S. OZAWA AND S. ROPPONGI
KODAI MATH. J.
18 (1995), 315—327

SINGULAR VARIATION OF DOMAINS AND CONTINUITY PROPERTY OF EIGENFUNCTION FOR SOME SEMI-LINEAR ELLIPTIC EQUATIONS

SHIN OZAWA AND SUSUMU ROPPONGI

1. Introduction

Let M be a bounded domain in $\boldsymbol{R}^3$ with smooth boundary ∂M. Let w be a fixed point in M. By $B(\varepsilon; w)$ we denote the ball of center w with radius ε. We remove $\overline{B(\varepsilon; w)}$ from M and we put $M_\varepsilon = M \backslash \overline{B(\varepsilon; w)}$. We write $B(\varepsilon; w) = B_\varepsilon$.

Fix $k \geqq 0$ and $p \in (1, 5)$. We put

$$\lambda(\varepsilon) = \inf_{X_\varepsilon}\left(\int_{M_\varepsilon} |\nabla u|^2 dx + k\int_{\partial M_\varepsilon} u^2 d\sigma\right), \tag{1.1$_\varepsilon$}$$

where

$$X_\varepsilon = \{u \in H^1(M_\varepsilon) : \|u\|_{L^{p+1}(M_\varepsilon)} = 1,\ u = 0 \text{ on } \partial M,\ u \geqq 0 \text{ in } M_\varepsilon\}.$$

Then, we know that there exists at least one solution u_ε which attains $(1.1)_\varepsilon$. It satisfies

$$\begin{aligned} &-\Delta u_\varepsilon = \lambda(\varepsilon) u_\varepsilon^p \quad \text{in} \quad M_\varepsilon \\ &\frac{\partial u_\varepsilon}{\partial \nu_x} + k u_\varepsilon = 0 \quad \text{on} \quad \partial B_\varepsilon \\ &u_\varepsilon = 0 \quad \text{on} \quad \partial M. \end{aligned} \tag{1.2}$$

Here $\partial/\partial\nu_x$ denotes the derivative along the exterior normal direction.

One of the main results of this paper is the following.

THEOREM 1. *Fix $p \in (1, 5)$. Then, there exists a constant C independent of ε such that*

$$\sup_{u_\varepsilon \in S_\varepsilon} \sup_{x \in M_\varepsilon} |u_\varepsilon(x)| \leqq C < +\infty,$$

where S_ε is the set of positive solutions of (1.2) which minimize $(1.1)_\varepsilon$.

Next we treat the asymptotic behaviours of $\lambda(\varepsilon)$ and positive solutions u_ε

Received January 25, 1994; revised August 19, 1994.

of (1.2) which minimize $(1.1)_\varepsilon$. We put

$$\lambda(0)=\inf_X \int_M |\nabla u|^2 dx\,, \tag{1.3}$$

where

$$X=\{u\in H_0^1(M)\,;\ \|u\|_{L^{p+1}(M)}=1,\quad u\geqq 0 \text{ in } M\}.$$

Then, there exists at least one solution u_0 which attains (1.3) and satisfies

$$\begin{aligned} -\Delta u_0&=\lambda(0)u_0{}^p \quad \text{in } M \\ u_0&=0 \quad \text{on } \partial M. \end{aligned} \tag{1.4}$$

We have the following theorems.

THEOREM 2. *Fix $p\in(1, 5)$. Then, there exists a constant C independent of ε such that*

$$|\lambda(\varepsilon)-\lambda(0)|\leqq C\varepsilon^{1/2}$$

holds for any sufficiently small $\varepsilon>0$.

THEOREM 3. *Fix $p\in(1, 5)$. Assume that the minimizer u_0 of (1.3) is unique. Then,*

$$\sup_{x\in M_\varepsilon} |u_\varepsilon(x)-u_0(x)| \longrightarrow 0 \qquad as \quad \varepsilon \longrightarrow 0$$

holds for any $u_\varepsilon\in S_\varepsilon$.

Remarks. When M is a bounded domain in $\boldsymbol{R}^2$, Theorem 1 is proved in Ozawa-Roppongi [10].

When M is a ball, the uniqueness of the minimizer of (1.3) is shown in Gidas, Ni, and Nirenberg [4]. See also Dancer [2]. On the other hand, we do not know whether the minimizer u_ε of $(1.1)_\varepsilon$ is unique or not in general and even in the case when M is a ball. When the Robin boundary condition on ∂B_ε in (1.2) is replaced by the zero Dirichlet condition, the uniqueness of u_ε is proved in Dancer [3] for any sufficiently small $\varepsilon>0$ under the assumptions that the minimizer u_0 of (1.3) is unique, and that $Ker(\Delta+p\lambda(0)u_0^{p-1})=\{0\}$.

For related topics, the reader may be referred to Lin [5], Osawa-Ozawa [6], Ozawa [7], [8], [9].

Section 2 contains preliminary material. We give the proof of Theorems 1, 2 and 3 in sections 3, 4 and 5, respectively. In Appendix we give an extension lemma for a function on M_ε to M. We will follow the established practice of using the same letter C (with or without subscript) to denote different constants independent of ε.

