)\xi_0^4\xi_1+\cdots+q_5(\sigma)\xi_1^5=0\text{ in }P\},$$

where $q_0,\cdots,q_5$ are homogeneous polynomials of degree two in $\sigma_0,\sigma_1,\cdots,\sigma_{n1}$. In this defining equation of M, we put

$$\sigma_0=a_{00}\xi_0+a_{01}\xi_1,\ \sigma_1=a_{10}\xi_0+a_{11}\xi_1,\ \sigma_2=a_{20}\xi_0+a_{21}\xi_1,$$

$$\sigma_{30}=a_3\xi_0,\ \sigma_{31}=a_3\xi_1,\ \cdots,\ \sigma_{n0}=a_n\xi_0,\ \sigma_{n1}=a_n\xi_1,$$

where $a_{00},a_{01},\cdots,a_n$ are constants. Then we obtain an equation

$$Q_0(a)\xi_0^7+Q_1(a)\xi_0^6\xi_1+\cdots+Q_7(a)\xi_1^7=0,$$

where $Q_0,\cdots,Q_7$ are homogeneous polynomials of degree two in $(a):=(a_{00}, a_{01},\cdots,a_n)$. If $n\geqq5$, then $Q_0(a)=\cdots=Q_7(a)=0$ has a non-trivial solution. We fix such a solution (a) and define a rational map $\alpha:\boldsymbol{P}^1_\xi\to\boldsymbol{P}^{2n-2}_\sigma$ by

$$\alpha(\xi_0:\xi_1):=(a_{00}\xi_0+a_{01}\xi_1:a_{10}\xi_0+a_{11}\xi_1:a_{20}\xi_0+a_{21}\xi_1$$
$$:a_3\xi_0:a_3\xi_1:\cdots:a_n\xi_0:a_n\xi_1).$$

If α is not a morphism, then $a_{00}:a_{10}:a_{20}=a_{01}:a_{11}:a_{21}$ and $a_3=\cdots=a_n=0$. Since (a) is non-trivial, the equations

$$\sigma_0:\sigma_1:\sigma_2=a_{00}:a_{10}:a_{20}=a_{01}:a_{11}:a_{21},\ \sigma_{30}=\sigma_{31}=\cdots=\sigma_{n0}=\sigma_{n1}=0$$

determine a point z on $\boldsymbol{P}^{2n-2}_\sigma$. Let Z be the fiber of $\boldsymbol{P}^1_\xi\times\boldsymbol{P}^{2n-2}_\sigma\to\boldsymbol{P}^{2n-2}_\sigma$ over z. Then we have $Z\subset M$ by the definition of Z, hence $0<LZ=HZ=(H_\sigma-H_\xi)Z=-1$. This is a contradiction, thus α is a morphism. Let Γ be the graph of α. Then $\Gamma\subset M$ by the definition of α, hence $0<L\Gamma=H\Gamma=(H_\sigma-H_\xi)\Gamma$. However, since $H_\sigma\Gamma=H_\xi\Gamma=1$, this is a contradiction too. Hence we have proved that $n\leqq4$, thus $\mathcal{E}\cong\mathcal{O}(-1,-1,-1,0)$ or $\mathcal{O}(-1,-1,-1,0,0)$. If $\mathcal{E}\cong\mathcal{O}(-1,-1,-1,0)$, then $P\cong\{(\xi_0:\xi_1)\times(\sigma_0:\sigma_1:\sigma_2:\sigma_{30}:\sigma_{31})\in\boldsymbol{P}^1_\xi\times\boldsymbol{P}^4_\sigma|\xi_0:\xi_1=\sigma_{30}:\sigma_{31}\}$. Thus the projection $\mu:P\to\boldsymbol{P}^4_\sigma$ is the blowing-up of $\boldsymbol{P}^4_\sigma$ with center $W:=\{\sigma_{30}=\sigma_{31}=0$ in $\boldsymbol{P}^4_\sigma\}\cong\boldsymbol{P}^2$. Since the exceptional divisor E of μ is a member of $|H_\sigma-H_\xi|$, we have $M\in|7H_\sigma-5E|$. Hence M is the strict transform of a hypersurface of degree seven in $\boldsymbol{P}^4_\sigma$, which has singularities with multiplicity five along W.

(2.8.2) When $\mathcal{E}\cong\mathcal{O}(-2,-1,0,\cdots,0)$, we claim that $n\leqq4$. The following argument is similar to (2.8.1). We have

$$P\cong\left\{\begin{array}{l}(\xi_0:\xi_1)\times(\sigma_0:\sigma_{10}:\sigma_{11}:\sigma_{20}:\sigma_{21}:\sigma_{22}:\cdots:\sigma_{n0}:\sigma_{n1}:\sigma_{n2})\in \boldsymbol{P}^1_\xi\times\boldsymbol{P}^{3n-1}_\sigma\\ \quad|\,\xi_0:\xi_1=\sigma_{10}:\sigma_{11}=\sigma_{20}:\sigma_{21}=\sigma_{21}:\sigma_{22}=\cdots=\sigma_{n0}:\sigma_{n1}=\sigma_{n1}:\sigma_{n2}\end{array}\right\},$$

$H=H_\sigma-2H_\xi$, and $M\in|2H_\sigma+3H_\xi|$. Thus $M=\{q_0(\sigma)\xi_0^3+q_1(\sigma)\xi_0^2\xi_1+q_2(\sigma)\xi_0\xi_1^2+q_3(\sigma)\xi_1^3=0 \text{ in } P\}$, where $q_0, \cdots, q_3$ are quadric polynomials in (σ). We put

$$\sigma_0=a_{00}\xi_0^2+a_{01}\xi_0\xi_1+a_{02}\xi_1^2,\ \sigma_{10}=\xi_0(a_{10}\xi_0+a_{11}\xi_1),\ \sigma_{11}=\xi_1(a_{10}\xi_0+a_{11}\xi_1),$$

$$\sigma_{20}=a_2\xi_0^2,\ \sigma_{21}=a_2\xi_0\xi_1,\ \sigma_{22}=a_2\xi_1^2,\ \cdots,\ \sigma_{n0}=a_n\xi_0^2,\ \sigma_{n1}=a_n\xi_0\xi_1,\ \sigma_{n2}=a_n\xi_1^2.$$

Then from the defining equation of M above, we obtain an equation

$$Q_0(a)\xi_0^7+Q_1(a)\xi_0^6\xi_1+\cdots+Q_7(a)\xi_1^7=0,$$

where $Q_0, \cdots, Q_7$ are quadric polynomials in $(a):=(a_{00}, a_{01}, \cdots, a_n)$. If $n\geqq 5$, then $Q_0(a)=\cdots=Q_7(a)=0$ has a non-trivial solution (a). We fix it and define a rational map $\alpha: \boldsymbol{P}^1_\xi\to\boldsymbol{P}^{3n-1}_\sigma$ by

$$\alpha(\xi_0:\xi_1):=(a_{00}\xi_0^2+a_{01}\xi_0\xi_1+a_{02}\xi_1^2:\xi_0(a_{10}\xi_0+a_{11}\xi_1):\xi_1(a_{10}\xi_0+a_{11}\xi_1):$$
$$a_2\xi_0^2:a_2\xi_0\xi_1:a_2\xi_1^2:\cdots:a_n\xi_0^2:a_n\xi_0\xi_1:a_n\xi_1^2).$$

If α is not a morphism, then $a_2=\cdots=a_n=0$ and for some $(c_0:c_1)\in\boldsymbol{P}^1_\xi$, we have $a_{10}c_0+a_{11}c_1=0$ and $a_{00}c_0^2+a_{01}c_0c_1+a_{02}c_1^2=0$. In the case $a_{10}=a_{11}=0$, let Z be the fiber of $\boldsymbol{P}^1_\xi\times\boldsymbol{P}^{3n-1}_\sigma\to\boldsymbol{P}^{3n-1}_\sigma$ over $z:=(1:0:\cdots:0)$. Then we have $Z\subset M$, hence $0<LZ=HZ=(H_\sigma-2H_\xi)Z=-2$. This is a contradiction, thus $a_{10}\neq 0$ or $a_{11}\neq 0$.

In this case, $a_{00}\xi_0^2+a_{01}\xi_0\xi_1+a_{02}\xi_1^2$ is devided by $a_{10}\xi_0+a_{11}\xi_1$ in $\boldsymbol{C}[\xi_0, \xi_1]$; we denote by $b_0\xi_0+b_1\xi_1$ its quotient. We put

$$Z=\{\sigma_0=b_0\sigma_{10}+b_1\sigma_{11},\ \sigma_{20}=\cdots=\sigma_{n2}=0 \text{ in } P\}.$$

Then $\dim Z=1$ and $Z\subset M$ by the definition of Z, hence $0<LZ=HZ=(H_\sigma-2H_\xi)Z$. However, since $H_\sigma Z=1$ and $H_\xi Z=1$, this is a contradiction too. Thus α is a morphism.

Let Γ be the graph of α. We have $\Gamma\subset M$ and then $0<L\Gamma=H\Gamma=(H_\sigma-2H_\xi)\Gamma$. However, since $H_\sigma\Gamma=2$ and $H_\xi\Gamma=1$, this is also a contradiction. Hence we have proved that $n\leqq 4$, thus $\mathcal{E}\cong\mathcal{O}(-2, -1, 0, 0)$ or $\mathcal{O}(-2, -1, 0, 0, 0)$.

(2.8.3) When $\mathcal{E}\cong\mathcal{O}(-3, 0, \cdots, 0)$, we claim that $n\leqq 4$ as before. P is isomorphic to

$$\left\{\begin{array}{l}(\xi_0:\xi_1)\times(\sigma_0:\sigma_{10}:\sigma_{11}:\sigma_{12}:\sigma_{13}:\cdots:\sigma_{n0}:\sigma_{n1}:\sigma_{n2}:\sigma_{n3})\in\boldsymbol{P}^1_\xi\times\boldsymbol{P}^{4n}_\sigma\,|\\ \xi_0:\xi_1=\sigma_{10}:\sigma_{11}=\sigma_{11}:\sigma_{12}=\sigma_{12}:\sigma_{13}=\cdots=\sigma_{n0}:\sigma_{n1}=\sigma_{n1}:\sigma_{n2}=\sigma_{n2}:\sigma_{n3}\end{array}\right\},$$

$H=H_\sigma-3H_\xi$, and $M\in|2H_\sigma+H_\xi|$. Thus $M=\{q_0(\sigma)\xi_0+q_1(\sigma)\xi_1=0 \text{ in } P\}$, where q_0 and q_1 are quadric polynomials in (σ). We put

$$\sigma_0 = a_{00}\xi_0^3 + a_{01}\xi_0^2\xi_1 + a_{02}\xi_0\xi_1^2 + a_{03}\xi_1^3,$$

$$\sigma_{10} = a_1\xi_0^3,\ \sigma_{11} = a_1\xi_0^2\xi_1,\ \sigma_{12} = a_1\xi_0\xi_1^2,\ \sigma_{13} = a_1\xi_1^3,\ \cdots,$$

$$\sigma_{n0} = a_n\xi_0^3,\ \sigma_{n1} = a_n\xi_0^2\xi_1,\ \sigma_{n2} = a_n\xi_0\xi_1^2,\ \sigma_{n3} = a_n\xi_1^3.$$

Then from the defining equation of M above, we obtain an equation

$$Q_0(a)\xi_0^7 + Q_1(a)\xi_0^6\xi_1 + \cdots + Q_7(a)\xi_1^7 = 0,$$

where $Q_0, \cdots, Q_7$ are quadric polynomials in $(a) := (a_{00}, a_{01}, \cdots, a_n)$. If $n \geqq 5$, then $Q_0(a) = \cdots = Q_7(a) = 0$ has a non-trivial solution (a). We fix it and define a rational map $\alpha: \boldsymbol{P}_\xi^1 \to \boldsymbol{P}_\sigma^{4n}$ by

$$\alpha(\xi_0 : \xi_1) := (a_{00}\xi_0^3 + a_{01}\xi_0^2\xi_1 + a_{02}\xi_0\xi_1^2 + a_{03}\xi_1^3 : a_1\xi_0^3 : a_1\xi_0^2\xi_1 : a_1\xi_0\xi_1^2 : a_1\xi_1^3 :$$
$$\cdots : a_n\xi_0^3 : a_n\xi_0^2\xi_1 : a_n\xi_0\xi_1^2 : a_n\xi_1^3).$$

If α is not a morphism, then $a_1 = \cdots = a_n = 0$. Let Z be the fiber of $\boldsymbol{P}_\xi^1 \times \boldsymbol{P}_\sigma^{4n} \to \boldsymbol{P}_\sigma^{4n}$ over $z := (1 : 0 : \cdots : 0)$. We have $Z \subset M$ and then $0 < LZ = HZ = (H_\sigma - 3H_\xi)Z = -3$. This is a contradiction, hence α is a morphism. Let Γ be the graph of α. We have $\Gamma \subset M$ and then $0 < L\Gamma = H\Gamma = (H_\sigma - 3H_\xi)\Gamma$. However, since $H_\sigma\Gamma = 3$ and $H_\xi\Gamma = 1$, this is a contradiction too. Hence we have proved that $n \leqq 4$, thus $\mathcal{E} \cong \mathcal{O}(-3, 0, 0, 0)$ or $\mathcal{O}(-3, 0, 0, 0, 0)$.

(2.9) Now we study the case $d=2$. We have $e=-2$, $b=6$, and $M \in |2H + 6H_\xi|$. By (2.7), $\mathcal{E} \cong \mathcal{O}(-2, 0, \cdots, 0)$ or $\mathcal{O}(-1, -1, 0, \cdots, 0)$.

(2.9.1) When $\mathcal{E} \cong \mathcal{O}(-1, -1, 0, \cdots, 0)$, we have $n \leqq 4$ as in (2.8.1). Hence $\mathcal{E} \cong \mathcal{O}(-1, -1, 0, 0)$ or $\mathcal{O}(-1, -1, 0, 0, 0)$.

(2.9.2) When $\mathcal{E} \cong \mathcal{O}(-2, 0, \cdots, 0)$, we have $n \leqq 4$ as in (2.8.2). Hence $\mathcal{E} \cong \mathcal{O}(-2, 0, 0, 0)$ or $\mathcal{O}(-2, 0, 0, 0, 0)$.

(2.10) Suppose that $d=3$. Then $e=-1$, $b=5$, and $M \in |2H + 5H_\xi|$. From (2.7), we have $\mathcal{E} \cong \mathcal{O}(-2, 0, \cdots, 0, 1)$, $\mathcal{E} \cong \mathcal{O}(-1, -1, 0, \cdots, 0, 1)$, or $\mathcal{E} \cong \mathcal{O}(-1, 0, \cdots, 0)$.