2. Preliminary lemmas

LEMMA 2.1. *Fix $\xi\in(0, 1)$ and $\alpha\in H^{\xi}(S^2)$. Then, there exists at least one solution of*

$$\Delta v_\varepsilon(x)=0 \qquad x\in \boldsymbol{R}^3\backslash\bar{B}_\varepsilon \tag{2.1}$$

$$\frac{\partial v_\varepsilon}{\partial \nu_x}(x)+kv_\varepsilon(x)=\alpha(\omega), \qquad x=w+\varepsilon\omega\in\partial B_\varepsilon\ (\omega\in S^2) \tag{2.2}$$

satisfying

$$\max_{x\in M_\varepsilon}|v_\varepsilon(x)|\leqq C\varepsilon\|\alpha\|_{H^{\xi}(S^2)}. \tag{2.3}$$

Proof. Without loss of generality, we may assume that $w=0$. We put $x=r\omega$ ($\omega\in S^2$) and $\omega=(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ $(0\leqq\theta<\pi, 0\leqq\varphi<2\pi)$. Let $P_n(z)$ be the Legendre polynomial and $P_n^m(z)$ be the associated Legendre function, that is,

$$P_n^m(z)=(1-z^2)^{m/2}\frac{d^m}{dz^m}P_n(z), \qquad (|z|\leqq 1,\ 0\leqq m\leqq n).$$

It is well known that $\{P_n^m(\cos\theta)\cos m\varphi,\ P_n^m(\cos\theta)\sin m\varphi\ ;\ 0\leqq m\leqq n\}_{n=0}^\infty$ is a complete orthogonal system of $L^2(S^2)$ consisting of eigenfunction of the Laplace-Beltrami operator Δ_{S^2} whose eigenvalues are $-n(n+1)$, $n=0, 1, 2, \cdots$.

Furthermore, we have the Parseval relation

$$\sum_{n=0}^\infty(2n+1)^{-1}\Big(a_{n,0}^2+\sum_{m=1}^n((n+m)!/2(n-m)!)(a_{n,m}^2+b_{n,m}^2)\Big) \tag{2.4}$$
$$=(4\pi)^{-1}\|\alpha\|_{L^2(S^2)}^2$$

for $\alpha(\omega)$ with the Fourier expansion

$$\alpha(\omega)=\sum_{n=0}^\infty Y_n(\theta, \varphi),$$

where

$$Y_n(\theta, \varphi)=\sum_{n=0}^\infty(a_{n,m}\cos m\varphi+b_{n,m}\sin m\varphi)P_n^m(\cos\theta). \tag{2.5}$$

We put

$$v_\varepsilon(x)=\sum_{n=0}^\infty\Big(\sum_{m=0}^n(s_{n,m}\cos m\varphi+t_{n,m}\sin m\varphi)P_n^m(\cos\theta)\Big)r^{-(n+1)}.$$

We see that

$$\frac{\partial v_\varepsilon}{\partial \nu_x}(x)+kv_\varepsilon(x)_{|x\in\partial B_\varepsilon}=\alpha(\omega)$$

$$=\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}(a_{n,m}\cos m\varphi+b_{n,m}\sin m\varphi)P_n^m(\cos\theta)\Big)$$

implies

$$\begin{cases} a_{n,m}=\varepsilon^{-(n+2)}(n+1+k\varepsilon)s_{n,m} \\ b_{n,m}=\varepsilon^{-(n+2)}(n+1+k\varepsilon)t_{n,m} \end{cases}$$

for $0\leqq m\leqq n$, $n\geqq 0$. Then we have

$$v_\varepsilon(x)=\varepsilon\sum_{n=0}^{\infty}(\varepsilon/r)^{n+1}(n+1+k\varepsilon)^{-1}Y_n(\theta,\varphi), \tag{2.6}$$

and it satisfies (2.1) and (2.2). By (2.5), and by using the Schwarz inequality and the relation

$$P_n(\cos\theta)^2+\sum_{m=1}^{n}(2(n-m)!/(n+m)!)P_n^m(\cos\theta)^2=1,$$

we see that

$$|Y_n(\theta,\varphi)|^2\leqq a_{n,m}^2+\sum_{m=1}^{n}(n+m)!/2(n-m)!)(a_{n,m}^2+b_{n,m}^2). \tag{2.7}$$

From (2.6) and (2.7), we have

$$|v_\varepsilon(x)|\leqq C\varepsilon^2r^{-1}\Big(\sum_{n=0}^{\infty}(\varepsilon/r)^{2n}(n+1)^{-1-\xi}\Big)^{1/2}K(\xi)^{1/2}\leqq C_\xi\varepsilon K(\xi)^{1/2} \tag{2.8}$$

for $x\in M_\varepsilon$, $\xi\in(0,1)$, where

$$K(\xi)=\sum_{n=0}^{\infty}(2n+1)^{-1}n^{\xi}\Big(a_{n,0}^2+\sum_{m=1}^{n}((n+m)!/2(n-m)!)(a_{n,m}^2+b_{n,m}^2)\Big).$$

By (2.4) and observing that j-th eigenvalue of $-\Delta_{S^2}\sim Cj$ as $j\to\infty$, we can easily see that $K(\xi)^{1/2}$ is equivalent to the norm $\|\alpha\|_{H^\xi(S^2)}$. Thus we get (2.3) from (2.8). q.e.d.

By Lemma 2.1 and the same repeating construction of the function $v_\varepsilon^{(n)}$ as in Ozawa [7, Proposition 1, pp. 260-262], we have the following.

LEMMA 2.2. *Fix* $\xi\in(0,1)$. *Assume that* $u_\varepsilon\in C^\infty(M_\varepsilon)$ *is harmonic in* M_ε, $u_\varepsilon=0$ *on* ∂M *and*

$$\frac{\partial u_\varepsilon}{\partial\nu_x}(x)+ku_\varepsilon(x)=L(\omega)\qquad x=w+\varepsilon\omega\in\partial B_\varepsilon(\omega\in S^2).$$

Then,

$$\|u_\varepsilon\|_{L^\infty(M_\varepsilon)}\leqq C\varepsilon\|L\|_{H^\xi(S^2)}$$

holds.

Next we want to show the following.