(2.10.1) When $\mathcal{E} \cong \mathcal{O}(-1, 0, \cdots, 0)$, we have $n \leqq 4$ as in (2.8.1). Hence $\mathcal{E} \cong \mathcal{O}(-1, 0, 0, 0)$ or $\mathcal{O}(-1, 0, 0, 0, 0)$.

(2.10.2) When $\mathcal{E} \cong \mathcal{O}(-1, -1, 0, \cdots, 0, 1)$, we have $n \leqq 4$ by the argument in [F2; (3.23.2)] which is similar to (2.8.1). Hence $\mathcal{E} \cong \mathcal{O}(-1, -1, 0, 1)$ or $\mathcal{O}(-1, -1, 0, 0, 1)$.

(2.10.3) When $\mathcal{E} \cong \mathcal{O}(-2, 0, \cdots, 0, 1)$, we have $n \leqq 4$ as in (2.8.2) and (2.10.2). Hence $\mathcal{E} \cong \mathcal{O}(-2, 0, 0, 1)$ or $\mathcal{O}(-2, 0, 0, 0, 1)$.

The next lemma is useful for $d \geqq 4$.

(2.11)
LEMMA. *When* $d\geqq 4$, -1 *does not appear twice in* $\{e_0, \cdots, e_n\}$.

We can prove this lemma by the argument in [F2; (3.18)].

(2.12) Now we study the case $d=4$. We have $e=0$, $b=4$, and $M\in|2H+4H_\xi|$. By (2.7) and (2.11), $\mathcal{E}\cong\mathcal{O}(-1, 0, \cdots, 0, 1)$ or $\mathcal{O}(0, \cdots, 0)$.

(2.12.1) When $\mathcal{E}\cong\mathcal{O}(-1, 0, \cdots, 0, 1)$, we have $n\leqq 4$ as in (2.10.2). Hence $\mathcal{E}\cong\mathcal{O}(-1, 0, 0, 1)$ or $\mathcal{O}(-1, 0, 0, 0, 1)$.

(2.12.2) When $\mathcal{E}\cong\mathcal{O}(0, \cdots, 0)$, by the argument in [F2; (3.23.1)], we have $n\leqq 4$, $P\cong \boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^n$, $\mathrm{Bs}|L|=\phi$, and the morphism $\varphi: M\to\boldsymbol{P}_\sigma^n$ defined by $|L|$ is a finite morphism of degree four. Conversely, a general member M of $|2H_\sigma+4H_\xi|$ on P does not contain any fiber of the projection $P\to\boldsymbol{P}_\sigma^n$, thus $L:=H_M$ is ample and (M, L) is a polarized manifold of the above type.

The next lemma is useful for $d\geqq 5$.

(2.13)
LEMMA. $e_0\geqq -1$ *when* $d\geqq 5$.

We can prove this lemma by the argument in [F2; (3.19)].

Similarly we obtain the following two lemmas.

(2.14)
LEMMA. $e_0\geqq 0$ *when* $d\geqq 7$.

(2.15)
LEMMA. $e_0\geqq 1$ *when* $d\geqq 9$.

(2.16) Now we study the case $d=5$. We have $e=1$, $b=3$, and $M\in|2H+3H_\xi|$. By (2.11) and (2.13), $\mathcal{E}\cong\mathcal{O}(-1, 0, \cdots, 0, 2)$, $\mathcal{O}(-1, 0, \cdots, 0, 1, 1)$, or $\mathcal{O}(0, \cdots, 0, 1)$.

(2.16.1) When $\mathcal{E}\cong\mathcal{O}(-1, 0, \cdots, 0, 2)$, we have $n\leqq 3$ similarly as in (2.10.2), hence $\mathcal{E}\cong\mathcal{O}(-1, 0, 0, 2)$. Furthermore $\mathrm{Bs}|L|$ is one point as in [F2; (3.23.2)].

(2.16.2) When $\mathcal{E}\cong\mathcal{O}(-1, 0, \cdots, 0, 1, 1)$, we have $n\leqq 4$ and $\mathrm{Bs}|L|$ is one point as in (2.16.1). Thus $\mathcal{E}\cong\mathcal{O}(-1, 0, 1, 1)$ or $\mathcal{O}(-1, 0, 0, 1, 1)$.

(2.16.3) When $\mathcal{E}\cong\mathcal{O}(0, \cdots, 0, 1)$, by the argument in [F2; (3.24)], we have $n\leqq 4$ and $|L|$ makes M the normalization of a hypersuface of degree five in $\boldsymbol{P}^{n+1}$, which has triple points along a $\boldsymbol{P}^2$ in $\boldsymbol{P}^{n+1}$.

(2.17) Suppose that $d=6$. We have $e=2$, $b=2$, and $M\in|2H+2H_\xi|$. From (2.7), (2.11), and (2.13), we have $\mathcal{E}\cong\mathcal{O}(-1, 0, \cdots, 0, 1, 1, 1)$, $\mathcal{O}(0, \cdots, 0, 1, 1)$, $\mathcal{O}(0, \cdots, 0, 2)$.

(2.17.1) When $\mathcal{O}\cong\mathcal{E}(-1, 0, \cdots, 0, 1, 1, 1)$, we show that $n=3$ similarly as in (2.7). Natural surjections $\mathcal{E}\to\mathcal{O}(e_0, \cdots, e_{n-1})$, $\mathcal{E}\to\mathcal{O}(e_0, \cdots, e_{n-2}, e_n)$, and $\mathcal{E}\to\mathcal{O}(e_0, \cdots, e_{n-3}, e_{n-1}, e_n)$ give prime divisors $D_1:=\boldsymbol{P}(\mathcal{O}(e_0, \cdots, e_{n-1}))$, $D_2:=\boldsymbol{P}(\mathcal{O}(e_0, \cdots, e_{n-2}, e_n))$, and $D_3:=\boldsymbol{P}(\mathcal{O}(e_0, \cdots, e_{n-3}, e_{n-1}, e_n))$ respectively. A natural surjection $\mathcal{E}\to\mathcal{O}(e_0, \cdots, e_{n-3})$ gives a subvariety $W:=\boldsymbol{P}(\mathcal{O}(e_0, \cdots, e_{n-3}))$ of $P=\boldsymbol{P}(\mathcal{E})$. We have $D_1\in|H-e_nH_\xi|$, $D_2\in|H-e_{n-1}H_\xi|$, $D_3\in|H-e_{n-2}H_\xi|$, and $W=D_1\cap D_2\cap D_3$ as schemes. Since H_W is not ample, we have $W\not\subset M$, hence $\dim(M\cap W)=n-3$ and $0<L^{n-3}\{M\cap W\}=H^{n-3}(2H+2H_\xi)(H-H_\xi)^3=2e-4=0$ if $n\geqq4$. This is a contradiction, thus we have $n=3$ and $\mathcal{E}\cong\mathcal{O}(-1, 1, 1, 1)$. By the argument in [F2; (3.26)], M is a double covering of $\boldsymbol{P}^1_\xi\times\boldsymbol{P}^2_\sigma$ and its branch locus is a smooth member of $|4H_\xi+2H_\sigma|$. We also have $L=[H_\xi+H_\sigma]_M$.

(2.17.2) When $\mathcal{E}\cong\mathcal{O}(0, \cdots, 0, 1, 1)$, we have $n\leqq4$ as in (2.16.3), hence $\mathcal{E}\cong\mathcal{O}(0, 0, 1, 1)$ or $\mathcal{O}(0, 0, 0, 1, 1)$. We show the existence of (M, L). When $\mathcal{E}\cong\mathcal{O}(0, 0, 1, 1)$, we have $P\cong\{(\xi_0:\xi_1)\times(\sigma_0:\sigma_1:\sigma_{20}:\sigma_{21}:\sigma_{30}:\sigma_{31})\in\boldsymbol{P}^1_\xi\times\boldsymbol{P}^5_\sigma|\xi_0:\xi_1=\sigma_{20}:\sigma_{21}=\sigma_{30}:\sigma_{31}\}$ and $H=H_\sigma$. Let M be a general member of $|2H_\sigma+2H_\xi|$ and put $L=[H_\sigma]_M$. Then $\mathrm{Bs}|L|=\phi$ and the restriction of $P\to\boldsymbol{P}^5_\sigma$ to M is the morphism φ defined by $|L|$. If $\varphi: M\to\varphi(M)$ is not finite, M contains a fiber Z of $P\to\boldsymbol{P}^5_\sigma$ over one point z on the line $l:=\{\sigma_{20}=\sigma_{21}=\sigma_{30}=\sigma_{31}=0$ in $\boldsymbol{P}^5_\sigma\}$. Using homogeneous polynomials q_0, q_1, and q_2 of degree two in (σ), we can describe that $M=\{q_0(\sigma)\xi_0^2+q_1(\sigma)\xi_0\xi_1+q_2(\sigma)\xi_1^2=0$ in $P\}$. Then $Z\subset M$ if and only if $q_0(z)=q_1(z)=q_2(z)=0$. Thus if we choose q_0, q_1, and q_2 generally to satisfy that $l\cap\{q_0(\sigma)=q_1(\sigma)=q_2(\sigma)=0$ in $\boldsymbol{P}^5_\sigma\}=\phi$, then φ becomes finite and L is ample. Similarly we can find an example of (M, L) when $\mathcal{E}\cong\mathcal{O}(0, 0, 0, 1, 1)$.

(2.17.3) When $\mathcal{E}\cong\mathcal{O}(0, \cdots, 0, 2)$, we have $n\leqq3$ as in (2.16.3), hence $\mathcal{E}\cong\mathcal{O}(0, 0, 0, 2)$. We can show the existence of (M, L) similarly as above.

When $d\geqq7$, the situation is much simpler.

(2.18)

LEMMA. *$Bs|L|=\phi$ and L is very ample when $d\geqq7$.*

We can prove this lemma similarly as in [F2; (3.31)]. This lemma tells us that our results overlap [I1; Theorem 4.3], but our method is different from his.

(2.19) Now we study the case $d=7$. We have $e=3$, $b=1$, and $M\in|2H+H_\xi|$. Furthermore $e_0\geqq0$ by (2.14), and $e_2\geqq1$ by the argument in [F2: (3.25)]. Hence $\mathcal{E}\cong\mathcal{O}(0, 0, 1, 2)$, $\mathcal{O}(0, 1, 1, 1)$, or $\mathcal{O}(0, 0, 1, 1, 1)$. In each case, (M, L) exists similarly as in (2.17.2). By the morphism defined by $|L|$, M is isomorphic to a manifold of degree seven in $\boldsymbol{P}^{n+3}$.

(2.20) Suppose that $d=8$. We have $e=4$, $b=0$, and $M\in|2H|$. Furthermore $e_0\geqq0$ by (2.14), and $e_1\geqq1$ by the argument in [F2; (3.26)]. Hence $\mathcal{E}\cong\mathcal{O}(0, 1, 1, 2)$, $\mathcal{O}(0, 1, 1, 1, 1)$, or $\mathcal{O}(1, 1, 1, 1)$.

(2.20.1) When $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 1)$, we have $P\cong \boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^3$, $H=H_\xi+H_\sigma$, and $M\in|2H_\sigma+2H_\xi|$. Hence M is a smooth divisor of bidegree (2, 2) on P. Conversely, let M be a general member of $|2H_\xi+2H_\sigma|$ and put $L=[H_\xi+H_\sigma]_M$. Since $\mathcal{E}$ is ample, L is ample and (M, L) is a polarized manifold of the above type.

(2.20.2) When $\mathcal{E}\cong\mathcal{O}(0, 1, 1, 1, 1)$, by the argument in [F2; (3.26)], M is a double covering of $\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^3$ and its branch locus is a smooth member of $|2H_\xi+2H_\sigma|$. We have also $L=[H_\xi+H_\sigma]_M$.

(2.20.3) Even when $\mathcal{E}\cong\mathcal{O}(0, 1, 1, 2)$, by the argument in [F2; (3.26)], we have a morphism $h: M\to\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^3$ and $L=h^*(H_\xi+H_\sigma)$. Since L is ample, $h: M\to h(M)$ is finite and $h(M)\in|a_1H_\xi+a_2H_\sigma|$ for some non-negative integers a_1 and a_2. Then $8=L^3=(\deg h)\cdot[H_\xi+H_\sigma]_{h(M)}^3=(\deg h)(a_1+3a_2)$. From the construction of h, we get $\deg h=2$ and $a_1=a_2=1$. Hence $h(M)\in|H_\xi+H_\sigma|$ and $M\to h(M)$ is a double covering.

(2.21) Suppose that $d=9$. We have $e=5$, $b=-1$, and $M\in|2H-H_\xi|$. Since $e_0\geqq 1$ by (2.15), $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 2)$ or $\mathcal{O}(1, 1, 1, 1, 1)$.

(2.21.1) When $\mathcal{E}\cong\mathcal{O}(1,1,1,1,1)$, similarly as in [F2; (3.27)], the restriction of the projection $P\cong\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^4\to\boldsymbol{P}_\sigma^4$ to M is a blowing-up of $\boldsymbol{P}_\sigma^4$ and its center is a complete intersection of two hyperquadrics in $\boldsymbol{P}_\sigma^4$.

(2.21.2) When $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 2)$, we have $P\cong\{(\xi_0:\xi_1)\times(\sigma_0:\sigma_1:\sigma_2:\sigma_{30}:\sigma_{31})\in\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^4|\xi_0:\xi_1=\sigma_{30}:\sigma_{31}\}$, hence P is the blowing-up of $\boldsymbol{P}_\sigma^4$ with center $\{\sigma_{30}=\sigma_{31}=0$ in $\boldsymbol{P}_\sigma^4\}$. The exceptional divisor E is $\{\sigma_{30}=\sigma_{31}=0$ in $P\}\in|H_\sigma-H_\xi|$, thus $M\in|3H_\sigma-E|$ and M is the strict transform of a smooth hypercubic in $\boldsymbol{P}_\sigma^4$.

(2.22) Suppose that $d=10$. We have $e=6$, $b=-2$, and $M\in|2H-H_\xi|$. Since $e_0\geqq 1$ by (2.15), $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 3)$, $\mathcal{O}(1, 1, 2, 2)$, $\mathcal{O}(1, 1, 1, 1, 2)$, or $\mathcal{O}(1, 1, 1, 1, 1, 1)$.

(2.22.1) When $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 1, 1, 1)$, we have $P\cong\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^5$, $H=H_\xi+H_\sigma$, $M\in|2H_\sigma|$, and $L=[H_\xi+H_\sigma]_M$. Hence $M\cong\boldsymbol{P}_\xi^1\times Q$, where Q is a smooth hyperquadric in $\boldsymbol{P}_\sigma^5$.

(2.22.2) When $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 1, 2)$, by the argument in [F2; (3.28)], we have M is the blowing-up of a hyperquadric in $\boldsymbol{P}_\sigma^5$ and its center is a smooth quadric surface.