LEMMA 2.3. *Fix* $q\in[3/2, 2]$ *and let* $\xi=2-(3/q)$. *Then,*

$$\|u(\varepsilon\cdot)\|_{H^{\xi}(S^2)}\leqq C_q\varepsilon^{1-(3/q)}\|u\|_{W^{1,q}(M)} \tag{2.9}$$

holds for any $u\in W^{1,q}(M)$.

Here $\|u(\varepsilon\cdot)\|_{H^{\xi}(S^2)}$ *denotes the* $H^{\xi}(S^2)$*-norm of the function* $u(\varepsilon\omega)$ $(\omega\in S^2)$.

Proof. Fix $q\in[3/2, 2]$ and let ξ be as above. Then, the Sobolev embedding: $W^{1,q}(\boldsymbol{R}^3)\subset W^{(1/2)+\xi,2}(\boldsymbol{R}^3)$ holds (see, for example, Adams [1, Theorem 7.58, p. 218]). Since the trace operator: $W^{(1/2)+\xi,2}(\boldsymbol{R}^3)\to H^{\xi}(S^2)$ is continuous,

$$\|v\|_{H^{\xi}(S^2)}\leqq C\|v\|_{W^{1,q}(\boldsymbol{R}^3)} \tag{2.10}$$

holds for any $v\in W^{1,q}(\boldsymbol{R}^3)$.

We take an arbitrary $u\in W^{1,q}(M)$ and take $\varphi\in C^{\infty}(\boldsymbol{R}^3)$ satisfying $0\leqq\varphi\leqq 1$, $\varphi\equiv 1$ on $B_{2\varepsilon}$, $\varphi\equiv 0$ on $\boldsymbol{R}^3\backslash\bar{B}_{3\varepsilon}$ and $|\nabla\varphi|\leqq C\varepsilon^{-1}$. We put $v_{\varepsilon}(x)=u(\varepsilon x)\varphi(\varepsilon x)$. Then, $v_{\varepsilon}\in W_0^{1,q}(B_3)$. We extend v_{ε} to $\boldsymbol{R}^3$ by defining $v_{\varepsilon}=0$ on $\boldsymbol{R}^3\backslash B_3$. Then, $v_3\in W^{1,q}(\boldsymbol{R}^3)$ and

$$\begin{aligned}\|v_{\varepsilon}\|_{L^q(\boldsymbol{R}^3)}^q&=\int_{B_3}|u(\varepsilon x)\varphi(\varepsilon x)|^q dx\\&=\varepsilon^{-3}\int_{B_{3\varepsilon}}|u(y)\varphi(y)|^q dy\leqq\varepsilon^{-3}\int_{B_{3\varepsilon}}|u(y)|^q dy\,.\end{aligned}$$

Here we used the transformation of co-ordinates: $y=\varepsilon x$. Let $r=3q/(3-q)$. Then, by the Sobolev embedding, $\|u\|_{L^r(M)}\leqq C\|u\|_{W^{1,q}(M)}$ holds. Using Hölder's inequality, we have

$$\begin{aligned}\int_{B_{3\varepsilon}}|u(y)|^q dy&\leqq\Big(\int_{B_{3\varepsilon}}|u(y)|^r dy\Big)^{q/r}\Big(\int_{B_{3\varepsilon}}1^{3/q}dy\Big)^{q/3}\\&\leqq C\varepsilon^q\|u\|_{L^r(M)}^q\leqq C\varepsilon^q\|u\|_{W^{1,q}(M)}^q\,.\end{aligned} \tag{2.11}$$

Therefore,

$$\|v_{\varepsilon}\|_{L^q(\boldsymbol{R}^3)}^q\leqq C\varepsilon^{q-3}\|u\|_{W^{1,q}(M)}^q \tag{2.12}$$

holds.

On the other hand, $|\nabla v_{\varepsilon}(x)|=\varepsilon\varphi(\varepsilon x)(\nabla u)(\varepsilon x)+\varepsilon u(\varepsilon x)(\nabla\varphi)(\varepsilon x)$ and $|\nabla\varphi|\leqq C\varepsilon^{-1}$,

$$\begin{aligned}\|\nabla v_{\varepsilon}\|_{L^q(\boldsymbol{R}^3)}^q&\leqq C\varepsilon^q\int_{B_3}|(\nabla u)(\varepsilon x)|^q dx+C\int_{B_3}|u(\varepsilon x)|^q dx\\&\leqq C\varepsilon^{q-3}\int_{B_{3\varepsilon}}|(\nabla u)(y)|^q dy+C\varepsilon^{-3}\int_{B_{3\varepsilon}}|u(y)|^q dy\end{aligned}$$

hold. Using (2.11) in the second term of the right hand side of the above inequality, we have

(2.13) $$\|\nabla v_\varepsilon\|^q_{L^q(R^3)}\leqq C\varepsilon^{q-3}\|\nabla u\|^q_{L^q(M)}+C\varepsilon^{q-3}\|u\|^q_{W^{1,q}(M)}\leqq C\varepsilon^{q-3}\|v\|^q_{W^{1,q}(M)}.$$

From (2.12) and (2.13),

(2.14) $$\|v_\varepsilon\|_{W^{1,q}(R^3)}\leqq C\varepsilon^{1-(3/q)}$$

holds.

Notice that $v_\varepsilon(x)=u(\varepsilon x)$ for $x\in S^2$. Therefore, by (2.14) and using (2.10) with $v=v_\varepsilon$, we can get (2.9). q.e.d.