(2.22.3) When $\mathcal{E}\cong\mathcal{O}(1, 1, 2, 2)$, we have $P\cong\{(\xi_0:\xi_1)\times(\sigma_0:\sigma_1:\sigma_{20}:\sigma_{21}:\sigma_{30}:\sigma_{31})\in\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^5|\xi_0:\xi_1=\sigma_{20}:\sigma_{21}=\sigma_{30}:\sigma_{31}\}$, $H=H_\xi+H_\sigma$, $M\in|2H_\sigma|$, and $L=[H_\xi+H_\sigma]_M$. Since $\mathcal{E}$ is ample, H is ample and then L is ample for any general member M of $|2H_\sigma|$. Because of (2.18), M is embedded in $\boldsymbol{P}^9$ as a manifold of degree nine by the morphism defined by $|L|$. On the other hand, the restriction of the projection $\mu: P\to\boldsymbol{P}_\sigma^5$ to M is the morphism defined by $|L-$

$H_\xi|$, and M is birationally mapped onto $\mu(M)$. We have $10=L^3=3[H_\xi]_M[H_\sigma]_M^2+[H_\sigma]_M^3$ and $[H_\xi]_M[H_\sigma]_M^2=2$ since $M\to\boldsymbol{P}_\xi^1$ is a hyperquadric fibration. Thus the degree of $\mu(M)$ is four. Furthermore, since $\mu(P)=\{\sigma_{20}\sigma_{31}-\sigma_{30}\sigma_{21}=0 \text{ in } \boldsymbol{P}_\sigma^5\}$ and $M\in|2H_\sigma|$, $\mu(M)$ is a complete intersection of two hyperquadrics in $\boldsymbol{P}_\sigma^5$. Even when $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 3)$, we have the same result as above.

(2.23) Suppose that $d=11$. We have $e=7$, $b=-3$, and $M\in|2H-3H_\xi|$. Since $e_0\geqq 1$ by (2.15), and since $n=3$ by (2.5), $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 4)$, $\mathcal{O}(1, 1, 2, 3)$, or $\mathcal{O}(1, 2, 2, 2)$.

(2.23.1) When $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 4)$, we claim that (M, L) does not exist. Assume that (M, L) exists. A natural surjection $\mathcal{E}\to\mathcal{O}(1, 1, 1)$ gives a prime divisor $W:=\boldsymbol{P}(\mathcal{O}(1, 1, 1))$ on P. We have $W\cong\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^2$, $H_W=H_\xi+H_\sigma$, and $W\not\subset M$, hence $[M]_W=M\cap W\in|2H_W-3H_\xi|=|2H_\sigma-H_\xi|$. This is a contradiction, thus we have proved the claim.

(2.23.2) Even when $\mathcal{E}\cong\mathcal{O}(1, 1, 2, 3)$, we can show that (M, L) does not exist. We have $P\cong\{(\xi_0:\xi_1)\times(\sigma_0:\sigma_1:\sigma_{20}:\sigma_{21}:\sigma_{30}:\sigma_{31}:\sigma_{32})\in\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^6|\xi_0:\xi_1=\sigma_{20}:\sigma_{21}=\sigma_{30}:\sigma_{31}=\sigma_{31}:\sigma_{32}\}$ and $H=H_\sigma+H_\xi$. Assume that there exists a smooth member M of $|2H_\sigma-H_\xi|$. Then there is an exact sequence of normal bundles

$$0\longrightarrow\mathcal{N}_{B/M}\longrightarrow\mathcal{N}_{B/P}\longrightarrow[\mathcal{N}_{M/P}]_B\longrightarrow 0,$$

where $B:=\mathrm{Bs}|2H_\sigma-H_\xi|=\{\sigma_{20}=\sigma_{21}=\sigma_{30}=\sigma_{31}=\sigma_{32}=0 \text{ in } P\}\cong\boldsymbol{P}(\mathcal{O}(1, 1))$. Since B is the complete intersection of $D_1:=\{\sigma_{20}=\sigma_{21}=0 \text{ in } P\}\cong\boldsymbol{P}(\mathcal{O}(1, 1, 3))$ and $D_2:=\{\sigma_{30}=\sigma_{31}=\sigma_{32}=0 \text{ in } P\}\cong\boldsymbol{P}(\mathcal{O}(1, 1, 2))$, we have $\mathcal{N}_{B/P}\cong[\mathcal{N}_{D_1/P}]_B\oplus[\mathcal{N}_{D_2/P}]_B\cong[H_\sigma-H_\xi]_B\oplus[H_\sigma-2H_\xi]_B$. Also we have $\mathcal{N}_{M/P}\cong[2H_\sigma-H_\xi]_B$. Then the morphism $\varphi: [H_\sigma-H_\xi]_B\oplus[H_\sigma-2H_\xi]_B\to[2H_\sigma-H_\xi]_B$ corresponding to $\mathcal{N}_{B/P}\to[\mathcal{N}_{M/P}]_B$ is given by some $\varphi_1\in H^0(B, [H_\sigma]_B)$ and $\varphi_2\in H^0(B, [H_\sigma+H_\xi]_B)$. Since $[H_\sigma]_B[H_\sigma+H_\xi]_B=1$, φ_1 and φ_2 have a common zero point, at which φ is not surjective. This is a contradiction and (M, L) does not exist.

(2.23.3) When $\mathcal{E}\cong\mathcal{O}(1, 2, 2, 2)$, we can show the existence of (M, L). We have $P\cong\{(\xi_0:\xi_1)\times(\sigma_0:\sigma_{10}:\sigma_{11}:\sigma_{20}:\sigma_{21}:\sigma_{30}:\sigma_{31})\in\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^6|\xi_0:\xi_1=\sigma_{10}:\sigma_{11}=\sigma_{20}:\sigma_{21}=\sigma_{30}:\sigma_{31}\}$ and $H=H_\sigma+H_\xi$. Putting $U_i=\{\xi_i\neq 0 \text{ in } P\}$ and $V_j=\{\sigma_j\neq 0 \text{ in } P\}$, we take a rational section $s_1:=\{(U_i\cap V_j, \sigma_0^2/\xi_0\cdot\xi_i/\sigma_j^2)\}_{i,j}$ of $2H_\sigma-H_\xi$. Note that $h^0(P, 2H-3H_\xi)=h^0(\boldsymbol{P}_\xi^1, S^2(\mathcal{E})\otimes[-3H_\xi])=15$. Let $f_1, \cdots, f_{15}$ be rational functions on P such that

$$f_1=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{10}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{11}}{\xi_1},\qquad f_2=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{20}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{21}}{\xi_1},$$

$$f_3=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{30}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{31}}{\xi_1},\qquad f_4=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}^2}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{11}}{\xi_1},$$

$$f_5=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{11}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{11}^2}{\xi_1},\qquad f_6=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{20}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{21}}{\xi_1},$$

$$f_7=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{21}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{11}\sigma_{21}}{\xi_1},\qquad f_8=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{30}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{31}}{\xi_1},$$

$$f_9=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{31}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{11}\sigma_{31}}{\xi_1},\qquad f_{10}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{20}^2}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{21}}{\xi_1},$$

$$f_{11}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{21}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{21}^2}{\xi_1},\qquad f_{12}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{30}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{31}}{\xi_1},$$

$$f_{13}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{31}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{21}\sigma_{31}}{\xi_1},\qquad f_{14}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{30}^2}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{30}\sigma_{31}}{\xi_1},$$

$$f_{15}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{30}\sigma_{31}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{31}^2}{\xi_1}.$$

Then $\boldsymbol{C}\langle f_1, \cdots, f_{15}\rangle$, the vector space spanned by $f_1, \cdots, f_{15}$ over $\boldsymbol{C}$, is isomorphic to $H^0(P, 2H_\sigma-H_\xi)$ by mapping each f_i to $f_i\cdot s_1$. Thus we can describe

$$|2H_\sigma-H_\xi|=\{\mathrm{div}(f\cdot s_1)\,|\,f\in\boldsymbol{C}\langle f_1, \cdots, f_{15}\rangle-0\},$$

where $\mathrm{div}(f\cdot s_1)$ is an effective divisor defined by a regular section $f\cdot s_1$ of $2H_\sigma-H_\xi$. Since $\mathrm{Bs}|2H_\sigma-H_\xi|=\{\sigma_{10}=\sigma_{11}=\cdots=\sigma_{31}=0 \text{ in } P\}\cong\boldsymbol{P}_\xi^1\times\{(1:0:\cdots:0)\}$, if we take $f=\sum_{i=1}^{15}c_if_i\in\boldsymbol{C}\langle f_1, \cdots, f_{15}\rangle$ with $(c_1, c_2, c_3)\neq(0, 0, 0)$, $\mathrm{div}(f\cdot s_1)$ is nonsingular along $\mathrm{Bs}|2H_\sigma-H_\xi|$. Thus a general member M of $|2H_\sigma-H_\xi|$ is smooth by Bertini's theorem. For such M, $L:=H_M$ is ample since $\mathcal{E}$ is ample, hence (M, L) is a polarized manifold as desired. Furthermore, similarly as in (2.16.3), $|L-H_\xi|$ makes M a desingularization of a variety of degree five in $\boldsymbol{P}_\sigma^6$.

(2.24) Suppose that $d=12$. We have $e=8$, $b=-4$, and $M\in|2H-4H_\xi|$. Since $e_0\geqq 1$ by (2.15), and since $n=3$ by (2.5), $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 5)$, $\mathcal{O}(1, 1, 2, 4)$, $\mathcal{O}(1, 1, 3, 3)$, $\mathcal{O}(1, 2, 2, 3)$, or $\mathcal{O}(2, 2, 2, 2)$.

(2.24.1) When $\mathcal{E}\cong\mathcal{O}(2, 2, 2, 2)$, we have $P\cong\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^3$, $H=H_\sigma+2H_\xi$, $M\in|2H_\sigma|$, and $L=[H_\sigma+H_\xi]_M$. Hence $M\cong\boldsymbol{P}_\xi^1\times Q$, where Q is a smooth quadric surface in $\boldsymbol{P}_\sigma^3$. Since $Q\cong\boldsymbol{P}_\mu^1\times\boldsymbol{P}_\lambda^1$, we have $M\cong\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\mu^1\times\boldsymbol{P}_\lambda^1$ and $L=2H_\xi+H_\mu+H_\lambda$.

(2.24.2) When $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 5)$, (M, L) does not exist by the argument in (2.23.1).

(2.24.3) Even when $\mathcal{E}\cong\mathcal{O}(1, 1, 2, 4)$, we can show that (M, L) does not exist similarly as in (2.23.2).

(2.24.4) When $\mathcal{E}\cong\mathcal{O}(1, 2, 2, 3)$, we can show the existence of (M, L) similarly as in (2.23.3). In fact, we have $P\cong\{(\xi_0:\xi_1)\times(\sigma_0:\sigma_{10}:\sigma_{11}:\sigma_{20}:\sigma_{21}:\sigma_{30}:\sigma_{31}:\sigma_{32})\in\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^7\,|\,\xi_0:\xi_1=\sigma_{10}:\sigma_{11}=\sigma_{20}:\sigma_{21}=\sigma_{30}:\sigma_{31}=\sigma_{31}:\sigma_{32}\}$, $H=H_\sigma+H_\xi$, and $h^0(P, 2H-4H_\xi)=h^0(\boldsymbol{P}_\xi^1, S^2(\mathcal{E})\otimes[-4H_\xi])=11$. We take a rational section $s_2:=\{(U_i\cap V_j, \sigma_0^2/\xi_0^2\cdot\xi_i^2/\sigma_j^2)\}_{i,j}$ of $2H_\sigma-2H_\xi$, where U_i and V_j are the same as in (2.23.3). Let $f_1, \cdots, f_{11}$ be rational functions on P such that

$$f_1=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{30}}{\xi_0^2},\quad f_2=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{10}^2}{\xi_0^2},\quad f_3=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{20}}{\xi_0^2},\quad f_4=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{30}}{\xi_0^2},$$

$$f_5=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{31}}{\xi_0^2},\quad f_6=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{20}^2}{\xi_0^2},\quad f_7=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{30}}{\xi_0^2},\quad f_8=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{31}}{\xi_0^2},$$

$$f_9=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{30}^2}{\xi_0^2},\quad f_{10}=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{30}\sigma_{31}}{\xi_0^2},\quad f_{11}=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{30}\sigma_{32}}{\xi_0^2}.$$

Then $H^0(P, 2H_\sigma-2H_\xi)\cong \boldsymbol{C}\langle f_1, \cdots, f_{11}\rangle$ and $\mathrm{Bs}|2H_\sigma-2H_\xi|=\boldsymbol{P}_\xi^1\times\{(1:0:\cdots:0)\}$. For any $f=\sum_{i=1}^{11}c_if_i$ with $c_1\neq 0$, $\mathrm{div}(f\cdot s_2)$ is nonsingular along $\mathrm{Bs}|2H_\sigma-2H_\xi|$, thus a general member M of $|2H_\sigma-2H_\xi|$ is smooth. Putting $L=H_M$, we obtain a polarized manifold (M, L) as desired. In this case, $|L-H_\xi|$ makes M a desingularization of a variety of degree six in $\boldsymbol{P}^7$.

(2.24.5) Even when $\mathcal{E}\cong\mathcal{O}(1, 1, 3, 3)$, we can show the existence of (M, L) similarly. We have $P\cong\{(\xi_0:\xi_1)\times(\sigma_0:\sigma_1:\sigma_{20}:\sigma_{21}:\sigma_{22}:\sigma_{30}:\sigma_{31}:\sigma_{32})\in \boldsymbol{P}_\xi^1\times \boldsymbol{P}_\sigma^7|\xi_0:\xi_1=\sigma_{20}:\sigma_{21}=\sigma_{21}:\sigma_{22}=\sigma_{30}:\sigma_{31}=\sigma_{31}:\sigma_{32}\}$ and $H^0(P, 2H_\sigma-2H_\xi)\cong \boldsymbol{C}\langle f_1, \cdots, f_{13}\rangle$, where

$$f_1=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{20}}{\xi_0^2},\quad f_2=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{30}}{\xi_0^2},\quad f_3=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_1\sigma_{20}}{\xi_0^2},\quad f_4=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_1\sigma_{30}}{\xi_0^2},$$

$$f_5=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{20}^2}{\xi_0^2},\quad f_6=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{21}}{\xi_0^2},\quad f_7=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{21}^2}{\xi_0^2},\quad f_8=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{30}}{\xi_0^2},$$

$$f_9=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{31}}{\xi_0^2},\quad f_{10}=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{21}\sigma_{31}}{\xi_0^2},\quad f_{11}=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{30}^2}{\xi_0^2},\quad f_{12}=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{30}\sigma_{31}}{\xi_0^2},$$

$$f_{13}=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{31}^2}{\xi_0^2}.$$

Since $\mathrm{Bs}|2H_\sigma-2H_\xi|=\{\sigma_{20}=\sigma_{21}=\cdots=\sigma_{32}=0 \text{ in } P\}$, if we take $f=\sum_{i=1}^{13}c_if_i$ with $c_1c_4-c_2c_3\neq 0$, then $\mathrm{div}(f\cdot s_2)$ is nonsingular along $\mathrm{Bs}|2H_\sigma-2H_\xi|$. Thus a general member M of $|2H_\sigma-2H_\xi|$ is smooth. Putting $L=H_M$, we obtain a polarized manifold (M, L) as desired, and $|L-H_\xi|$ makes M a desingularization of a variety of degree six in $\boldsymbol{P}^7$.