3. Proof of Theorem 1

Let $G_\varepsilon(x, y)$ be the Green function of the Laplacian in M_ε satisfying

$$-\Delta_x G_\varepsilon(x, y)=\delta(x-y), \qquad x, y\in M_\varepsilon$$

$$G_\varepsilon(x, y)=0, \qquad x\in\partial M, y\in M_\varepsilon$$

$$\frac{\partial}{\partial\nu_x}G_\varepsilon(x, y)+kG_\varepsilon(x, y)=0, \qquad x\in\partial B_\varepsilon, y\in M_\varepsilon,$$

Let $G(x, y)$ be the Green function of the Laplacian in M under the zero Dirichlet condition on ∂M. We put

$$(Gf)(x)=\int_M G(x, y)f(y)dy,$$

$$(G_\varepsilon f)(x)=\int_{M_\varepsilon} G_\varepsilon(x, y)f(y)dy.$$

For the sake of simplicity we write $\|\cdot\|_{L^r(M)}$, $\|\cdot\|_{L^r(M_\varepsilon)}$ as $\|\cdot\|_r$, $\|\cdot\|_{r,\varepsilon}$, respectively for $r\in[1, \infty]$.

We have the following.

LEMMA 3.1. *Fix $q\in(3/2, 2]$ and $f\in L^q(M_\varepsilon)$. Then,*

(3.1) $$\|G_\varepsilon f-G\tilde{f}\|_{\infty,\varepsilon}\leqq C\varepsilon^{2-(3/q)}\|f\|_{q,\varepsilon}$$

holds. Here $\tilde{f}$ denotes the extension of f to M in Appendix of this paper.

Proof. Without loss of generality we may assume that $w=0$. We put $v_\varepsilon(x)=(G_\varepsilon f-G\tilde{f})(x)$ for $x\in M_\varepsilon$. Then, $\Delta v_\varepsilon=0$ in M_ε, $v_\varepsilon=0$ on ∂M and

$$\Big(\frac{\partial v_\varepsilon}{\partial\nu_x}+kv_\varepsilon\Big)(x)=-\Big(\frac{\partial}{\partial\nu_x}G\tilde{f}+kG\tilde{f}\Big)(\varepsilon\omega) \qquad x=\varepsilon\omega\in\partial B_\varepsilon(\omega\in S^2).$$

Let ξ be as in Lemma 2.3. Then $\xi\in(0, 1)$. Thus, by Lemmas 2.2 and 2.3,

(3.2) $$\|v_\varepsilon\|_{\infty,\varepsilon}\leqq C\varepsilon\Big\|\Big(\frac{\partial}{\partial\nu_x}G\tilde{f}+kG\tilde{f}\Big)(\varepsilon\cdot)\Big\|_{H^\xi(S^2)},$$

(3.3) $$\|G\tilde{f}(\varepsilon\cdot)\|_{H^{\xi}(S^2)}\leqq C\varepsilon^{1-(3/q)}\|G\tilde{f}\|_{W^{1,q}(M)}$$

and

(3.4) $$\|\nabla G\tilde{f}(\varepsilon\cdot)\|_{H^{\xi}(S^2)}\leqq C\varepsilon^{1-(3/q)}\|\nabla G\tilde{f}\|_{W^{1,q}(M)}$$

hold.

Since

$$\left(\frac{\partial}{\partial\nu_x}G\tilde{f}\right)(\varepsilon\omega)=\nu_x\cdot(\nabla G\tilde{f})(\varepsilon\omega)=-\omega\cdot(\nabla G\tilde{f})(\varepsilon\omega)\quad x=\varepsilon\omega\in\partial B_\varepsilon\ (\omega\in S^2),$$

$$\left|\left(\frac{\partial}{\partial\nu_x}G\tilde{f}\right)(\varepsilon\omega)\right|\leqq|(\nabla G\tilde{f})(\varepsilon\omega)|$$

and

$$\begin{aligned}&\left|\left(\frac{\partial}{\partial\nu_x}G\tilde{f}\right)(\varepsilon\omega)-\left(\frac{\partial}{\partial\nu_x}G\tilde{f}\right)(\varepsilon\omega')\right|\\&=|\omega\cdot(\nabla G\tilde{f})(\varepsilon\omega)-\omega'\cdot(\nabla G\tilde{f})(\varepsilon\omega')|\\&=|\omega\cdot\{(\nabla G\tilde{f})(\varepsilon\omega)-(\nabla G\tilde{f})(\varepsilon\omega')\}+(\omega-\omega')\cdot(\nabla G\tilde{f})(\varepsilon\omega')|\\&\leqq|(\nabla G\tilde{f})(\varepsilon\omega)-(\nabla G\tilde{f})(\varepsilon\omega')|+|\omega-\omega'|\,|(\nabla G\tilde{f})(\varepsilon\omega')|\end{aligned}$$

hold for any $\omega,\omega'\in S^2$. Thus we have

(3.5) $$\begin{aligned}&\left\|\left(\frac{\partial}{\partial\nu_x}G\tilde{f}\right)(\varepsilon\cdot)\right\|_{H^{\xi}(S^2)}^2\\&=\int_{S^2}\left|\left(\frac{\partial}{\partial\nu_x}G\tilde{f}\right)(\varepsilon\omega)\right|^2d\omega+\iint_{S^2\times S^2}\left|\left(\frac{\partial}{\partial\nu_x}G\tilde{f}\right)(\varepsilon\omega)-\left(\frac{\partial}{\partial\nu_x}G\tilde{f}\right)(\varepsilon\omega')\right|^2\\&\quad\cdot|\omega-\omega'|^{-2-2\xi}d\omega d\omega'\\&\leqq\int_{S^2}|(\nabla G\tilde{f})(\varepsilon\omega)|^2d\omega+2\iint_{S^2\times S^2}|(\nabla G\tilde{f})(\varepsilon\omega')|^2|\omega-\omega'|^{-2\xi}d\omega d\omega'\\&\quad+2\iint_{S^2\times S^2}|(\nabla G\tilde{f})(\varepsilon\omega)-(\nabla G\tilde{f})(\varepsilon\omega')|^2|\omega-\omega'|^{-2-2\xi}d\omega d\omega'.\end{aligned}$$