(2.25) Summarizing the results above, we obtain the following.

THEOREM. *Let (M, L) be a polarized manifold of the type* (1.2). *Then $g(C)$, the genus of C, is* 0 *or* 1, *$\mathcal{E}:=\Phi_*\mathcal{O}_M[L]$ is a locally free sheaf on C, $M\in |2H(\mathcal{E})+B_{P(\mathcal{E})}|$ for some line bundle B on C, and $L=[H(\mathcal{E})]_M$. Putting $d=L^n$, $e=c_1(\mathcal{E})$, and $b=\deg B$, we have the following results.*

When $g(C)=1$, we have $1\leq d\leq 6$, $e=d-2$, $b=4-d$, and

(i) *if $d=1$ or 2, then $\mathcal{E}$ is not ample;*

(ii) *if $d=3$ or 4, then $\mathcal{E}$ is ample as long as it is indecomposable;*

(iii) *if $d=5$ or 6, then $\mathcal{E}$ is ample.*

When g(C) we have $C\cong \boldsymbol{P}^1_\xi$, $1\leqq d\leqq 12$, $e=d-4$, $b=8-d$, $M\in|2H(\mathcal{E})+bH_\xi|$, *and their lists are in the table below.*

d	$\mathcal{E}$	(M, L)
1	$\mathcal{O}(-3, 0, 0, 0)$ $\mathcal{O}(-3, 0, 0, 0, 0)$ $\mathcal{O}(-2, -1, 0, 0)$ $\mathcal{O}(-2, -1, 0, 0, 0)$ $\mathcal{O}(-1, -1, -1, 0)$ $\mathcal{O}(-1, -1, -1, 0, 0)$	The existence is uncertain.
2	$\mathcal{O}(-2, 0, 0, 0)$ $\mathcal{O}(-2, 0, 0, 0, 0)$ $\mathcal{O}(-1, -1, 0, 0)$ $\mathcal{O}(-1, -1, 0, 0, 0)$	The existence is uncertain.
3	$\mathcal{O}(-2, 0, 0, 1)$ $\mathcal{O}(-2, 0, 0, 0, 1)$ $\mathcal{O}(-1, -1, 0, 1)$ $\mathcal{O}(-1, -1, 0, 0, 1)$ $\mathcal{O}(-1, 0, 0, 0)$ $\mathcal{O}(-1, 0, 0, 0, 0)$	The existence is uncertain.
4	$\mathcal{O}(-1, 0, 0, 1)$ $\mathcal{O}(-1, 0, 0, 0, 1)$ $\mathcal{O}(0, 0, 0, 0)$ $\mathcal{O}(0, 0, 0, 0, 0)$	The existence is uncertain. The existence is uncertain. $\|L\|$ makes M a quadruple covering of $\boldsymbol{P}^3$. $\|L\|$ makes M a quadruple covering of $\boldsymbol{P}^4$.
5	$\mathcal{O}(-1, 0, 0, 2)$ $\mathcal{O}(-1, 0, 1, 1)$ $\mathcal{O}(-1, 0, 0, 1, 1)$ $\mathcal{O}(0, 0, 0, 1)$ $\mathcal{O}(0, 0, 0, 0, 1)$	$\mathrm{Bs}\|L\|$ is a point. $\mathrm{Bs}\|L\|$ is a point. $\mathrm{Bs}\|L\|$ is a point. $\|L\|$ makes M the normalization of a hypersurface of degree five in $\boldsymbol{P}^4$. $\|L\|$ makes M the normalization of a hypersurface of degree five in $\boldsymbol{P}^5$.

d	$\mathcal{E}$	(M, L)
6	$\mathcal{O}(-1, 1, 1, 1)$	M is a double covering of $\boldsymbol{P}^1_\xi \times \boldsymbol{P}^2_\sigma$ with branch locus being a smooth divisor of bidegree (4,2). $L=[H_\xi+H_\sigma]_M$.
	$\mathcal{O}(0, 0, 1, 1)$	Exist.
	$\mathcal{O}(0, 0, 0, 1, 1)$	Exist.
	$\mathcal{O}(0, 0, 0, 2)$	Exist.
7	$\mathcal{O}(0, 0, 1, 2)$	Exist.
	$\mathcal{O}(0, 1, 1, 1)$	Exist.
	$\mathcal{O}(0, 0, 1, 1, 1)$	Exist.
8	$\mathcal{O}(0, 1, 1, 1, 1)$	M is a double covering of $\boldsymbol{P}^1_\xi \times \boldsymbol{P}^3_\sigma$ with branch locus being a smooth divisor of bidegree (2,2). $L=[H_\xi+H_\sigma]_M$.
	$\mathcal{O}(0, 1, 1, 2)$	M is a double covering of a divisor of bidegree (1, 1) on $\boldsymbol{P}^1_\xi \times \boldsymbol{P}^3_\sigma$. $L=[H_\xi+H_\sigma]_M$.
	$\mathcal{O}(1, 1, 1, 1)$	M is a smooth divisor of bidegree (2,2) on $\boldsymbol{P}^1_\xi \times \boldsymbol{P}^3_\sigma$. $L=[H_\xi+H_\sigma]_M$.
9	$\mathcal{O}(1, 1, 1, 1, 1)$	M is the blowing-up of $\boldsymbol{P}^4_\sigma$ with center being a complete intersection of two hyperquadrics. $L=[H_\xi+H_\sigma]_M$.
	$\mathcal{O}(1, 1, 1, 2)$	M is the strict transform of a smooth hypercubic in $\boldsymbol{P}^4_\sigma$ by the blowing-up of $\boldsymbol{P}^4_\sigma$ with center being a $\boldsymbol{P}^2$. $L=[H_\xi+H_\sigma]_M$.
10	$\mathcal{O}(1, 1, 1, 1, 1, 1)$	$M \cong \boldsymbol{P}^1_\xi \times Q$, where Q is a smooth hyperquadric in $\boldsymbol{P}^5_\sigma$. $L=[H_\xi+H_\sigma]_M$.
	$\mathcal{O}(1, 1, 1, 1, 2)$	M is the blowing-up of a hyperquadric in $\boldsymbol{P}^5_\sigma$ with center being a smooth quadric surface. $L=[H_\xi+H_\sigma]_M$.
	$\mathcal{O}(1, 1, 2, 2)$	M is a desingularization of a complete intersection of two hyperquadrics in $\boldsymbol{P}^5_\sigma$. $L=[H_\xi+H_\sigma]_M$.
	$\mathcal{O}(1, 1, 1, 3)$	M is a desingularization of a complete intersection of two hyperquadrics in $\boldsymbol{P}^5_\sigma$. $L=[H_\xi+H_\sigma]_M$.
11	$\mathcal{O}(1, 2, 2, 2)$	$\lvert L-H_\xi \rvert$ makes M a desingularization of a three-dimensional variety of degree five in $\boldsymbol{P}^6$.
12	$\mathcal{O}(1, 1, 3, 3)$	$\lvert L-H_\xi \rvert$ makes M a desingularization of a three-dimensional variety of degree six in $\boldsymbol{P}^7$.
	$\mathcal{O}(1, 2, 2, 3)$	$\lvert L-H_\xi \rvert$ makes M a desingularization of a three-dimensional variety of degree six in $\boldsymbol{P}^7$.
	$\mathcal{O}(2, 2, 2, 2)$	$M \cong \boldsymbol{P}^1_\xi \times \boldsymbol{P}^1_\mu \times \boldsymbol{P}^1_\lambda$ and $L=2H_\xi+H_\mu+H_\lambda$.