Since $\xi\in(0,1)$, we can easily see

(3.6) $$\int_{S^2}|\omega-\omega'|^{-2\xi}d\omega=\frac{4^{\xi}}{1-\xi}\pi$$

for any $\omega'\in S^2$. From (3.5) and (3.6), we have

(3.7) $$\begin{aligned}&\left\|\left(\frac{\partial}{\partial\nu_x}G\tilde{f}\right)(\varepsilon\cdot)\right\|_{H^{\xi}(S^2)}^2\\&\leqq C_\xi\left(\int_{S^2}|(\nabla G\tilde{f})(\varepsilon\omega)|^2d\omega+\iint_{S^2\times S^2}|(\nabla G\tilde{f})(\varepsilon\omega)-(\nabla G\tilde{f})(\varepsilon\omega')|^2|\omega-\omega'|^{-2-2\xi}d\omega d\omega'\right)\end{aligned}$$

$$=C_{\xi}\|(\nabla G\tilde{f})(\varepsilon\cdot)\|^2_{H^{\xi}(S^2)}.$$

Notice that $\|G\tilde{f}\|_{W^{2,q}(M)}\leqq C\|\tilde{f}\|_q\leqq C\|f\|_{q,\varepsilon}$ hold by *a priori* estimate and Lemma A in Appendix. Thus, by (3.3), (3.4) and (3.7),

$$\left\|\left(\frac{\partial}{\partial\nu_x}G\tilde{f}+kG\tilde{f}\right)(\varepsilon\cdot)\right\|_{H^{\xi}(S^2)} \tag{3.8}$$

$$\leqq C\varepsilon^{1-(3/q)}(\|\nabla G\tilde{f}\|_{W^{1,q}(M)}+k\|G\tilde{f}\|_{W^{1,q}(M)})$$

$$\leqq C\varepsilon^{1-(3/q)}\|G\tilde{f}\|_{W^{2,q}(M)}\leqq C\varepsilon^{1-(3/q)}\|f\|_{q,\varepsilon}$$

hold. From (3.2) and (3.8), we get (3.1). q.e.d.

Now we are in a position to prove Theorem 1. We take an arbitrary $u_\varepsilon\in S_\varepsilon$. We fix $q\in(3/2, 2]$. Then, by the Sobolev embedding: $W^{2,q}(M)\subset C(\overline{M})$ and *a priori* estimate,

$$\|G\tilde{u}^p_\varepsilon\|_{\infty,\varepsilon}\leqq C\|G\tilde{u}^p_\varepsilon\|_{W^{2,q}(M)}\leqq C\|\tilde{u}^p_\varepsilon\|_q \tag{3.9}$$

hold. Notice that $u_\varepsilon=\lambda(\varepsilon)G_\varepsilon u^p_\varepsilon$ and $0\leqq\lambda(\varepsilon)\leqq C$. Thus, by Lemma 3.1 and (3.9), we have

$$\|u_\varepsilon\|_{\infty,\varepsilon}\leqq\|\lambda(\varepsilon)(G_\varepsilon u^p_\varepsilon-G\tilde{u}^p_\varepsilon)+\lambda(\varepsilon)G\tilde{u}^p_\varepsilon\|_{\infty,\varepsilon} \tag{3.10}$$

$$\leqq C\|G_\varepsilon u^p_\varepsilon-G\tilde{u}^p_\varepsilon\|_{\infty,\varepsilon}+C\|G\tilde{u}^p_\varepsilon\|_{\infty,\varepsilon}$$

$$\leqq C(\varepsilon^{2-(3/q)}+1)\|\tilde{u}^p_\varepsilon\|_{q,\varepsilon}\leqq C\|u_\varepsilon\|^p_{pq,\varepsilon}.$$

At first we treat the case $p\in(1, 2)$. We put $q=(p+1)/p$. Then, $q\in(3/2, 2)$. We recall that $\|u_\varepsilon\|_{p+1,\varepsilon}=1$. Therefore, by (3.10), $\|u_\varepsilon\|_{\infty,\varepsilon}\leqq C\|u_\varepsilon\|^p_{p+1,\varepsilon}=C$ hold.

Next we treat the case $p\in[2, 5)$. Since $(p+1)/(p-1)>3/2$, we can take $q\in(3/2, 2]$ so that $(p+1)/(p-1)>q$. Notice that $q>3/2>(p+1)/p$. Thus we have the interpolation inequality:

$$\|u_\varepsilon\|_{pq,\varepsilon}\leqq\|u_\varepsilon\|^a_{p+1,\varepsilon}\cdot\|u_\varepsilon\|^{1-a}_{\infty,\varepsilon}, \tag{3.11}$$

where $a=(p+1)/(pq)$. By (3.10), (3.11) and the fact that $\|u_\varepsilon\|_{p+1,\varepsilon}=1$,

$$\|u_\varepsilon\|_{\infty,\varepsilon}\leqq C\|u_\varepsilon\|^p_{pq,\varepsilon}\leqq C\|u_\varepsilon\|^{\tau}_{\infty,\varepsilon}$$

hold for $\tau=(1-a)p=p-(p+1)/q$. Since $(p+1)>(p-1)>q$, $\tau<1$ holds. This implies $\|u_\varepsilon\|_{\infty,\varepsilon}\leqq C$.

Thus we get the desired Theorem 1.