§ 3. The case of a Veronese fibration over a curve

In this section we study the case (1.3), using the argument in [F ; (13.10)].

(3.1) Put $H=K+2L$, then $\mathcal{E}:=\Phi_*\mathcal{O}_M[H]$ is a locally free sheaf of rank three on C and (M, H) is the scroll of $\mathcal{E}$. We have $L=2H+\Phi^*B$ for some $B\in\mathrm{Pic}(C)$. Similarly as before, we put $d=L^3$, $e=c_1(\mathcal{E})$, $b=\deg B$ and denote by $g(C)$ the genus of C. Then $e\geqq 0$, $e+b=1$, and $d=8e+12b$. By the canonical bundle formula, we obtain that $K^C+\det\mathcal{E}+2B=0$, hence $2g(C)-2+e+2b=0$. From these results, $(e, d)=(0, 12)$ or $(2, 4)$.

(3.2) When $(e, d)=(0, 12)$, we have $b=1$ and $g(C)=0$, hence $C\cong \boldsymbol{P}^1$, $B=\mathcal{O}(1)$, and $\mathcal{E}\cong\mathcal{O}(e_1)\oplus\mathcal{O}(e_2)\oplus\mathcal{O}(e_3)$ for $e_1, e_2, e_3\in\boldsymbol{Z}$. For each $1\leqq i\leqq 3$, a natural surjection $\mathcal{E}\to\mathcal{O}(e_i)$ gives a section Z_i of Φ and $H_{Z_i}=\mathcal{O}(e_i)$. Since $e_1+e_2+e_3=e=0$ and $L_{Z_i}=\mathcal{O}(2e_i+1)$ is ample, we have $e_1=e_2=e_3=0$ and $\mathcal{E}\cong\mathcal{O}_C^{\oplus 3}$, thus $M\cong\boldsymbol{P}^1_\xi\times\boldsymbol{P}^2_\sigma$ and $L=H_\xi+2H_\sigma$.

(3.3) When $(e, d)=(2, 4)$, we have $b=-1$ and $g(C)=1$. Hence C is an elliptic curve and $\det\mathcal{E}+2B=0$ since $K^C=\mathcal{O}_C$. Let Q be any quotient bundle of $\mathcal{E}$. If rank $Q=1$, then $Z:=\boldsymbol{P}_C(Q)$ is a section of Φ and $HZ=c_1(Q)$. Then $c_1(Q)\geqq 1$ since $0<LZ=2c_1(Q)-1$. If rank $Q=2$, then $D:=\boldsymbol{P}_C(Q)\in|H-\Phi^*\mathcal{F}|$, where $\mathcal{F}$ is the kernel of $\mathcal{E}\to Q$. Since $0<L^2D=4(1-c_1(\mathcal{F}))$, we have $c_1(Q)=e-c_1(\mathcal{F})\geqq 2$. In both cases we have $(\mathrm{rank}\ Q)\cdot c_1(\mathcal{E})<(\mathrm{rank}\mathcal{E})\cdot c_1(Q)$, hence $\mathcal{E}$ is stable. Conversely, let $\mathcal{E}$ be a semistable vector bundle on C with rank $\mathcal{E}=3$ and $c_1(\mathcal{E})=2$. We put $M=\boldsymbol{P}_C(\mathcal{E})$, $H=H(\mathcal{E})$ and let $\Phi: M\to C$ be the bundle map. By the semistability criterion in [Mi ; (3.1)], $3H-\Phi^*(\det\mathcal{E})$ is nef. Since C is an elliptic curve, we can find some $B\in\mathrm{Pic}(C)$ satisfying $\det\mathcal{E}+2B=0$. Then $3(2H+\Phi^*B)=2(3H+\Phi^*(2B))-\Phi^*B$ is ample. Hence $L:=2H+\Phi^*B$ is ample and (M, L) is a polarized manifold of the type (1.3).

(3.4) Summing up, we obtain the following theorem.

THEOREM. *Let (M, L) be a polarized of the type* (1.3). *We put $d=L^3$ and denote by $g(C)$ the genus of C. Then (M, L) is one of the following two types.*

(I) *$g(C)=0$, heuce $C\cong\boldsymbol{P}^1_\xi$; $d=12$, $M\cong\boldsymbol{P}^1_\xi\times\boldsymbol{P}^2_\sigma$, and $L=H_\xi+2H_\sigma$.*

(II) *$g(C)=1$ and $M\cong\boldsymbol{P}_C(\mathcal{E})$, where $\mathcal{E}:=\Phi_*\mathcal{O}_M[K+2L]$ is a stable vector bundle of rank three on C with $c_1(\mathcal{E})=2$; $d=4$ and $L=2H(\mathcal{E})+\Phi^*B$, where $B\in$ Pic(C) with* $\det\mathcal{E}+2B=0$.

REFERENCES

[BeLP] M. BELTRAMETTI, A. LANTERI AND M. PALLESCHI, Algebraic surfaces containing an ample divisor of arithmetic genus two, Ark. Mat., 25 (1987), 189-210.

[BiLL] A. BIANCOFIORE, A. LANTERI AND E. L. LIVORNI, Ample and spanned vector bundles of sectional genera three, Math. Ann., **291** (1991), 87-101.

[F] T. FUJITA, Classification Theories of Polarized Varieties, London Math. Soc. Lecture Note Ser., **155**, Cambridge Univ. Press, 1990.

[F1] T. FUJITA, On polarized manifolds whose adjoint bundles are not semipositive, Algebraic Geometry Sendai 1985, Adv. Stud. Pure Math., **10**, Kinokuniya, 1987, 167-178.

[F2] T. FUJITA, Classification of polarized manifolds of sectional genus two, Algebraic Geometry and Commutative Algebra (Hijikata et al., eds.), Kinokuniya, 1987, 73-98.

[H1] R. HARTSHORNE, Ample vector bundles on curves, Nagoya Math. J., **43** (1971), 73-89.

[H2] R. HARTSHORNE, Algebraic Geometry, Graduate Texts in Math., **52**, Springer, 1977.

[I1] P. IONESCU, Embedded projective varieties of small invariants, Algebraic Geometry Bucharest 1982, Lect. Notes in Math., **1056**, Springer, 1984, 142-186.

[I2] P. IONESCU, Generarized adjunction and applications, Math. Proc. Cambridge Philos. Soc., **99** (1986), 457-472.

[Ma] H. MAEDA, On polarized surfaces of sectional genus three, Sci. Papers College Arts Sci. Univ. Tokyo, **37** (1987), 103-112.

[Mi] Y. MIYAOKA, The Chern classes and Kodaira dimension of a minimal variety, Algebraic Geometry Sendai 1985, Adv. Stud. Pure Math., **10**, Kinokuniya, 1987, 449-476.

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