Remark. Since $\|u_\varepsilon\|_{\infty,\varepsilon}\leqq C$ holds, we have the following by using Lemma 3.1 with $f=u^p_\varepsilon$ and $q=2$.

$$\|G_\varepsilon u^p_\varepsilon-G\tilde{u}^p_\varepsilon\|_{\infty,\varepsilon}\leqq C\varepsilon^{1/2} \tag{3.12}$$

4. Proof of Theorem 2

Since $u_0\cdot\|u_0\|_{p+1,\varepsilon}^{-1}\in X_\varepsilon$, we see

$$\lambda(\varepsilon)\leqq\|u_0\|_{p+1,\varepsilon}^{-2}\left(\int_{M_\varepsilon}|\nabla u_0|^2dx+k\int_{\partial B_\varepsilon}u_0^2d\sigma\right) \tag{4.1}$$

by $(1.1)_\varepsilon$. Notice that $u_0\in C^1(\overline{M})$, $\|u_0\|_{p+1,\varepsilon}=1$ and $\lambda(0)=\|\nabla u_0\|_2^2$. Therefore,

$$\|u_0\|_{p+1,\varepsilon}^{p+1}=1-\int_{B_\varepsilon}u_0^{p+1}dx=1+O(\varepsilon^3), \tag{4.2}$$

$$\int_{M_\varepsilon}|\nabla u_0|^2dx=\lambda(0)-\int_{B_\varepsilon}|\nabla u_0|^2dx=\lambda(0)+O(\varepsilon^3), \tag{4.3}$$

and

$$\int_{\partial B_\varepsilon}u_0^2d\sigma=O(\varepsilon^2) \tag{4.4}$$

hold. Summing up (4.1), (4.2), (4.3) and (4.4), we have the following.

$$\lambda(\varepsilon)\leqq\lambda(0)+C(k\varepsilon^2+\varepsilon^3) \tag{4.5}$$

We take $\psi_\varepsilon\in C^\infty(\boldsymbol{R}^3)$ satisfying $0\leqq\psi_\varepsilon\leqq1$, $\psi_\varepsilon=1$ on $\boldsymbol{R}^3\backslash B_{2\varepsilon}$, $\psi_\varepsilon=0$ on B_ε, and $|\nabla\psi_\varepsilon|\leqq C\varepsilon^{-1}$. Since $(\psi_\varepsilon u_\varepsilon)\cdot\|\psi_\varepsilon u_\varepsilon\|_{p+1}^{-1}\in X$, we see

$$\lambda(0)\leqq\|\psi_\varepsilon u_\varepsilon\|_{p+1}^{-2}\int_M|\nabla(\psi_\varepsilon u_\varepsilon)|^2dx \tag{4.6}$$

by (1.3). We recall that $\|u_\varepsilon\|_{p+1,\varepsilon}=1$. Thus, we have

$$\|\psi_\varepsilon u_\varepsilon\|_{p+1}^{p+1}=\int_{M_\varepsilon}u_\varepsilon^{p+1}dx+\int_{M_\varepsilon}(\psi_\varepsilon^{p+1}-1)u_\varepsilon^{p+1}dx=1+O(\varepsilon^3). \tag{4.7}$$

On the other hand, we see

$$\int_M|\nabla(\psi_\varepsilon u_\varepsilon)|^2dx=I_1(\varepsilon)+I_2(\varepsilon)+I_3(\varepsilon),$$

where

$$I_1(\varepsilon)=\int_M\psi_\varepsilon^2|\nabla u_\varepsilon|^2dx,$$

$$I_2(\varepsilon)=2\int_M\psi_\varepsilon u_\varepsilon\nabla\psi_\varepsilon\cdot\nabla u_\varepsilon dx,$$

$$I_3(\varepsilon)=\int_M u_\varepsilon^2|\nabla\psi_\varepsilon|^2dx.$$

We recall $(1.1)_\varepsilon$ and Theorem 1. Thus, we have

$$I_1(\varepsilon)\leqq\int_{M_\varepsilon}|\nabla u_\varepsilon|^2dx=\lambda(\varepsilon)-k\int_{\partial B_\varepsilon}u_\varepsilon^2d\sigma\leqq\lambda(\varepsilon)\leqq C,$$

$I_3(\varepsilon)\leqq C\varepsilon$ and $|I_2(\varepsilon)|\leqq\{I_1(\varepsilon)I_3(\varepsilon)\}^{1/2}\leqq C\varepsilon^{1/2}$. Summing up these facts, we have

$$\int_M|\nabla(\phi_\varepsilon u_\varepsilon)|^2dx=\lambda(\varepsilon)+O(\varepsilon^{1/2}). \tag{4.8}$$

From (4.6), (4.7) and (4.8), we see that $\lambda(0)\leqq\lambda(\varepsilon)+C\varepsilon^{1/2}$. Combining this with (4.5), we get Theorem 2.

5. Proof of Theorem 3

At first we want to show the following.

LEMMA 5.1. *Let $\tilde{u}_\varepsilon$ be an extension of u_ε to M as in Appendix. Assume that the minimizer u_0 of* (1.3) *is unique. Then,*

$$\tilde{u}_\varepsilon\longrightarrow u_0 \quad \textit{strongly in } H_0^1(M) \quad \textit{as} \quad \varepsilon\longrightarrow 0.$$

Proof. Since $\tilde{u}_\varepsilon=u_0$ a.e. in M_ε,

$$\|\tilde{u}_\varepsilon\|_{p+1}^{p+1}=\|u_\varepsilon\|_{p+1}^{p+1}+\int_{B_\varepsilon}u_\varepsilon{}^{p+1}dx=1+O(\varepsilon^3),$$

and

$$\int_{M_\varepsilon}|\nabla\tilde{u}_\varepsilon|^2dx=\int_{M_\varepsilon}|\nabla u_\varepsilon|^2dx+\int_{B_\varepsilon}|\nabla\tilde{u}_\varepsilon|^2dx$$

hold. By $(1.1)_\varepsilon$, Theorems 1 and 2, we see

$$\int_{M_\varepsilon}|\nabla u_\varepsilon|^2dx=\lambda(\varepsilon)-k\int_{\partial B_\varepsilon}u_\varepsilon{}^2d\sigma=\lambda(0)+O(\varepsilon^{1/2}).$$

On the other hand, $\|\nabla\tilde{u}_\varepsilon\|_{L^2(M)}^2\leqq C$ holds from Theorem 1 and (A.3) of Lemma A in Appendix. Thus, we have

$$\int_{B_\varepsilon}|\nabla\tilde{u}_\varepsilon|^2dx=o(1) \quad \text{as} \quad \varepsilon\longrightarrow 0.$$

Summing up these facts, we get the following.

$$\|\tilde{u}_\varepsilon\|_{p+1}\longrightarrow 1 \quad \text{as} \quad \varepsilon\longrightarrow 0 \tag{5.1}$$

$$\|\nabla\tilde{u}_\varepsilon\|_2^2\longrightarrow\|\nabla u_0\|_2^2=\lambda(0) \quad \text{as} \quad \varepsilon\longrightarrow 0 \tag{5.2}$$

Next we want to show the following.

$$\tilde{u}_\varepsilon\longrightarrow u_0 \quad \text{weakly in} \quad H_0^1(M) \quad \text{as} \quad \varepsilon\longrightarrow 0 \tag{5.3}$$

Assume that (5.3) does not hold. Then, there exist $\eta>0$, $F\in(H_0^1(M))^*$, and a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ satisfying $\varepsilon_n \downarrow 0 (n\to\infty)$ such that

$$|F(\tilde{u}_{\varepsilon_n})-F(u_0)|\geqq\eta \tag{5.4}$$

holds. Since $\{\tilde{u}_{\varepsilon_n}\}$ is bounded in $H_0^1(M)$, there exist a subsequence $\{\tilde{u}_{\varepsilon_{n'}}\}$ and $v\in H_0^1(M)$ satisfing

$$\begin{aligned} \tilde{u}_{\varepsilon_{n'}} &\longrightarrow v \quad \text{weakly in} \quad H_0^1(M) \\ \tilde{u}_{\varepsilon_{n'}} &\longrightarrow v \quad \text{strongly in} \quad L^{p+1}(M) \\ \tilde{u}_{\varepsilon_{n'}} &\longrightarrow v \quad \text{a.e. in} \quad M. \end{aligned} \tag{5.5}$$

Since $\tilde{u}_{\varepsilon_{n'}}\geqq 0$ a.e. in M, $v\geqq 0$ a.e. in M. From (5.1) and (5.2), $\|\tilde{u}_{\varepsilon_{n'}}\|_{p+1}\to 1$ and $\|\nabla\tilde{u}_{\varepsilon_{n'}}\|_2^2\to\|\nabla u_0\|_2^2=\lambda(0)$ as $n'\to\infty$. Thus, by (5.5), we have $\|v\|_{p+1}=1$ and

$$\|\nabla v\|_2\leqq\liminf_{n'\to\infty}\|\nabla\tilde{u}_{\varepsilon_{n'}}\|_2\leqq\|\nabla u_0\|_2=\lambda(0)^{1/2}.$$

Here we used the lower semi-continuity of the H_0^1-norm. Therefore we have $v\in X$ and $\lambda(0)\leqq\|\nabla v\|_2^2\leqq\|\nabla u_0\|_2^2\leqq\lambda(0)$. Hence v is a minimizer of (1.3). Thus, $v=u_0$ must hold by the assumption. Letting $n=n'\to\infty$ in (5.4), we have $0=|F(v)-F(u_0)|\geqq\eta$. This contradicts $\eta>0$. Therefore we get (5.3).

From (5.2), (5.3) and the uniform convexity of H_0^1, we get the desired result.

q.e.d.

Now we are in a position to prove Theorem 3. Since $u_\varepsilon=\lambda(\varepsilon)G_\varepsilon u_\varepsilon{}^p$ and $u_0=\lambda(0)Gu_0{}^p$ hold, we have

$$u_\varepsilon(x)-u_0(x)=\sum_{i=1}^{3}J_i(\varepsilon\,;\,x) \qquad x\in M_\varepsilon,$$

where

$$\begin{aligned} J_1(\varepsilon\,;\,x)&=\lambda(\varepsilon)(G_\varepsilon u_\varepsilon^p-G\tilde{u}_\varepsilon^p)(x), \\ J_2(\varepsilon\,;\,x)&=\lambda(\varepsilon)G(\tilde{u}_\varepsilon^p-u_0^p)(x), \\ J_3(\varepsilon\,;\,x)&=(\lambda(\varepsilon)-\lambda(0))Gu_0^p(x). \end{aligned}$$

We recall that $0<\lambda(\varepsilon)\leqq C$. Thus, by (3.12) and Theorem 2, $\|J_1(\varepsilon\,;\,\cdot)\|_{\infty,\varepsilon}\leqq C\varepsilon^{1/2}$ and $\|J_3(\varepsilon\,:\,\cdot)\|_{\infty,\varepsilon}\leqq C\varepsilon^{1/2}\|Gu_0^p\|_{\infty,\varepsilon}\leqq C\varepsilon^{1/2}$ hold. Furthermore, by the Sobolev embedding: $W^{2,6}(M)\subset C^0(M)$ and *a priori* estimate,

$$\|G(\tilde{u}_\varepsilon^p-u_0^p)\|_\infty\leqq C\|G(\tilde{u}_\varepsilon^p-u_0^p)\|_{W^{2,6}(M)}\leqq C\|\tilde{u}_\varepsilon^p-u_0^p\|_{L^6(M)}$$

hold. Thus, by using Theorem 1 and Lemma 5.1,

$$\begin{aligned} \|J_2(\varepsilon\,;\,\cdot)\|_{\infty,\varepsilon}&\leqq C\|\tilde{u}_\varepsilon^p-u_0^p\|_{L^6(M)} \\ &\leqq C\|\tilde{u}_\varepsilon-u_0\|_{L^6(M)}\sup_{\varepsilon>0}\max(\|u_0\|_\infty^{p-1},\ \|\tilde{u}_\varepsilon\|_\infty^{p-1}) \end{aligned}$$

$$\leqq C\|\tilde{u}_\varepsilon-u_0\|_{H_0^1(M)}=o(1).$$

Summing up these facts, we get the desired Theorem 3.

6. Appendix

Let M, M_ε be as in Introduction. Then we have the following.

LEMMA A. *For a function u on M_ε, there exists a function $\tilde{u}$ satisfying the following*:

(A.1) $$\tilde{u}(x)=u(x) \text{ a.e. in } M_\varepsilon,$$

(A.2) $$\|\tilde{u}\|_{L^s(M)}\leqq C\|u\|_{L^s(M_\varepsilon)} \qquad (1\leqq s\leqq\infty)$$

holds for any $u\in L^s(M_\varepsilon)$.

(A.3) $$\|\tilde{u}\|_{H^1(M)}\leqq C\|u\|_{H^1(M_\varepsilon)}+C\varepsilon^{1/2}\|u\|_{L^\infty(M_\varepsilon)}$$

holds for any $u\in H^1(M_\varepsilon)\cap L^\infty(M_\varepsilon)$.

Proof. Without loss of generality, we may assume that $w=0$. For a function u on M_ε, we put

$$\tilde{u}(x)=\begin{cases} u(x) & x\in M_\varepsilon \\ u(\varepsilon^2 x|x|^{-2})\eta_\varepsilon(x) & x\in B_\varepsilon, \end{cases}$$

where $\eta_\varepsilon(x)\in C^\infty(\boldsymbol{R}^3)$ satisfies $0\leqq\eta_\varepsilon\leqq 1$, $\eta_\varepsilon=1$ on $\boldsymbol{R}^3\backslash\bar{B}_{\varepsilon/2}$, $\eta_\varepsilon=0$ on $B_{\varepsilon/4}$ and $|\nabla\eta_\varepsilon|\leqq 8\varepsilon^{-1}$. Notice that both $\eta_\varepsilon(\varepsilon^2 x|x|^{-2})$ and $(\nabla\eta_\varepsilon)(\varepsilon^2 x|x|^{-2})$ vanish on $\boldsymbol{R}^3\backslash B_{4\varepsilon}$. Then, by using the Kelvin transformation of co-ordinates: $y=\varepsilon^2 x|x|^{-2}$, we have

$$\int_{B_\varepsilon}|\tilde{u}(x)|^s dx=\int_{\boldsymbol{R}^3\backslash B_\varepsilon}|u(y)|^s\eta_\varepsilon(\varepsilon^2 y|y|^{-2})^s(\varepsilon|y|^{-1})^6 dy$$

$$\leqq\int_{M_\varepsilon}|u(y)|^s dy \qquad (1\leqq s<\infty),$$

where the term $(\varepsilon|y|^{-1})^6$ comes from the absolute value of the determinant of the Jacobian of the Kelvin transformation. And we have

$$\int_{B_\varepsilon}|\nabla\tilde{u}(x)|^2 dx=C\int_{B_\varepsilon}|u(\varepsilon^2 x|x|^{-2})|^2|(\nabla\eta_\varepsilon)(x)|^2 dx$$

$$+C\int_{B_\varepsilon}(\varepsilon|x|^{-1})^4|(\nabla u)(\varepsilon^2 x|x|^{-2})|^2|\eta_\varepsilon(x)|^2 dx$$

$$\leqq C\varepsilon^4\int_{M_\varepsilon}|u(y)|^2|y|^{-6}dy+C\int_{M_\varepsilon}|(\nabla u)(y)|^2 dy$$

$$\leqq C\varepsilon\|u\|_{L^{\infty}(M_{\varepsilon})}^{2}+C\int_{M_{\varepsilon}}|(\nabla u)(y)|^{2}dy .$$

Thus we get the desired result. q.e.d.

REFERENCES

[1] R. ADAMS, Sobolev Spaces, Academic Press, New York, 1975.

[2] E. N. DANCER, The effect of domain shape on the number of positive solutions of certain nonlinear equations, J. Differential Equations, **74** (1988), 120-156.

[3] E. N. DANCER, On the positive solutions of some weakly nonlinear equations on annular region, Math. Z., **206** (1991), 551-562.

[4] B. GIDAS, W. M. NI AND L. NIRENBERG, Symmetry and related properties via the maximum principle, Comm. Math. Phys., **68** (1979), 209-243.

[5] S. S. LIN, Semilinear elliptic equation on singularly perturbed domains, Comm. Partial Differential Equations, **16** (1991), 617-645.

[6] T. OSAWA AND S. OZAWA, Singular variation of nonlinear eigenvalues, Proc. Japan Acad., **69A** (1993), 217-218.

[7] S. OZAWA, Spectra of domains with small spherical Neumann boundary, J. Fac. Sci. Univ. Tokyo Sect. IA, **30** (1983), 259-277.

[8] S. OZAWA, L^{∞} boundedness of nonlinear eigenfunction under singular variation of domains, preprint.

[9] S. OZAWA, Singular variation of the ground state eigenvalue for a sem-ilinear elliptic equation, Tôhoku Math. J., **45** (1993), 359-368.

[10] S. OZAWA AND S. ROPPONGI, Singular variation of domains and L^{∞} boundedness of eigenfunction for some semi-linear elliptic equations, Proc. Japan Acad., **70A** (1994), 67-70.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
TOKYO INSTITUTE OF TECHNOLOGY
OH-OKAYAMA, MEGURO-KU, TOKYO, 152
JAPAN